UNIVERSITY OF TORONTO
CSC238 (Day Session), St. George Campus

Term Test 1, October 2002

Duration: 50 min.
Aids allowed: NONE

Student Number: |_|_|_|_|_|_|_|_|_|

Last Name: ____________________

First Name: ____________________

Tutorial Section: (circle one)	MB128 Anna	RS208 Mark	RS310 Lee	GB304 Hamza

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This test consists of 3 questions on 4 pages (including this one). *When you receive the signal to start, please make sure that your copy of the examination is complete.* Answer each question directly on the examination paper, in the space provided, and *use the reverse side of the pages for rough work.* (If you need more space for one of your solutions, use the reverse side of the page and indicate **clearly** which part of your work should be marked.)

Be aware that concise, well thought-out answers will be rewarded over long rambling ones. Also, unreadable answers will be given zero (0) so write legibly. In your answers, you may use any theorems or facts given during the course, without proof or justification, *as long as you state them clearly*. You must prove any other theorems and justify any other facts needed for your solutions.

# 1:	______/10
# 2:	______/10
# 3:	______/10
TOTAL:	______/30

Good Luck! PLEASE HAND IN

Question 1. [10 MARKS]

Consider the following recursive definition of function f:

$$f(n) = \begin{cases} 40 & \text{if } n = 1 \\ 150 & \text{if } n = 2 \\ 3f(n-1) + 2f(n-2) + 4^n & \text{if } n > 2 \end{cases}$$

Prove that for $n \geq 1$, $f(n) \leq 10 \times 4^n$.

Solution: Let's define $P(n)$: as "$f(n) \leq 10 \times 4^n$". We prove by induction on n that $P(n)$ holds for $n \geq 1$.

Basis: For $n = 1$, $f(n) = 40 \leq 10 \times 4^1$.
For $n = 2$, $f(n) = 150 < 10 \times 16 = 10 \times 4^2$.

Ind. Hyp.: Let $i > 2$ be an arbitrary integer and assume that for all $2 \leq j < i$, $P(j)$ is true.

Ind. Step: We prove that $P(i)$ is true. Since $i > 2$, by definition of f:

$$\begin{aligned} f(i) &= 3f(i-1) + 2f(i-2) + 4^i \\ &\leq 3 \times 10 \times 4^{i-1} + 2 \times 10 \times 4^{i-2} + 4^i \qquad \text{by Ind. Hyp.} \\ &= 30 \times 4^{i-1} + 20 \times 4^{i-2} + 4^i \\ &= 30 \times 4^{i-1} + 5 \times 4^{i-1} + 4^i \\ &= 35 \times 4^{i-1} + 4^i \\ &\leq 36 \times 4^{i-1} + 4^i \\ &= 9 \times 4^i + 4^i \\ &= 10 \times 4^i \end{aligned}$$

as wanted.

Question 2. [10 MARKS]

Consider the following program with the given precondition:

Precondition: x, y, and z are natural numbers.

```
1.  while x ≠ 0 or y ≠ 0 or z ≠ 0 do
2.      if x > 0 then
3.          x := x − 1
4.      else if y > 0 then
5.          y := y − 1
6.          x := x + 2
7.      else if z > 0 then
8.          z := z − 1
9.          y := y + 2
10.         x := x + 2
11.     end if
```

Assuming the precondition holds, prove that the program terminates. (You do *NOT* need to prove partial correctness). **Hint:** It might be easier to consider x, y, and z as digits of a number in base 3.

Solution: As suggested in the hint, if we consider x, y, and z as digits of a number in base 3, then this program is basically a downward counter. So let's define $E = 9z + 3y + x$. We show that E is always a natural number and E_k's form a decreasing sequence.

Define $S(i)$ as: *"If the loop is executed at least i times, then (a) E_i is a natural number and (b) if $i > 0$ then $E_i < E_{i-1}$"*. We prove by induction on i that $S(i)$ is true for $i \geq 0$.

