University of Toronto
Scarborough Campus
December 18, 1998
9:10 am – 12:00 noon

CSC B38F Final Examination

Aids allowed: One 8.5×11 handwritten, non-photocopied 'cheat sheet'

Duration: Two hours and fifty minutes

- There should be ***11 pages*** in this exam booklet, including this cover page.
- Please check that you have a complete exam.
- Do all problems.
- Put ***all answers in this booklet***, in the spaces provided.
- For rough work, use the backs of the pages; ***these will not be marked***.
- Good luck!

Name ______________________________

Student Number ______________________________

Problem	Marks Rec'ved	Marks Worth
1.		20
2.		20
3.		10
4.		20
5.		10
6.		10
7.		10
TOTAL		100

QUESTION 1. (20 marks)

For each assertion below, state whether it is true or false (by circling the corresponding word), and justify your answer in the space provided. ***No credit without correct justification.***

a. $n^2 + 100n \in \Theta(n^2)$. **TRUE / FALSE**

b. If C and C' are sets of Boolean connectives so that $C \subseteq C'$, and C is a complete set, then C' is also a complete set. **TRUE / FALSE**

c. $\forall x\,(P(x) \to Q(x))$ is logically equivalent to $\neg\exists x\,(P(x) \wedge \neg Q(x))$. **TRUE / FALSE**

d. If L is a regular language and $L' \subseteq L$, then L' is also a regular language. **TRUE / FALSE**

QUESTION 2. (20 marks)

If $A[1..n]$ is a sorted array of integers and x is an integer that appears in A, the program below determines (in variable *first*) the ***largest*** index i such that $A[i] = x$. More precisely, the precondition and postcondition of the program are:

Precondition: n is a positive integer.
For all u, v such that $1 \leq u \leq v \leq n$, $A[u] \leq A[v]$.
For some u such that $1 \leq u \leq n$, $A[u] = x$.

Postcondition: $1 \leq first \leq n$, $A[first] = x$, and if $first < n$ then $A[first + 1] > x$.

```
first := 1; last := n
while first ≠ last do
        mid := (first + last + 1) div 2
        if A[mid] > x then
                last := mid − 1
        else
                first := mid
        end if
end while
```

a. (5 marks) State an appropriate loop invariant (call it P) which, together with the loop termination condition, implies the postcondition. For this part simply ***state*** P; you need not prove anything, yet.

b. (5 marks) ***Prove*** that your statement P in part (a), together with the loop termination condition, implies the postcondition.

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c. (10 marks) ***Prove*** that your statement P in part (a) is indeed a loop invariant with respect to the given precondition. You need not prove termination. In your proof you may use, ***without proof***, the following fact: *for any* $f, \ell \in \mathbb{N}$, *if* $f < \ell$ *then* $f < (f + \ell + 1)\ \mathbf{div}\ 2 \leq \ell$.

QUESTION 3. (10 marks)

Prove that $\{\leftrightarrow, \wedge\}$ is ***not*** a complete set of connectives.

Hint: Consider the truth assignment in which all propositional variables are true.

QUESTION 4. (20 marks)

Consider the following sentence F in predicate logic:

$$\Big(P(\boldsymbol{a}) \wedge \forall x\, \forall y\, \big((P(x) \wedge S(x,y)) \rightarrow P(y)\big)\Big) \rightarrow \forall x\, P(x)$$

Answer each of the following questions, and justify your answer.

a. Let $\mathcal{S}_1$ be a structure with domain $\mathbb{N}$, where the meaning of the constant symbol $\boldsymbol{a}$ is 0 (i.e., $\boldsymbol{a}^{\mathcal{S}_1} = 0$), and the meaning of $S(x,y)$ is the predicate "$y = x + 1$".

Is F true in $\mathcal{S}_1$ for ***every*** possible predicate $P^{\mathcal{S}_1} \subseteq \mathbb{N}$?

b. Let $\mathcal{S}_2$ be a structure with domain $\mathbb{Z}$, where the meaning of $\boldsymbol{a}$ is 0, and the meaning of $S(x,y)$ is the predicate "$y = x + 1$".

Is F true in $\mathcal{S}_2$ for ***every*** possible predicate $P^{\mathcal{S}_2} \subseteq \mathbb{Z}$?

c. Let $\mathcal{S}_3$ be a structure with domain $\mathbb{R}^{\geq 0}$, the set of nonnegative reals, where the meaning of $\boldsymbol{a}$ is 0, and the meaning of $S(x, y)$ is the predicate "$y = x + 1$".

