Week 10: Pigeonhole Principle, Functions

Agenda:

- Pigeonhole principle
- Composition of functions

Reading:

- Textbook: Sections 7.3-7.4

The Pigeonhole Principle:

- If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.

  Equivalently, a function from a finite set A to a smaller finite set B cannot be one-to-one.

- Example 1: In a group of 13 people, at least two are born in the same month.

- Example 2: if $S \subset \mathbb{Z}^+$ and $|S| = 21$ then S contains at least two integers with the same reminder upon division by 20. The reason is the set of reminders for 20 is $\{0, 1, \ldots, 19\}$ and has size 20.

- Example 3: Prove that if 51 integers are selected from the set $\{1, 2, 3, \ldots, 100\}$, then there are two integers such that one divides the other.

  Solution: we can group the integers into pairs: $\{1, 100\}, \{2, 99\}, \{3, 98\}, \ldots$. There are 50 pairs and so the 51 integers must contain both integers from a pair.

- Example 4: The decimal expansion of a rational either terminates or repeats.

  e.g. 2.15, $1.\overline{31}$.

  Consider the decimal expansion of $\frac{4}{33} = 0.12121212\ldots$. The digits 12 repeat infinitely.

  In general, to compute the decimal expansion of $\frac{a}{b}$ we have to do a series of division which ends when we get a reminder of 0, or will continue. The point is the set of possible reminders are $\{0, 1, \ldots, b-1\}$. So eventually either we get a reminder of 0 and the decimal expansion terminates, or we will get a repeated reminder. In this case we get a period.

  For example $\frac{5}{12} = 0.41\overline{6}$.

The Generalized Pigeonhole Principle:

- A function f from one finite set A to a smaller finite set B, for any positive integer k, if $|A| > k|B|$, then there must exists an element in B which has at least $k + 1$ preimages.

- Example 5: In a group of 22 people at least 4 are born on the same day of week.

  Proof: If at most 3 are born on each day, there can be at most $3 \times 7 = 21$ people. This is using the contrapositive form of Generalized Pigeonhole Principle.

- Example 6: example 7.3.6 from the text.

- Example 7: Given 10 positive integers from $\{2, \ldots, 900\}$ that are pairwise relatively prime to each other prove that at least one of them is prime.

  Proof: Consider the sequence of primes: $2, 3, 5, 7, 11, 13, 19, 23, 29, 31, \ldots$. Suppose by way of contradiction that none of the 10 integers is prime. So they have two or more (possibly equal) primes in their factorization. Since they are all relatively prime to each other, they cannot have any prime factor in common. Since there are only 9 primes from 2 to 29, among the 10 integers, at least one of them has both of its prime factors at least 31; thus it must be larger than 900, a contradiction.

Composition of Functions:

- Given $f : A \to B$ and $g : B \to C$, define $g \circ f : A \to C$ as

$$(g \circ f)(a) = g(f(a)), \forall a \in A$$

- Theorem: If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are one-to-one, then $g \circ f$ is one-to-one.

  Proof: Need to show that for all $x_1, x_2 \in A$, $(g \circ f)(x_1) = (g \circ f)(x_2) \longrightarrow x_1 = x_2$.

  Consider two arbitrary $x_1, x_2 \in A$ and assume that $(g \circ f)(x_1) = (g \circ f)(x_2)$.

This means: $g(f(x_1)) = g(f(x_2))$. Since g is 1-to-1, we must have $f(x_1) = f(x_2)$.

Since f is 1-to-1 this implies $x_1 = x_2$ as wanted.

- Theorem: If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are onto, then $g \circ f$ is onto.

  Proof: We need to show that $\forall z \in C, \exists x \in A$ s.t. $g(f(x)) = z$.

  Let $z \in C$ be an arbitrary element. Since g is onto, there is $y \in B$ s.t. $g(y) = z$.

  Since f is onto, there is $x \in A$, s.t. $f(x) = y$. Thus, $g(f(x)) = z$, as wanted.