Basis: If $i = 0$ then by precondition, all x, y, and z are non-negative integers and therefore E_0 is a natural number. Since $i = 0$, part (b) is trivially true.

Ind. Hyp.: Let $i \geq 0$ be an arbitrary integer and assume that $S(i)$ is true.

Ind. Step: We prove that $S(i+1)$ is true. Assume that the loop is executed at least $i+1$ times (otherwise $S(i+1)$ is trivially true). We consider three different cases:

- Case 1: Condition in line 2 is satisfied. In this case $x_{i+1} = x_i - 1$ and since $x_i > 0$, therefore $x_{i+1} \geq 0$. Also, $y_{i+1} = y_i$ and $z_{i+1} = z_i$ and by the Ind. Hyp. are natural numbers. So E_{i+1} is also natural. Furthermore, $E_{i+1} = 9z_{i+1} + 3y_{i+1} + x_{i+1} = 9z_i + 3y_i + x_i - 1 = E_i - 1$ and condition (b) is also satisfied.
- Case 2: Condition in line 4 is satisfied. In this case, $y_{i+1} = y_i - 1$ and $x_{i+1} = x_i + 2$ and $z_{i+1} = z_i$. Since $y_i > 0$, y_{i+1} is a natural number. Also, x_{i+1} and z_{i+1} are natural (using the Ind. Hyp.). Therefore, E_{i+1} is a natural number. Furthermore, $E_{i+1} = 9z_{i+1} + 3y_{i+1} + x_{i+1} = 9z_i + 3y_i - 3 + x_i + 2 = E_i - 1$, and therefore, $E_{i+1} < E_i$.
- Case 3: Condition in line 7 is satisfied. In this case, $z_{i+1} = z_i - 1$, $y_{i+1} = y_i + 2$, and $x_{i+1} = x_i + 2$. Since $z_i > 0$, z_{i+1} is a natural number. By the Ind. Hyp., x_i and y_i are natural, and so are x_{i+1} and y_{i+1}. Therefore, E_{i+1} is natural. Furthermore, $E_{i+1} = 9z_{i+1} + 3y_{i+1} + x_{i+1} = 9z_i - 9 + 3y_i + 6 + x_i + 2 = 9z_i + 3y_i + x_i - 1 = E_i - 1$.

Hence the value of E_i is always a natural number and decreases at each iteration. By the Well-Ordering principle, the sequence of $E_0, E_1, \ldots$ is finite, and so is the number of itearations of the loop.

Question 3. [10 MARKS]

Part (a) [5 MARKS]

Indicate (without proof) whether the following statements are true. (Hint: you may use the tips I gave in the class).

- $\frac{n^3}{10} + 5n^2 \in \Theta(n^3)$ **True**
- $(5n \log n)^3 \in \mathcal{O}(n^4)$ **True**
- $2^{3n} \in \mathcal{O}(6^n)$ **False**
- $n^3 - 2n^2 \log n \in \Omega(n^3)$ **True**
- $\frac{5n^2}{\log n} \in \mathcal{O}(n \log n)$ **False**

Part (b) [5 MARKS]

Prove that $n - \sqrt{n} \in \Omega(n)$.

Solution: Let $n_0 = 4$ and $c = \frac{1}{2}$. We show that for all $n \geq 4 : n - \sqrt{n} \geq \frac{1}{2} \times n$.

$$\begin{aligned}
n &\geq 4 && \text{(since } n \text{ is positive, we can multiply both sides by } n\text{)} \\
n \times n &\geq 4n && \text{(since both sides are greater than 1 we can take the sqrt both sides)} \\
n &\geq 2\sqrt{n} && \text{(add } n \text{ to both sides)} \\
2n &\geq 2\sqrt{n} + n \\
2n - 2\sqrt{n} &\geq n \\
n - \sqrt{n} &\geq \frac{1}{2}n,
\end{aligned}$$

as wanted.

Total Marks = 30