Is F true in $\mathcal{S}_3$ for ***every*** possible predicate $P^{\mathcal{S}_3} \subseteq \mathbb{R}^{\geq 0}$?

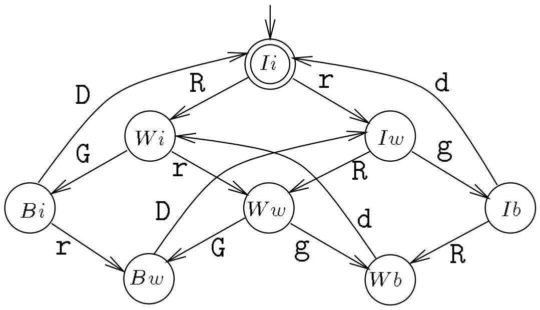
d. Let $\mathcal{S}_4$ be a structure with domain $\mathbb{N}$, where the meaning of $\boldsymbol{a}$ is 0, and the meaning of $S(x, y)$ is the predicate "$y = x + 2$".

Give an example of a predicate $P^{\mathcal{S}_4} \subseteq \mathbb{N}$ such that F becomes false.

QUESTION 5. (10 marks)

The figure below is taken from the course notes. It depicts a DFSA that represents the language of acceptable behaviours of a simple two workstation system sharing (with mutual exclusion) a common printer. Here $\mathtt{R}, \mathtt{G}, \mathtt{D}$ (respectively $\mathtt{r}, \mathtt{g}, \mathtt{d}$) denote the ***request***, ***grant*** and ***done*** events for workstation 1 (respectively, workstation 2).

Recall that each state of the DFSA is a pair of workstation states, whose intuitive meaning is as follows: I (respectively i) means that workstation 1 (respectively 2) is "idle" with respect to using the printer; W (respectively w) means that workstation 1 (respectively 2) is "waiting" for permission to use the printer; and B (respectively b) means that workstation 1 (respectively 2) is "busy" using the printer.



Suppose we want to restrict the acceptable behaviours so as to model the situation where workstation 1 always has priority over workstation 2; that is, if workstation 1 has requested the printer before permission has been granted to workstation 2, then permission must be granted to workstation 1 (and workstation 1 must be done) before permission can be granted to workstation 2.

a. (3 marks) Specify a string $x \in \{\mathtt{R}, \mathtt{G}, \mathtt{D}, \mathtt{r}, \mathtt{g}, \mathtt{d}\}^*$ which is an acceptable behaviour for the system depicted in the figure above but is not an acceptable behaviour for the system in which priority is given to workstation 1.

b. (7 marks) Explain how to modify the figure (or redraw it, if your modifications are too complicated) so as to represent the acceptable behaviours for the system with priority to workstation 1. (You need not prove that your modified (or redrawn) figure is correct — but it should be!)

QUESTION 6. (10 marks)

a. (3 marks) Give a regular expression that denotes the language

$$L = \{w : w \in \{0,1\}^* \text{ and } w \text{ contains 0100 as a substring}\}$$

(You need not prove that your regular expression is correct — but it should be!)

b. (7 marks) Draw the state diagram for some DFSA which accepts the language

$$\overline{L} = \{w : w \in \{0,1\}^* \text{ and } w \text{ does } \textbf{\textit{not}} \text{ contain 0100 as a substring}\}$$

(You need not prove that your DFSA is correct — but it should be!)

QUESTION 7. (10 marks)

Let L be any language over $\{0, 1\}$.

a. (5 marks) Consider the following operation on languages:

$$\textbf{substring}(L) = \{w : w \in \{0,1\}^* \text{ and } w \text{ contains some } x \in L \text{ as a substring}\}$$

Show that if L is regular then so is $\textbf{substring}(L)$.

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b. (5 marks) Consider the following operation on languages L and L':

$\mathbf{ifthen}(L, L') = \{w :$ if w contains some $x \in L$ as a substring then w contains some $y \in L'$ as a substring$\}$

Show that if L and L' are regular languages then so is $\mathbf{ifthen}(L, L')$.
Hint: Recall that $(P \rightarrow P')$ LEQV $(\neg P \vee P')$, and use part (a).