Week 8: Counting

Agenda:

- Basic Definitions, addition and multiplication rules
- Combinations, permutations

Reading:

- Textbook: Sections 6.1-6.5

A first word on probability:

- A **random process** is a process in which an **outcome** from a set of possible outcomes must occur, but not sure of which one

- **sample space** S: is the set of all possible outcomes of a random process

- An **event** E is a subset of S

  e.g. when rolling a die, the sample space is $\{1, 2, 3, 4, 5, 6\}$. The possibility of having a 1 or 3 is the event (corresponding to subset $\{1, 3\}$).

- Equal likelihood of occurrence: when S is finitely many

  $Pr(E) = \dfrac{|E|}{|S|}$: the **probability** that E occurs (one outcome of E appears)

- For example, rolling a fair die or tossing a fair coin [“fair” $\longleftrightarrow$ “equal likelihood”]

- **Theorem**. If m and n are integers and $m \leq n$, there there are $n - m + 1$ integers from m to n inclusive.

  For example,

  - how many three-digit integers there are?
    from 100 to 999 there are 900 integers.

  - how many three-digit integers are divisible by 5?
    among those 900, there are $\lfloor 900/5 \rfloor = 180$ multiples of 5.

  - What is the probability that a randomly chosen three-digit integer is divisible by 5?
    The answer will be 180/900.

- The multiplication rule:

If an operation consists of k steps, and

- the first step can be performed in n_1 ways,
- the second step can be performed in n_2 ways, regardless of how the first step was performed,
- ...,
- the k-th step can be performed in n_k ways, regardless of how the preceding steps were performed,

then the entire operation can be performed in $n_1 n_2 \ldots n_k$ ways.

- Example:
  Alberta license plate is of the form CCC-DDD, where C is a letter and D is a digit:

  $26 \times 26 \times 26 \times 10 \times 10 \times 10$

- Restriction: no repeated letters, neither repeated digits:

  $26 \times 25 \times 24 \times 10 \times 9 \times 8$

- A **permutation** of a set of objects is an **ordering** of the objects in a row

- **Theorem**. For any integer $n \geq 1$, the number of permutations of a set of n elements is

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

  [$0! = 1$]

- Permutations around a circle: How many different ways can a group of n people sit around a round table if the rotations of one particular way of sitting are all considered equivalent to it?

  $n!/n = (n-1)!$.

- An r-**permutation** of a set of n objects is an **ordered** selection of r objects taken from the n objects.

The number of r-permutations of a set of n objects is $P(n,r) = \dfrac{n!}{(n-r)!}$

- The addition rule:

  If a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, ..., A_k, then

  $$|A| = |A_1| + |A_2| + \ldots + |A_k|.$$

- Example:

  suppose each Alberta license plate consist of one, two, or three letters followed by three digits. How many different plates can be generated?

  Answer: Let A_1 be the set of those that have one letter, A_2 those that have 2, and A_3 be those with exactly three letters. Then $|A_1| + |A_2| + |A_3| = 26 \times 10^3 + 26^2 \times 10^3 + 26^3 \times 10^3$.

- Difference rule: For two sets A and B (of a universe U), if $B \subseteq A$ then $|A - B| = |A| - |B|$.

- Example: Among license plates of the form CCC-DDD how many have a repeated letter or repeated digit?

  It is easier to count the total number of license plates, and those that have no repeated letter or digit and the subtract it from the total.

  $(26 \times 26 \times 26 - 26 \times 25 \times 24) \times 10 \times 10 \times 10$

- If S is a finite sample space and A is an event in S, then

  $$Pr(A^c) = 1 - Pr(A).$$

- The inclusion/exclusion rule

  If A, B, and C are finite sets, then

  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$

- Example How many positive integers ≤ 1000 are multiples of 3 or 5?

  Let A be the set of those that are multiple of 3; so $|A| = 333$

  If B is the set of those that are multiple of 5 then $|B| = 200$.

  $A \cap B$ are those that are multiple of 15; $|A \cap B| = 66$.

  So $|A \cup B| = 333 + 200 - 66 = 467$.

- An r-**combination** of a set of n elements — a subset of r of the n elements

  $(r \leq n)$

- $\binom{n}{r}$: the number of r-combinations

- Note that an r-permutation is **ordered**

- A key relationship: an r-combination corresponds to $r!$ r-permutations

  Therefore,

$$P(n,r) = r! \times \binom{n}{r}, \quad \text{or,}$$

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)! \times r!}$$

- Example: Consider a group of 7 people.

  How many ways can we select a group of 4?

  Answer: $\binom{7}{4}$.

  What if there is a couple that insist they have to be together always?

  Answer: It will be the number of groups of 4 without the couple, which is $\binom{5}{4}$, plus the number of groups that cinlude the couple which is $\binom{5}{2}$.

What if there are 3 men and 4 women and we want to select a group of 5 which involves 2 men and 3 women?

Answer: $\binom{3}{2} \times \binom{4}{3}$.

- $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$

  Proof: By the formula

  By "element argument":

  Consider all the subsets of size r from the set $\{1, 2, \ldots, n, n+1\}$. We partition these into two groups:

  G_1 consist of all those r-combinations which do not include $n+1$, so they are r-combinations of $\{1, \ldots, n\}$. Thus $|G_1| = \binom{n}{r}$.

  G_2 consist of all those r-cmbinations which do include $n+1$; each such set is obtained from an $(r-1)$-combination of $\{1, \ldots, n\}$ by adding element $n+1$. So $|G_2| = \binom{n}{(r-1)}$.

- More examples

  - The poker hand problem (Royal flush, straight flush, four of a kind, full house, flush, straight, three of a kind, two pairs, one pair, no pairs)

  - How many hands in total? $\binom{52}{5}$.

  - How many hands are full house?

    Answer: a full house is like $5, 5, 5, A, A$. We have 13 choices for the rank of the triple and then 12 choices for the rank of the pair (which has to be different from the rank of triple). For each then we have $\binom{4}{3}$ ways to select the triple and $\binom{4}{2}$ to select the pair:

    $$13 \times 12 \times \binom{4}{3} \times \binom{4}{2}$$

    .

  - For computations for other hands, check:
    http://en.wikipedia.org/wiki/Poker_ probability

- Permutations of a set with repeated elements:

  Suppose a set contains n objects, of which

  - n_1 are of type 1 and are indistinguishable from each other
  - n_2 are of type 2 and are indistinguishable from each other
  - $\ldots$
  - n_k are of type k and are indistinguishable from each other

  and $n = n_1 + n_2 + \ldots + n_k$.

  Then the number of **distinct** permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-n_2-\ldots-n_k}{n_k}$$

$$= \frac{n!}{n_1!n_2!n_3!\ldots n_k!}$$

- Example: number of different ways we can re-arrange the letters in MISSISSIPPI?

$$\frac{11!}{4!4!2!1!}$$

- An r-**combination with repetition allowed**, or **multi-set of size** r, of a set of n (distinct) elements is an unordered selection of elements taken from the set with repetition allowed

- Each element in the base set can be selected for unlimited (but $\leq r$) times

- Counting

  - Making r copies for each element in the base set ...
    Is $\binom{n\times r}{r}$ the correct answer?

    NO (why?)

  - The answer is $\binom{n+r-1}{r}$. For proof see Theorem 6.5.1 from the text.