

Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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How to play a Sum Game?

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- Review: global alphabeta can be too slow
- Incentives are great if they work
(e.g. decomposition search, later)
- Using incentives fails when:
 - incentives are incomparable
 - subgames are too complex to compute incentives
- What else can we do?
Use temperature/thermographs

Why Use Temperature/Thermographs?

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- It is much easier to compute thermographs (TG) than canonical forms
- Memory per game is constant in practice
Most TG have very few lines, 10 or more is rare
- Even knowing approximate temperatures can be a big help
Need an algorithm to compute them

Simple Sum Strategies - Hotstrat

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- Simple Sum Strategies
 - Use local data such as thermograph only
 - No global search
- Simplest idea: HotStrat
- Play in hottest subgame
ties broken arbitrarily
- Which move? See later

HotStrat Example

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- Sum game $G + H + K$
- $G = 10|8||2$
- $H = 5| - 3$
- $K = 5|1|| - 1| - 8$
- Temperatures
 - $t(G) = 7/2$
 - $t(H) = 4$
 - $t(K) = 15/4$
- H is hottest
 - HotStrat will play in H
 - for either Left or Right going first

HotStrat Example

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Sample play of sum $G + H + K$, assuming Right goes first:

- Move $H \rightarrow H^R = -3$
- $t(H^R) = -1$ since it is integer
- Now sum game is $G - 3 + K$, with $t(G) = 7/2, t(K) = 15/4$
- Left moves in K which is is hottest
- $K \rightarrow K^L = 5|1$
- $t(K^L) = 2$
- Sum $G - 3 + K^L$, G is hottest, Right moves from G to 2
- Sum $2 - 3 + K^L = -1 + K^L$, Left moves K^L to $K^{LL} = 5$
- Sum $-1 + 5 = 4$. Stop reached.

HotStrat Move Selection

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- Remember the construction of scaffold
- Each Left scaffold is the max of the taxed left options
- In general, at different t , different moves can contribute to the max
- HotStrat chooses an *orthodox* move that defines the scaffold of the thermograph at the *temperature* of the game
- Ties broken arbitrarily
Tie: different options have the same value at $t(G)$

Definition - Orthodox Move

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- $L_t(G)$ = Left scaffold at temperature t
- $R_t(G^L)$ = Right scaffold of option G^L at t
- Move to G^L is orthodox at temperature t (or t -orthodox) iff taxed right option equals left scaffold of G at t
- $R_t(G^L) - t = L_t(G)$
- Likewise for right options (tax: $+t$):
- G^R is orthodox iff $L_t(G^R) + t = R_t(G)$

Weakness of HotStrat

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- We have seen before that the hottest move is not always optimal
- How bad can it get?
- We would like an error bounded by the temperature of the hottest subgame
- Remember $t(G + H) \leq \max(t(G), t(H))$
- Unfortunately, HotStrat can lose much more
- It can be tricked into unbounded loss
Loss grows without a limit, as the number of subgames grows
- Note: this is worst-case performance. Not too bad in practice

Weakness of HotStrat - Example

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- $G = \{10||0| - 20\}|-4, t(G) = 2$
- $H = 10||9| - 9, t(H) = 1$
- $m(G) = -2, m(H) = 9$, so $m(G + H) = 7$
- Play $G + H$, Left goes first and follows HotStrat:
- G hottest, so $G \longrightarrow G^L = \{10||0| - 20\}, t(G^L) = 10$
- Right *does not follow HotStrat*, plays
 $H \longrightarrow H^R = \{9| - 9\}, t(H^R) = 9$
- G^L is hottest, so Left plays there to 10, then Right plays H^R to -9
- Total score $10 - 9 = 1$
- Both played the same number of moves

Weakness of HotStrat - Watching the Means

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- $G = \{10||0| - 20\}|-4, m(G) = -2$
- $H = 10||9| - 9, m(H) = 9$
- Left plays $G \rightarrow G^L = \{10||0| - 20\},$
- $m(G^L) = 0$, Left gains 2 points over $m(G) = -2$
- Right $H \rightarrow H^R = \{9| - 9\}$
- $m(H^R) = 0$, Right gains 9 points over $m(H) = 9$
- Left $G^L \rightarrow 10$, Left gains 10 points
- Right plays H^R to -9, Right gains 9 points
- Total gains Left - Right = $(2+10) - (9+9) = -6$ for Left
- Observe how means of subgames shifted: Right gained 9 two times, Left only got 2+10

