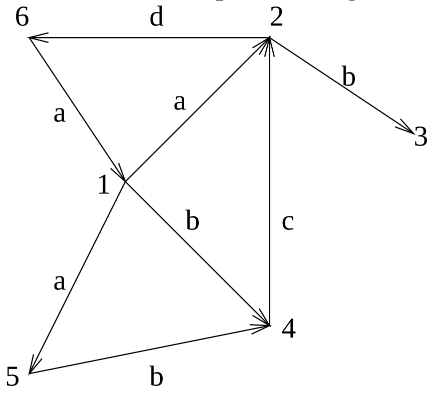
Tutorial notes for the Viterbi Algorithm problem

This problem is stated in CLR 15-5. In practice, Viterbi algorithm is used for speech recognition.

You are given a directed graph $G = (V, E)$ in which every edge is labeled by a letter from a finite alphabet Σ, that is there is a function $\sigma(u, v)$ that gives a label for the edge (u, v) for every pair $(u, v) \in E$. You are also given a distinguished vertex v_0 and a string $s =< l_1, ..., l_k >$ of symbols from Σ. Design a DP algorithm to check whether there is a path starting at v_0 labeled with s.

For example, let $\Sigma = \{a, b, c, d, e\}$, $v_0 = 1$ and $s = abc$ and consider the following graph. It has a path $\{1, 5, 4, 2\}$ labeled with s.



Now we want to design a dynamic programming algorithm for finding whether there is a path from v_0 labeled with s.

Step 1: Let $0 \leq i \leq k$ and $1 \leq v \leq n$ (that is, i indexes the letters of s and v the vertices). Then define $A(i, v) = 1$ if there is a path from v_0 to v labeled with the prefix of length i of s, and $A(i, v) = 0$ otherwise. The final answer is "yes" if there is v such that $A(k, v) = 1$.

Step 2: Initialize with $A(0, v_0) = 1$ and for $v \neq v_0$ $A(0, v) = 0$. Now the recurrence becomes:

$$A(i, v) = \max\{A(i-1, u) | (u, v) \in E \text{ and } \sigma(u, v) = l_i\}$$

That is, to set the value of $A(i, v)$ we check whether there is an edge labeled with l_i leading to $A(i, v)$ from a vertex that was reached in the previous step. If there is u such that $A(i-1, u) = 1$, then it is possible to arrive to u by a path labeled with the prefix of length $i-1$ of s; the condition $(u, v) \in E$ states that (u, v) is an edge and the condition $\sigma(u, v) = l_i$ states that this edge is labeled with l_i. Note that there can be several paths leading to a vertex, but we are only interested in an existence of a path, not their number. This is why the recurrence has max in it: if there is a path, $A(i, v)$ becomes 1, otherwise $A(i, v) = 0$.

The array for the example above is:

i \ v	1	2	3	4	5	6
0	1	0	0	0	0	0
2	0	1	0	0	1	0
2	0	0	1	1	0	0
3	0	1	0	0	0	0

So there is a path from vertex 1 labeled with s which ends at vertex 2.

Step 3: Skipped. As usual, just compute the array and then find a non-zero value in the last row. If succeed, output "path exists", otherwise output "no path" . For this problem, it is simplest to encode the graph by its adjacency matrix with labels instead of 0/1 (that is, if a matrix M encodes the graph, $M(u, v) = \sigma(u, v)$ if there is an edge (u, v), and some special symbol not in Σ if there is no edge (u, v) in the graph.)

Step 4: Suppose now that there is a path, and we want to output the vertices that constitute the path. The idea is just to retrace the steps. That is, starting with a non-zero entry in the last row, find an edge with the correct label that leads to a vertex v for which $A(i-1, v)$. The most straightforward algorithm outputs vertices in reverse order; make a recursive procedure in style of *PrintOpt* in the notes for scheduling to output the vertices in the correct order.