UNIVERSITY OF TORONTO
CSC238 (Day Session), St. George Campus

Term Test 2, November 2002

Duration: 50 min.
Aids allowed: NONE

Student Number: |_|_|_|_|_|_|_|_|_|

Last Name: ____________________

First Name: ____________________

Tutorial Section: **(circle one)**	MB128 Anna	RS208 Mark	RS310 Lee	GB304 Hamza

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This test consists of 3 questions on 5 pages (including this one). *When you receive the signal to start, please make sure that your copy of the examination is complete.* Answer each question directly on the examination paper, in the space provided, and *use the reverse side of the pages for rough work.* (If you need more space for one of your solutions, use the reverse side of the page and indicate **clearly** which part of your work should be marked.)

Be aware that concise, well thought-out answers will be rewarded over long rambling ones. Also, unreadable answers will be given zero (0) so write legibly. In your answers, you may use any theorems or facts given during the course, without proof or justification, *as long as you state them clearly.* You must prove any other theorems and justify any other facts needed for your solutions.

# 1:	______/10
# 2:	______/10
# 3:	______/10
Bonus	______/ 3
TOTAL:	______/30

Good Luck! PLEASE HAND IN

Question 1. [10 MARKS]

Consider the following recursive definition of function f:

$$f(n) = \begin{cases} 4 & \text{if } n = 1 \\ 5f(\lfloor \frac{n}{3} \rfloor) + 7f(\lfloor \frac{n}{6} \rfloor) + 2n^2 & \text{if } n > 1 \end{cases}$$

Without using the general theorem for divide and conquer recurrences, prove that $f(n) \in \mathcal{O}(n^2)$. You should present a constant c and show that $f(n) \leq c \cdot n^2$ for all $n \geq 1$.

Answer: The function should have been defined for $n = 0$, because for $n = 2$ we need $f(0)$. So let's assume that $f(0) = 0$. (You could make any reasonable assumption for $f(0)$ and solve the problem based on that assumption). We prove that $f(n) \in \mathcal{O}(n^2)$, by showing that $f(n) \leq 8n^2$ for $n \geq 0$, which also shows that $f(n) \leq 8n^2$ for $n \geq 1$. Define $Q(n)$: "$f(n) \leq 8n^2$". We show that $Q(n)$ holds for $n \geq 0$.

Basis: It is easy to see that $Q(0)$ and $Q(1)$ hold.

Ind. Hyp.: Let $k > 1$ be an arbitrary integer and assume that $P(k)$ holds for $0 \leq j < k$.

Ind. Step: Since $k > 1$:

$$\begin{aligned} f(k) &= 5f(\lfloor \frac{k}{3} \rfloor) + 7f(\lfloor \frac{k}{6} \rfloor) + 2k^2 \\ &\leq 5 \times 8(\lfloor \frac{k}{3} \rfloor)^2 + 7 \times 8(\lfloor \frac{k}{6} \rfloor)^2 + 2k^2 \qquad \text{(by Ind. Hyp.)} \\ &\leq 40(\frac{k}{3})^2 + 56(\frac{k}{6})^2 + 2k^2 \qquad \text{(by property of } \lfloor \enspace \rfloor) \\ &= \frac{40}{9}k^2 + \frac{56}{36}k^2 + 2k^2 \\ &= \frac{216}{36}k^2 + 2k^2 \\ &= 8k^2. \end{aligned}$$

Question 2. [10 MARKS]

Prove that $(p \leftrightarrow \neg q) \rightarrow (q \rightarrow \neg r)$ is logically equivalent to $r \rightarrow (q \rightarrow p)$. Write the name of every law that you use.

Answer:

