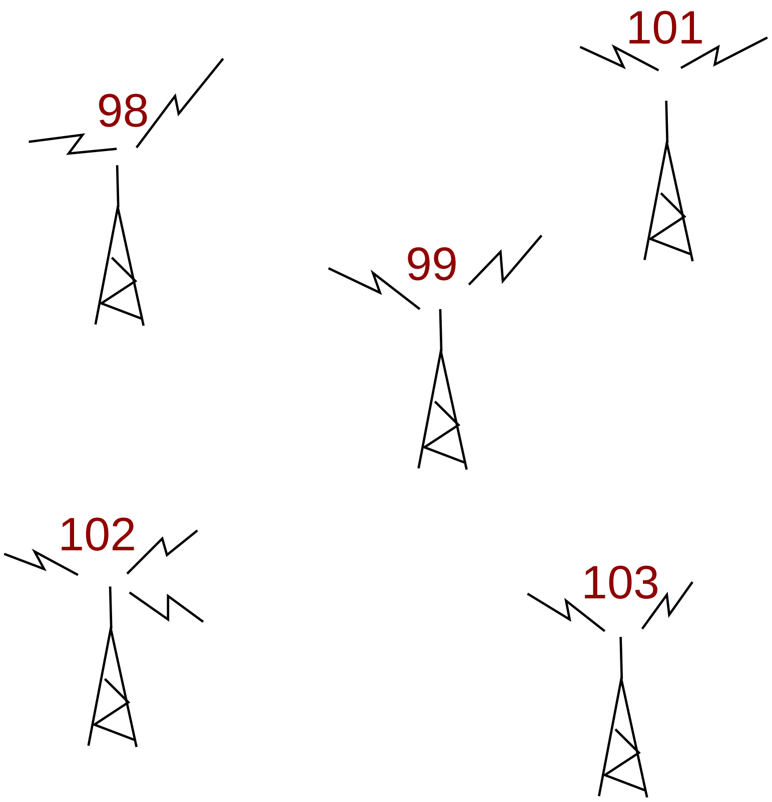
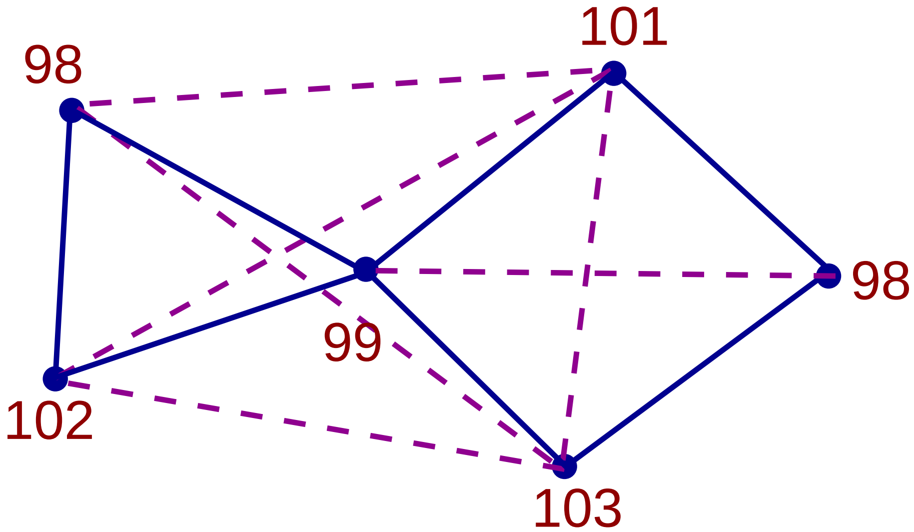
Frequency Channel Assignment on Planar Networks

