Tutorial notes for Hall's Theorem

Problem: Let $G = (V, E)$ be a bipartite graph where $|V| = 2n$, $V = L \cup R$ where $|L| = |R| = n$, and $E \subseteq L \times R$. A matching $M \subseteq E$ is said to be *perfect* if $|M| = n$. The goal of this problem is to prove a necessary and sufficient condition for the existence of a perfect matching in G. (This condition is called Hall's Theorem).

For a set $L' \subseteq L$, define the *neighborhood* of L' by $N(L') = \{v \in R | (u, v) \in E \text{ for some } u \in L'\}$. The following parts of this problem show that G has a perfect matching iff $|N(L') \geq |L'|$ for all $L' \subseteq L$.

(a) Prove that if G has a perfect matching, then $|N(L')| \geq |L'|$ for all $L' \subseteq L$.

(b) Consider the flow network associated with G described in class (and the text). This network contains the vertices and (directed) edges of G, as well as edges from a new vertex s to every vertex in L, as well as edges from every vertex in R to a new vertex t; all edges have capacity 1.

Let (S, T) be a cut of this network, and assume that the capacity of this cut $c(S, T)$ (the sum of capacities of the edges going from S to T, which in this case is the number of edges going from S to T) is less than n. Prove that $|N(S \cap L)| < |S \cap L|$.

(c) Use part (b) to prove that if $|N(L')| \geq |L'|$ for all $L' \subseteq L$, then G has a perfect matching.

Solution:

(a) Let M be a perfect matching for G, and let $L' \subseteq L$. Say that L' has k distinct members: $L' = \{x_1, \ldots, x_k\}$. Since M is a perfect matching, there exist k distinct points $y_1, \ldots, y_k$ in R such that $(x_i, y_i) \in M \subseteq E$ for each i. Therefore, $\{y_1, \ldots, y_k\} \subseteq N(L')$, so $|N(L') \geq k = |L'|$.

(b) Let (S, T) be a cut of the associated network, such that $c(S, T) < n$. Let $S_1 \subseteq L$ and $S_2 \subseteq R$ be such that $S = \{s\} \cup S_1 \cup S_2$, and let x be the number of edges from S_1 to $R - S_2$. This means that $N(S_1)$ contains at most x vertices that are not in S_2, so that $|N(S_1)| \leq |S_2| + x$. The set of edges going from S to T includes the $n - |S_1|$ edges going from s to $L - \{S_1\}$, plus the x edges going from S_1 to $R - S_2$, plus the $|S_2$ edges going from S_2 to t. So $n > c(S, T) = n - |S_1| + x + |S_2|$, so $|S_1| > |S_2| + x \geq |N(S_1)|$. So $|S_1| > |N(S_1)|$, which is what we were required to show.

(c) Assume that $|N(L')| \geq |L'|$ for every $L' \subseteq L$. Part (b) therefore implies that every cut of the associated network has capacity at least n. Since the minimum cut capacity equals the maximum flow value, and the maximum flow value equals the size of a largest matching in G, a largest matching must have size at least n. Since no matching can have size bigger than n, a largest matching must have size exactly n, and must therefore be a perfect matching.