	$(p \leftrightarrow \neg q) \rightarrow (q \rightarrow \neg r)$	
$LEQV$	$\neg(p \leftrightarrow \neg q) \vee (\neg q \vee \neg r)$	[$\rightarrow$ law, twice]
$LEQV$	$\neg(p \wedge \neg q \vee \neg p \wedge \neg\neg q) \vee (\neg q \vee \neg r)$	[$\leftrightarrow$ law]
$LEQV$	$\neg(p \wedge \neg q \vee \neg p \wedge q) \vee \neg q \vee \neg r$	[Double negation]
$LEQV$	$(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)) \vee \neg q \vee \neg r$	[DeMorgan's]
$LEQV$	$((\neg p \vee \neg\neg q) \wedge (\neg\neg p \vee \neg q)) \vee \neg q \vee \neg r$	[DeMorgan's, twice]
$LEQV$	$((\neg p \vee q) \wedge (p \vee \neg q)) \vee \neg q \vee \neg r$	[Double negation, twice]
$LEQV$	$\neg q \vee ((\neg p \vee q) \wedge (p \vee \neg q)) \vee \neg r$	[Commutativity of $\vee$]
$LEQV$	$(\neg q \vee \neg p \vee q) \wedge (\neg q \vee p \vee \neg q) \vee \neg r$	[Distributivity of $\vee$ over $\wedge$]
$LEQV$	$(\neg q \vee q \vee \neg p) \wedge (\neg q \vee \neg q \vee p) \vee \neg r$	[Commutativity of $\vee$]
$LEQV$	$(\neg q \vee \neg q \vee p) \vee \neg r$	[Identity]
$LEQV$	$(\neg q \vee p) \vee \neg r$	[Idempotency]
$LEQV$	$\neg r \vee (\neg q \vee p)$	[Commutativity of $\vee$]
$LEQV$	$r \rightarrow (q \rightarrow p)$	[$\rightarrow$ law, twice]

Bonus. [3 MARKS]

Recall the binary propositional connective ***nor***, denoted by $\downarrow$, which has the following truth table:

x	y	$x \downarrow y$
0	0	1
0	1	0
1	0	0
1	1	0

Briefly explain why $\{\downarrow\}$ is a complete set of connectives (you don't have to use induction).

Answer: We have already seen that $\{\neg, \vee\}$ is a complete set. So it is enough to show how to build $\neg p$ and $p \vee q$ using only $\downarrow$. It is easy to see from the truth table that

$$p \downarrow q \quad \text{LEQV} \quad \neg(p \vee q)$$

Let's call this the "$\downarrow$ law". Therefore:

- We get $\neg p$ by $p \downarrow p$, because:

$$\begin{array}{llll} p \downarrow p & \text{LEQV} & \neg(p \vee p) & [\text{by } \downarrow \text{ law}] \\ & \text{LEQV} & \neg p & [\text{Idempotency law}] \end{array}$$

- We get $p \vee q$ by $(p \downarrow q) \downarrow (p \downarrow q)$, because:

$$\begin{array}{llll} (p \downarrow q) \downarrow (p \downarrow q) & \text{LEQV} & \neg((p \downarrow q) \vee (p \downarrow q)) & [\text{by } \downarrow \text{ law}] \\ & \text{LEQV} & \neg(\neg(p \vee q) \vee \neg(p \vee q)) & [\text{by } \downarrow \text{ law}] \\ & \text{LEQV} & \neg(\neg(p \vee q)) & [\text{Idempotency law}] \\ & \text{LEQV} & p \vee q & [\text{double negation}] \end{array}$$

Question 3. [10 MARKS]

Recall that propositional variables can be used to represent binary numbers, as we had in assignment 3. Suppose we are given a three-bit natural number x and a four-bit natural number y, where x is represented by three propositional variables x_2, x_1, x_0, and y is represented by four propositional variables y_3, y_2, y_1, y_0. More precisely, for any truth assignment τ of these propositional variables (which gives a value 1 (representing true) or 0 (representing false) for each variable), $x = 4\tau(x_2) + 2\tau(x_1) + \tau(x_0)$ and $y = 8\tau(y_3) + 4\tau(y_2) + 2\tau(y_1) + \tau(y_0)$. Using propositional variables x_2, x_1, x_0 and y_3, y_2, y_1, y_0 and connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ write a propositional formula that is satisfied by exactly those truth assignments τ for which $y = x + 6$. (Solutions based on large truth tables will not get full marks).

Answer: Let's look at the bits of $x + 6$ starting from the least significant one. Since the binary representation of 6 is 110, therefore, when we add 6 to x, the least significant bit of the result is always equal to x_0. The second bit is always $\neg x_1$. The third one is 1 only if $x_2 = 0$ and $x_1 = 0$ or if $x_2 = 1$ and $x_1 = 1$ (i.e we have a carry from addition in previous bit). This means that the third bit is 1 when $x_2 \leftrightarrow x_1$ is true. The fourth bit is 1 only if $x_2 = 1$ or we had a carry from the previous bit, which means that $x_1 = 1$. Therefore, the whole formula would be:

$$(y_0 \leftrightarrow x_0) \wedge (y_1 \leftrightarrow \neg x_1) \wedge (y_2 \leftrightarrow (x_2 \leftrightarrow x_1)) \wedge (y_3 \leftrightarrow x_2 \vee x_1)$$

Total Marks = 30