Michael Molloy
Mohammad R. Salavatipour

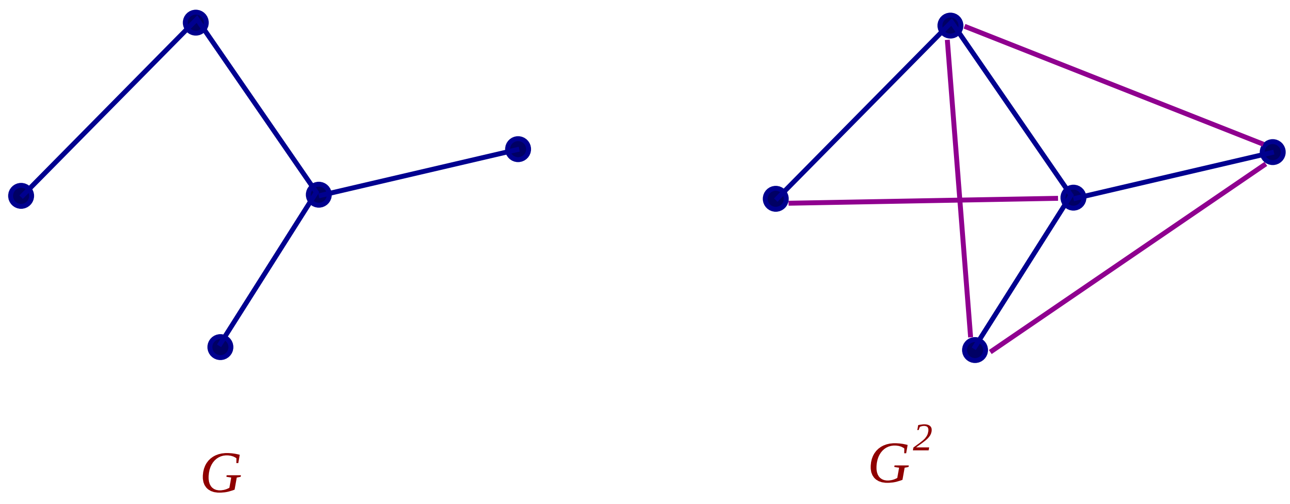
Department of Computer Science
University of Toronto


98
101
99
98
102
103


98
101
99
98
102
103

- $L(p,q)$-Labeling: A coloring C of G s.t. if:
  - u and v are adjacent: $|C(u) - C(v)| \geq p$
  - u and v at distance two: $|C(u) - C(v)| \geq q$
- λ_q^p: Min. number of colors needed in an $L(p,q)$-labeling.
- $\Delta(G)$: Max. degree of vertices of G.
- $N_G(v)$: set of neighbors of v.
- k-vertex, $\leq k$-vertex, $\geq k$-vertex: $d_G(v) = k$, $d_G(v) \leq k$ or $d_G(v) \geq k$.

- G^2: $(u, v) \in G^2 \iff \text{dist}_G(u, v) \leq 2$ and $u \neq v$.



- So $L(1,1)$-labeling $\equiv$ Coloring G^2, and

$$\lambda_1^1(G) = \chi(G^2)$$

Known upper bounds for $\chi(G^2)$ for planar G:

- $8\Delta(G) - 22$ [J93]
- $3\Delta(G) + 5$ [W96],
- $2\Delta(G) + 25$ [VM99],
- $\frac{9}{5}\Delta(G) + 1$, for large Δ [BBGLV01].

Best possible: $\lceil \frac{3}{2}\Delta(G) \rceil + 1$, [W77].

Theorem 1: $\chi(G^2) \leq \frac{5}{3}\Delta(G) + 61$ and a coloring can be found in $O(n^2)$

Proof: Complicated!

Theorem 2: $\chi(G^2) \leq \frac{21}{11}\Delta(G) + 50$.

Overview of the Algorithm

G is a given planar graph.

- Construct G' from G by removing a proper vertex v.

- Color G' recursively, and extend the coloring to v, by recoloring at most a constant number of vertices.

