Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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Preview - What's Next in Combinatorial Games

What's Next? Program for Next Few Weeks

- Types of games - some classifications
- Impartial games introduction (Example Nim)
- Impartial games algorithms, MEX rule, Lemoine and Viennot's algorithm, Sprouts and Cram results
- How to play sums of games; incentives of moves
- All-small games (Example Clobber)
- Temperature and thermographs
- Temperature-based algorithms

Types of Combinatorial Games

Review - Combinatorial Games

- $G = \{G^L | G^R\}$
- A game is given by sets of left and right options
- Recursively, each option is again a game
- Can build a whole theory from this one definition
- What games are there?
- Many ways to classify games
- Review what we have so far, then introduce more types

Types of Games - Finite vs Infinite

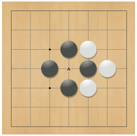
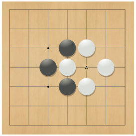
Finite vs infinite games

- We mostly ignore infinite games
- Example of an infinite game:
  - $G = \{G_1 | \}$
  - $G_1 = \{G_2 | \}$
  - $G_2 = \{G_3 | \}$
  - $G_3 = \{G_4 | \}$
  - ...
  - Infinitely many moves for Left

Types of Games - Loopy vs Loop-free

Games with loops ("loopy" games) vs loop-free games

- We mostly ignore games with loops
- Examples of loopy games
- $G = \{G|0\}$
  - Left can play back to the same position here
  - Right can break out of the loop, move to 0
- $G = \{G|G\}$ this one never ends...
- $G = \{1|H\}$, $H = \{G| - 1\}$
  - This is like a "ko" situation in Go
  - Players can move back and forth between G and H

More Classification

- From now, focus on finite, loop-free games
- Can classify them further
- Next step: numbers vs not-numbers
- Which games are (finite) numbers?

Games that are Integers

- So far: integers = free moves for one player
- $0 = \{\,|\,\}$, $1 = \{0\,|\,\}$, $-1 = \{\,|\,0\}$
- Recursive definition of integers:
  $n + 1 = \{n\,|\,\}$
  $-(n+1) = \{\,|\,-n\}$
- From having $n + 1$ free moves, player can "use up" one move and have n moves left

Fractions

- Fractions: other games behave like fractions
- In finite games, we can only construct fractions where the denominator is a power of 2
- "Dyadic" fractions of the form $\frac{n}{2^m}$ for integer n, m

Fractions Example

- Example: the game $0|1$
- Claim: $0|1$ "behaves like" the number 1/2
- Claim: $0|1 + 0|1 = 1$
- Let's do the proof - play the difference game $0|1 + 0|1 - 1$
- If this is a second player win, then:
- $0|1 + 0|1 - 1 = 0$
- $0|1 + 0|1 = 1$

Fractions Examples

- Definition: 0|1 = 1/2
- "Average" advantage = 1/2 move for Left
- If we have 2 copies of this game, Left gets exactly 1 more move than Right, $0|1 + 0|1 = 1$
- If we have 1000 copies of this game, Left gets exactly 500 more moves than Right, etc.
- In finite games, all fractions are "dyadic": they have denominator that is a power of 2.
- Examples: 1/2, 3/4, -5/16, ...

Construction of Fractions

- Theorem (Siegel Th. 3.6):
- Let $n \geq 1$ and m odd. Then

$$\frac{m}{2^n} = \{\frac{m-1}{2^n} | \frac{m+1}{2^n}\}$$

- Examples:
- $5/8 = \{4/8 \,|\, 6/8\} = \{1/2 \,|\, 3/4\}$
- $3/4 = \{2/4 \,|\, 4/4\} = \{1/2 \,|\, 1\}$
- $1/2 = \{0/2 \,|\, 2/2\} = \{0 \,|\, 1\}$
- $-5/16 = \{-6/16 \,|\, -4/16\} = \{-3/8 \,|\, -1/4\}$

Numbers and Outcome Classes

- All positive numbers: L-positions, Left wins
- All negative numbers: R-positions, Right wins
- 0: $\mathcal{P}$-position, second player wins
- No numbers are $\mathcal{N}$-positions (no first player wins)

More Classification - Games that are not Numbers

- Impartial games, Nimbers
- All-small games, infinitesimals
- Hot games

Impartial Games

- Impartial games are a special type of combinatorial games
- Both players always have the same options
- Example: Nim heap with n tokens, $*n$
- Impartial games have their own specialized algorithms
- We'll talk about them soon
- Theorem: all finite impartial games are *equal* to some nim heap
- Opposite of impartial: **partizan** games
- Partizan: players may have different options

Impartial Games and Outcome Classes

- No $\mathcal{L}$-positions
- No $\mathcal{R}$-positions
- $\mathcal{P}$-position: *0 = 0
- All others are $\mathcal{N}$-positions (first player wins)

Infinitesimals

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Preview - What's Next in Combinatorial Games

Types of Combinatorial Games

- Game G such that $-\epsilon < G < \epsilon$ for any number $\epsilon > 0$
- Examples so far: 0, *, Nimbers *n
- More examples: *up* = $\uparrow$ = $0|*$, *down* = $\downarrow$ = $*|0$
- Try to prove: $\downarrow < 0 < \uparrow$
- Also true: $-\epsilon < \downarrow$ and $\uparrow < \epsilon$
- Examples: tiny and miny: for integer n, $+_n = 0||0|-n$, $-_n = -(+_n) = n|0||0$

Future lecture: definition of infinitesimals by using "Leftscore" and "Rightscore"

Infinitesimals (2)

- Infinitesimals occur frequently in games such as **Amazons**, *cooled* Go (see later).
- Tinies and minies correspond to threats
- Threat: From tiny 0ll0l–n, miny nl0ll0 :
  - Move to $n|0$ or $0|-n$ threatens some (large) profit, n or $-n$, with the next move
  - *Usually*, opponent will reply to the threat and move to 0
- Outcome classes of infinitesimals: all four, e.g. $\uparrow$, $\downarrow$, 0, *

All-Small Games

- Clobber is an *all-small* partizan game -
- All values are infinitesimals
- Outcome classes: all four, e.g. $\uparrow$, $\downarrow$, 0, *

Hot Games

- Hot game: first player can earn extra moves by playing
- Moves in such games are more valuable than moves in infinitesimals, or in numbers
- Examples: $\{1|0\}, \{2|-2\}, \{5|-2\}, \{-1|-6\}, \{15|12\}$
- In all these examples, Left moves to a *larger* number than Right
- There are many other, more complex hot games
- Outcome classes: all except $\mathcal{P}$ (0 is a number, not hot)

Preview: Types of Games and Temperature

- Soon we will talk about the temperature of a game G
- A single number $t(G)$
- Roughly: how many points (= free moves) would we pay for the right to go first?
- Temperature is closely related to the type of game
  - Hot game: $t(G) > 0$
  - Most infinitesimals: $t(G) = 0$
  - Non-integer numbers (fractions): $-1 < t(G) < 0$
  - Integers: $t(G) = -1$

Summary and Outlook

- Covered the most important types of games
- For many specialized types, not all four outcome classes can happen
- Several of these types allow specialized algorithms
- We can analyze sumes of those games more efficiently
- Hard problem: play sums of hot games (or a mix of hot and other games)