Week 6: Induction

Agenda:

- More examples of simple induction
- Strong Induction

Reading:

- Textbook pages 227–244.

- Example: Prove that for all $n \geq 1$: $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 1$.

  Proof: Let $P(n)$ be the predicate: $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} < 1$. We prove $P(n)$ for all $n \geq 1$.

  Basis: for $n = 1$: $\frac{1}{2} < 1$; so $P(n)$ holds.

  Ind. Step: Let $k \geq 1$ be an arbitrary integer and assume that $P(k)$ holds; i.e. $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} < 1$. We prove $P(k+1)$ holds too.

  For $k+1$ the LHS of inequality in the predicate is:

$$\begin{aligned}\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k}\right) \\ &< \frac{1}{2} + \frac{1}{2} \times 1 = 1\end{aligned}$$

  because the sum in the (..) is the LHS of inequality for $P(k)$ which by induction hypothesis is less than 1.

- We can in fact prove something stronger than previous statement: for all $n \geq 1$: $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

  Proof: Let $P(n)$ be the predicate: $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$. We prove $P(n)$ for all $n \geq 1$.

  Basis: for $n = 1$: $\frac{1}{2} = 1 - \frac{1}{2}$; so $P(n)$ holds.

  Ind. Step: Let $k \geq 1$ be an arbitrary integer and assume that $P(k)$ holds; i.e. $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$. We prove $P(k+1)$ holds too.

  For $k+1$ the LHS of inequality in the predicate is:

$$\begin{aligned}\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \\ &= \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \\ &= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}}\end{aligned}$$

Some Common Errors:

- There is a gap between the basis and induction step.

- We "prove" the following predicate for all $n \geq 1$:
  Given any set of $n \geq 1$ different lines in the plane no two of which are parallel to each other, all lines must have one point in common.

  Basis: Obvious for $n = 1$. Even we can prove for $n = 2$: any two lines not parallel to each other have one (and exactly one) common point.

  Ind. Step: Suppose the statement is true for any set of k lines; we prove it for $k + 1$.

  The first k lines (by induction hypothesis) go through one common point.

  The last k lines go through one common point.

  The common point of the first k lines and the last k lines must be the same (otherwise all the lines between 2 and $k-1$ go through two different points; thus all are the same, contradicting the assumption that they are all different).

- What's wrong with the above "proof"?

  We implicitly assume that $k \geq 3$ in the induction step; but the basis only proves for values ≤ 2.

- Example: Prove that for all $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

  Let $P(n)$ be "$2^{2n} - 1$ is divisable by 3"; we prove $P(n)$ for all $n \geq 1$.

  Basis: $P(1)$ holds since $2^{2\times 1} - 1 = 3$ is divisible by 3.

  Ind Step: Let $k \geq 1$ be an arbitrary integer and suppose $P(k)$ holds, i.e. $2^{2k} - 1 = 3r$ for some integer r.

  Then $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} \cdot 4 - 1 = 3 \cdot 2^{2k} + 2^{2k} - 1 = 3 \cdot 2^{2k} + 3r = 3(2^{2k} + r)$.

Strong Mathematical Induction:

- **Principle (axiom)**: Let $P(n)$ be a property/predicate defined for integers n, and a a fixed integer
  - $P(a)$ is true
  - For all integers $k \geq a$, if $P(k)$ is true then $p(k+1)$ is true

  Then, "for all integers $n \geq a$, $P(n)$" is true.

- Proof by (the principle of) Mathematical Induction:
  - (basis step): $P(a)$ is true
  - (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $p(k+1)$ is true

Examples:

- Let $a_0, a_1, a_2, \ldots$ be a sequence defined by the following rules:

$$a_i = \begin{cases} 2 & 0 \leq i \leq 2 \\ a_{i-1} + a_{i-2} + a_{i-3} & i \geq 3 \end{cases}$$

  Prove that $a_n \leq 2^n$ for all $n \geq 2$.

  Proof: Let $P(n)$ be the predicate $a_n \leq 2^n$; we prove $P(n)$ for all $n \geq 2$.

  Basis: For $n = 2$: $a_2 = 2 \leq 2^1$; so $P(2)$ holds
  For $n = 3$: $a_3 = 2 + 2 + 2 = 6 \leq 2^3$; so $P(3)$ holds
  For $n = 4$: $a_4 = 2 + 2 + 6 = 10 \leq 2^4$; so $P(4)$ holds.

  Ind. Step: Let $k \geq 4$ be an arbitrary integer and assume that $P(i)$ holds for all $2 \leq i \leq k$. We prove $P(k+1)$ holds.

  Because $k \geq 4$ then $k+1 \geq 5$ and so $a_{k+1} = a_k + a_{k-1} + a_{k-2}$. Since $k \geq 4$, each of k, $k-1$, and $k-2$ are at least 2.

  Thus $a_{k+1} \leq 2^k + 2^{k-1} + 2^{k-2} = 2^{k-2}(2^2 + 2 + 1) = 2^{k-2} \cdot 7 \leq 2^{k-2} \cdot 2^3 = 2^{k+1}$.

  Thus $P(k+1)$ holds too.

Well-Ordering Principle:

- The **Well-Ordering Principle** for integers

  Let S be a set containing one or more integers all of which are greater than some fixed integer.

  Then S has a least element.

- For example, the set of prime numbers.

- The **Quotient-Remainder Theorem**

  Given any integer n and a positive integer d, there **exist unique** integers q and r such that

  $$n = dq + r \text{ and } 0 \leq r < d.$$

  Proof As in the text.