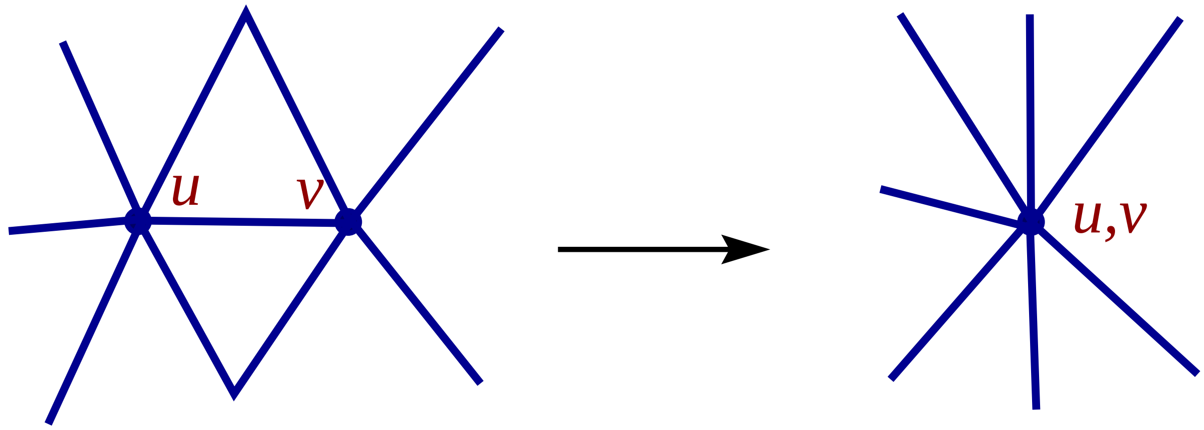
- Each iteration can be done in $O(n)$, where $n = |G|$.

- Overall running time is $O(n^2)$

- We assume that $\Delta(G) \geq 301$.

- We also assume that G is triangulated with min. degree at least 4.

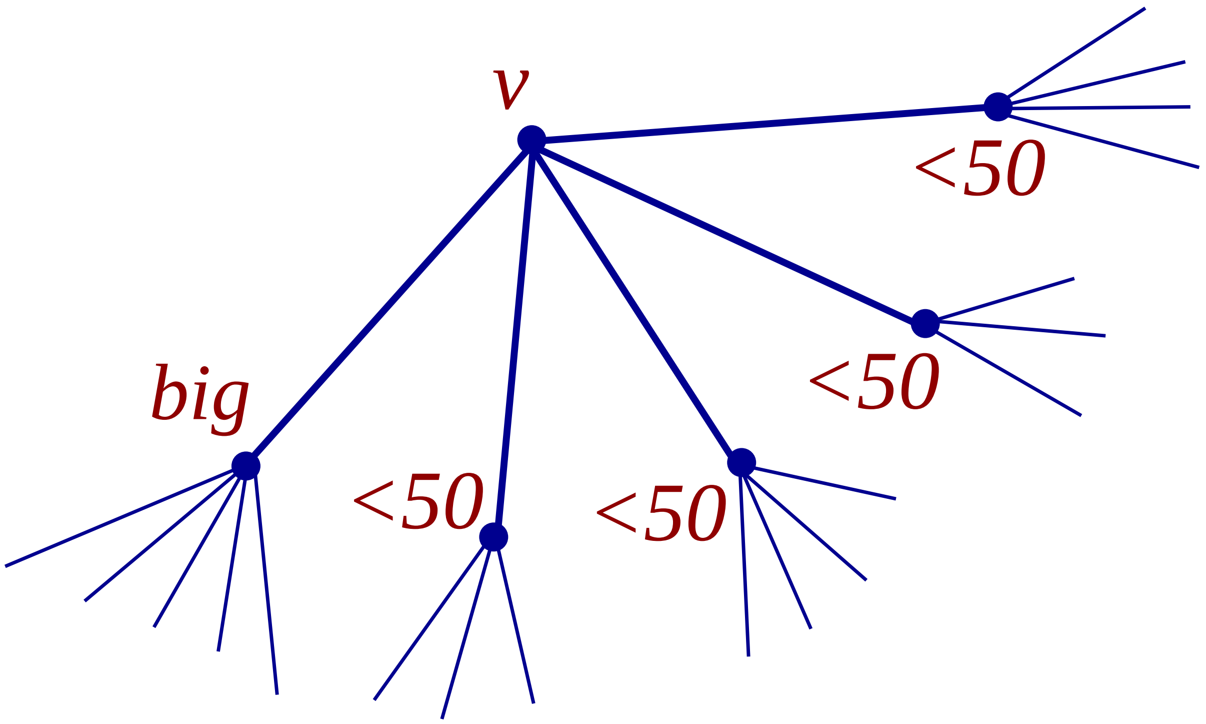
Lemma 1: if $(u, v) \in E(G)$, $d_G(u) + d_G(v) \leq \Delta + 3$, and $d_{G^2}(v) \leq \frac{21}{11}\Delta + 49$ then v is proper.