Discussion

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- $m(G + H) = 7, t(G + H) \leq 2$
- Left goes first, can expect to get to around $m+t = 9$
- However, Left only got to a stop of 1
- Left even lost 6 points compared to the mean, far more than the temperature
- And Left even went first!
- If we play n copies of $G + H$:
- We should expect to get around n times $m(G + H) = 7n$
- With the strategies above, all n copies will play out the same way.
- Left only gets a score of 1 per copy
- Loses about $6n$ points, grows to infinity with increasing n
- Unbounded loss

How to Fix HotStrat? - Watching the Means

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- $G = \{10||0| - 20\}|-4, m(G) = -2$
- $H = 10||9| - 9, m(H) = 9$
- Left plays $G \rightarrow G^L = \{10||0| - 20\},$
- $m(G^L) = 0$, Left gains 2 points
- Right $H \rightarrow H^R = \{9| - 9\}$
- $m(H^R) = 0$, Right gains 9 points
- **Left answers** $H^R \rightarrow H^{RL} = 9$
- $m(H^{RL}) = 9$, Left gains 9 points
- Right $G^L \rightarrow G^{LR} = \{0| - 20\}$
- $m(G^{LR}) = -10$, Right gains 10 points
- Left $G^{LR} \rightarrow 0$
- $m(0) = 0$, Left gains 10 points

How to Fix HotStrat? - Watching the Means

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- Total gains Left - Right = $(2+9+10) - (9+10) = +2$ for Left
- Compare with previous result.
- Left “defended” the mean of H , end result in H is $H^{RL} = 9$.
- Left gained 2 points from going first in G , $m(G) = -2$ and end result in G is $G^{LRL} = 0$

How to Fix HotStrat?

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- In the example, Left should answer Right's *temperature-increasing* move. $H^R \longrightarrow H^{RL} = 9$
- Then Right moves $G^L = \{10 || 0 | - 20\}$ to $G^{LR} = \{0 | - 20\}$
- Finally, Left moves to G^{LR} to 0, total score +9
- Compare with previous result. Left “defended” the mean of H .

Refinements of HotStrat

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- HotStrat+ (Müller and Li 2004)
 - play sente moves earlier
 - A little better in practice
 - no bounded loss either
- ThermoStrat (Winning Ways)
 - **bounded loss** (by temperature of second hottest game)
 - pretty complex
- **SenteStrat** (Berlekamp 1996)
 - Simpler than ThermoStrat
 - Also has **bounded loss**
 - Focus on this method here

Simple Sum Strategy - SenteStrat

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- SenteStrat (Berlekamp 1996):
 - Answer opponent *sente* moves in the same subgame
 - Play HotStrat otherwise
- How is sente defined here?
- Answer: relative to an *ambient* a :
the lowest temperature reached so far
- An opponent move is considered *sente* if:
 - They move any subgame G_i to a temperature $t > a$
 - In all such cases, SenteStrat replies in the same G_i
- SenteStrat is a simple algorithm with bounded loss

SenteStrat on Our Example

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- $G = \{10||0| - 20\}|-4, t(G) = 2$
- $H = 10||9| - 9, t(H) = 1$
- Ambient $a = \max(t(G), t(H)) = 2$
- Left plays HotStrat, $G \longrightarrow G^L = 10||0| - 20$
- Again Right tries $H \longrightarrow H^R = 9| - 9$
- $t(H^R) = 9 > a$
- Left treats this move as sente and answers in H^R
- $H^R \longrightarrow H^{RL} = 9$
- Play then continues the same as in good sequence above

Summary and Outlook

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- Algorithms for sums of “relatively simple” subgames
- We can compute means, temperatures, thermographs of subgames
- Discussed simple “no-search” algorithms: HotStrat, SenteStrat
- Adding a little bit of search helps in practice, see (Müller and Li 2004) paper