Week 11: Recursion

Agenda:

- Recursively defined sequences
- Solving recurrences

Reading:

- Textbook: Sections 8.1-8.3

Definitions for sequences:

- To write the first few terms with the expectation that the general pattern will be obvious

  For $a_0, a_1, a_2, \ldots$,

  $$3, 5, 7, \ldots$$

- To give an **explicit formula** for its n-th term

  $$a_n = \frac{(-1)^n}{n+1}, \text{ for all integers } n \geq 0.$$

- To define it recursively,
  - a **recurrence relation** to relate later terms to earlier terms
  - **initial conditions**, values of the first few terms

  $$a_k = \begin{cases} 1, & k = 0 \\ 3, & k = 1 \\ a_{k-1} + a_{k-2}, & k \geq 2 \end{cases}$$

- Example: Fibonacci numbers:

  $F_0 = F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.

- Example: number of partitions of a set of n elements into r subsets (Stirling number of the second kind): $S_{n,r}$

  For example, $S_{3,2} = 3$: $\{a_1, a_2\}\{a_3\}$, $\{a_2, a_3\}\{a_1\}$, $\{a_3, a_1\}\{a_2\}$

- There are two kinds of partitions, one of which always contains subset $\{a_n\}$ as one of the r parts

  so the remaining $r-1$ parts contain elements $\{a_1, \ldots, a_{n-1}\}$

  $\#$ = the number of ways $\{a_1, a_2, \ldots, a_{n-1}\}$ can be partitioned into $r-1$ subsets — $S_{n-1,r-1}$

- The other does NOT contain a subset $\{a_n\}$, or a_n is always with somebody else:

  Each partition of $\{a_1, a_2, \ldots, a_{n-1}\}$ into r subsets **corresponds to** r partitions ...

  $\# = r$ times the number of ways $\{a_1, a_2, \ldots, a_{n-1}\}$ partitioned into r subsets — $r \times S_{n-1,r}$

- Therefore,

$$S_{n,r} = S_{n-1,r-1} + r \times S_{n-1,r}, \quad n \geq r \geq 1$$

  $S_{n,1} = 1$, $S_{n,n} = 1$, for all $n \geq 1$.

Computing the n-th term in a sequence:

- If the explicit formula is given, such as $a_n = \frac{(-1)^n}{n+1}$, it is easy

- When the sequence is specified as a recurrence relation,
  - Method #1: by iteration:

    For example, $m_k = 2m_{k-1} + 1$, until reaching the base cases

    So, $m_{100} = 2m_{99} + 1 = 2(2m_{98} + 1) + 1 = 4m_{98} + 3 = \ldots$
  - We compute the **explicit formula** (out of the recurrence relation)

$$\begin{array}{rcl}
m_k & = & 2m_{k-1} + 1 \\
2m_{k-1} & = & 2^2 m_{k-2} + 2^1 \\
2^2 m_{k-2} & = & 2^3 m_{k-3} + 2^2 \\
\ldots & = & \ldots \\
2^{k-1} m_1 & = & 2^k m_0 + 2^{k-1} \quad (m_0 = 0) \\
\hline
m_k & = & 1 + 2^1 + 2^2 + \ldots + 2^{k-1} = 2^k - 1 \quad (k \geq 1)
\end{array}$$

- We have computed the explicit formula as $m_k = 2^k - 1$, for all $k \geq 1$, which also holds for $k = 0$

- Prove it by mathematical induction:

  Proof. (Base case $k = 0$): $m_0 = 2^0 - 1 = 0$, and so it holds for the base case.

  (Inductive phase): Assume $m_k = 2^k - 1$ for some $k \geq 0$. Then, by the recurrence relation we have $m_{k+1} = 2m_k + 1$. Therefore, $m_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$, which means the formula holds also for $k + 1$. By the principle of mathematical induction, the explicit formula holds for all $k \geq 0$.

- **Arithmetic sequence**: $a_k = a_{k-1} + d$ (or $a_k = a_0 + dk$)

- **Geometric sequence**: $a_k = a_{k-1} \times q$ (or $a_k = a_0 \times q^k$)

Second-order linear homogeneous recurrence relations with constant coefficients:

- **Definition:** they are of the form:

  $$a_k = Aa_{k-1} + Ba_{k-2}, \text{ for all integers } k \geq \text{ some fixed integer,}$$

  where A and B are fixed real numbers with $B \neq 0$

- Question: what if $B = 0$?