- Max degree does not increase and there are at most $\frac{21}{11}\Delta + 49$ colors at distance two of $v \implies$ we can color v.

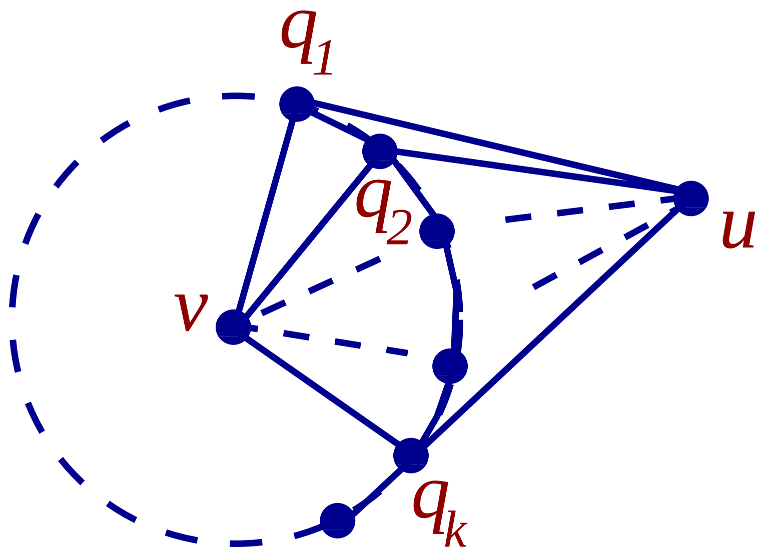
A vertex v is big if $d_G(v) \geq 50$.

Lemma 2: Every $\leq$5-vertex that is adjacent to at most one big vertex is proper.



- $d_{G^2}(v) \leq \Delta + 4 \times 49 < \frac{21}{11}\Delta + 50$
- $d_G(v) + 49 < \Delta + 3$,
- so by previous lemma, v is proper.

Sparse Segment: u, v are big and $q_1, \ldots, q_k$ are 4-vertices, $k \geq 3$



Lemma 3: If $k \geq \frac{1}{11} d_G(v) - 48$ then each of $q_2, q_3, \ldots, q_{k-1}$ is proper.

- Because $d_{G^2}(q_2) \leq d_G(v) + d_G(u) - k + 1 \leq \frac{21}{11}\Delta + 49$.

- By Lemma 1, q_2 is proper.

Discharging Method

- Prove at least one of the configurations of Lemmas 1-3 exists.

- Give $d_G(v) - 6$ units of charge to vertex v

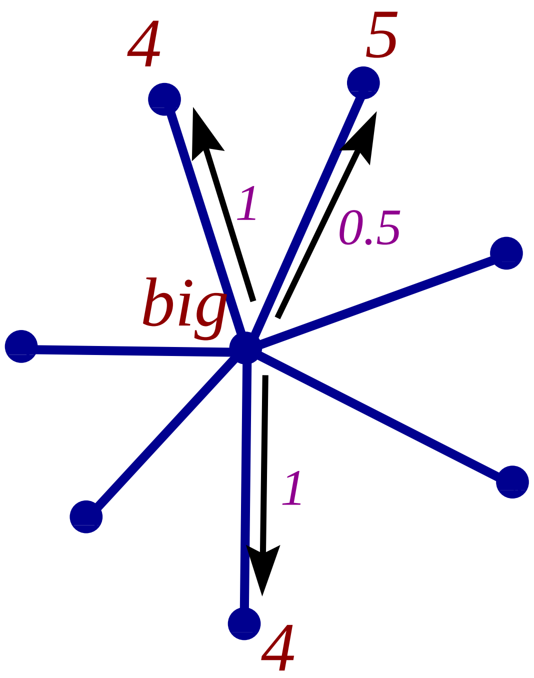
- By Euler formula, total charge is -12

- Charges of:

  - 4-vertices: -2

  - 5-vertices: -1

  - $\geq$6-vertices: $\geq$0



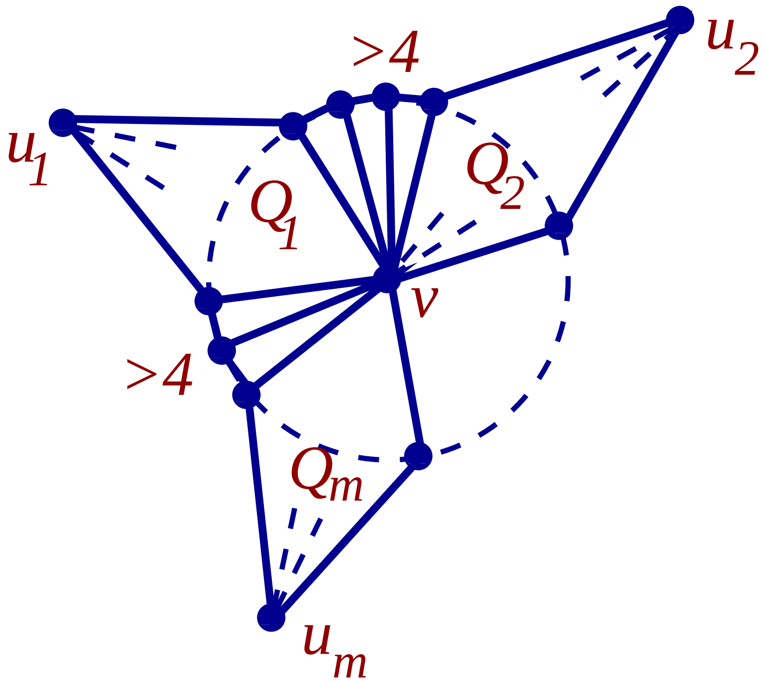
Discharging Rules: Every big vertex v sends:

- 1 to every 4-vertex neighbor.
- 0.5 to every 5-vertex neighbor.

Assuming no Config. of Lemma 2, 4- and 5-vertices are adjacent to two big vertices.

All non-big vertices have ≥ 0 Charge.

Let v be big, $Q_1, \ldots, Q_m$ maximal sparse segments.



- At least one $\geq$5-vertex between $Q_i \& Q_{i+1}$

- If $m \leq 11$, for some $i : Q_i \geq \frac{1}{11} d_G(v) - 48$, and Lemma 3 applies.

- if $m \geq 12$, v sends at most $d_G(v) - 6$ and v has ≥ 0 charge.

- if no proper vertex exists, then all vertices will have ≥ 0 charge.

- But the initial total charge was -12.

- There must be a vertex with negative charge; it must be proper.

Full version of paper is available at:

`http://www.cs.toronto.edu/~mreza/research.html`