- Both the Fibonacci numbers $\{F_0, F_1, F_2, \ldots\}$ and the Lucas numbers $\{L_0, L_1, L_2, \ldots\}$ are ...

- The **characteristic equation** of the recurrence relation is

  $$x^2 - Ax - B = 0$$

  $\{1, t, t^2, t^3, \ldots\}$, $t \neq 0$, satisfies the recurrence relation $\Longleftrightarrow$ t is a solution to the equation

  Note: the equation could have no real solutions ...

- The Distinct-Roots Theorem:

  Assume x_0 and x_1, $x_0 \neq x_1$, are two real solutions to the characteristic equation

- Both $\{x_0^n\}_{n=0}^{\infty}$ and $\{x_1^n\}_{n=0}^{\infty}$ satisfy the recurrence relation

  Given any numbers C and D, $\{Cx_0^n + Dx_1^n\}_{n=0}^{\infty}$ also satisfies the recurrence relation

- C and D are determined by the base cases (or the initial conditions)

- Example: The Fibonacci numbers:

  The characteristic equation is

$$x^2 - x - 1 = 0,$$

  which has solutions $x_0 = \frac{1+\sqrt{5}}{2} > 0$ and $x_1 = \frac{1-\sqrt{5}}{2}$

  So $F_n = Cx_0^n + Dx_1^n = C(\frac{1+\sqrt{5}}{2})^n + D(\frac{1-\sqrt{5}}{2})^n$.
  Using the fact that $F_0 = 1$ and $F_1 = 1$ we have:

  $F_0 = C(\frac{1+\sqrt{5}}{2})^0 + D(\frac{1-\sqrt{5}}{2})^0 = C + D = 1$

  Also: $F_1 = C(\frac{1+\sqrt{5}}{2} + D(\frac{1-\sqrt{5}}{2}) = 1$

  These two equations imply: $C = \frac{1+\sqrt{5}}{2\sqrt{5}}$ and $D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$

- Thus:
  $F_n = \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-(1-\sqrt{5})}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}$

- Example: Find explicit formula for the following recurrence:

  $a_0 = 1$, $a_1 = 3$, for $n \geq 2$: $a_n = 3a_{n-1} - 2a_{n-2}$.

  The char. equation is $x^2 - 3x + 2 = 0$. Thus $x = \frac{3\pm\sqrt{9-8}}{2}$, i.e. $x_0 = 2$, $x_1 = 1$.

  So $a_n = C2^n + D$.

$a_0 = 1 = C + D$, $a_1 = 3 = 2C + D$, So $C = 2$, $D = -1$. Thus $a_n = 2^{n+1} - 1$.

- The Single-Root Case: Assume x_0 is the only real solution to the characteristic equation

- Both $\{x_0^n\}_{n=0}^{\infty}$ and $\{nx_0^n\}_{n=0}^{\infty}$ satisfy the recurrence relation

  Given any numbers C and D, $\{Cx_0^n + Dnx_0^n\}_{n=0}^{\infty}$ also satisfies the recurrence relation

- C and D are determined by the base cases (or the initial conditions)

  For example, find the explicit formula for

$$P_n = \begin{cases} 2P_{n-1} - P_{n-2}, & n \geq 2, \\ 2, & n = 1, \\ 1, & n = 0. \end{cases}$$

  - $x_0 = x_1 = 1$
  - $P_n = C + Dn$
  - $C = 1$ and $C + D = 2$. Therefore, $P_n = 1 + n$, for all $n \geq 0$.

- Gambler's Ruin (Example 8.3.6 from the text).

- Example: $a_k = 4a_{k-2}$, $a_1 = 1$, $a_2 = 6$.

  $x^2 - 4 = 0$, so $x = 2$ and $x = -2$. Thus $a_k = C2^k + D(-2)^k$.

  $a_1 = 2C - 2D = 1$, $a_2 = 4C + 4D = 6$. Therefore, $C = 1$ and $D = \frac{1}{2}$.

  So $a_k = 2^k - (-2)^{k-1}$.