Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

Martin Müller

Department of Computing Science
University of Alberta
mmueller@ualberta.ca

Winter 2022

Rational Numbers, Simplest Number and Birthday

- Remember integers
- $0 = \{|\}$
- $1 = \{0|\}$, $2 = \{1|\}$, ..., $n + 1 = \{n|\}$
- $-1 = \{|0\}$, $-2 = \{| - 1\}$, ..., $-(n + 1) = \{| - n\}$
- We also constructed games that behave like fractions

One Half

- Remember the game we called 1/2: $G = \{0|1\}$
- It is a win for Left
- We proved that $G + G = 1$ by showing that $G + G - 1$ is a second player win
- Mean is 1/2
- The left and right incentives are -1/2
  - Incentive for Left: $G^L - G = 0 - (1/2) = -1/2$
  - Incentive for Right: $G - G^R = 1/2 - 1 = -1/2$
- Left loses 1/2 by playing to 0
- Right loses 1/2 by playing to 1
- The temperature of 1/2 is also -1/2

Dyadic Rationals

- Remember dyadic rational: fraction p/q where q is some power of 2
- General construction in CGT: integers m odd, $j > 0$

$$\frac{m}{2^j} = \frac{m-1}{2^j} \mid \frac{m+1}{2^j}$$

Incentives of Rational Numbers

We can easily compute the incentives

$$G = \frac{m}{2^j} = \frac{m-1}{2^j} \mid \frac{m+1}{2^j}$$

$$G^L - G = \frac{m-1}{2^j} - \frac{m}{2^j} = -\frac{1}{2^j}$$

$$G - G^R = \frac{m}{2^j} - \frac{m+1}{2^j} = -\frac{1}{2^j}$$

- All incentives of numbers are negative numbers
- The other implication also holds:
  - If the incentive is a negative number
  - Then the game is a number

Simplest Number Theorem

- What about games with other dyadic fractions as options?
- E.g. $\{3/8 \,|\, 13/16\}$
- The answer is the *simplest number* between G^L and G^R
- Theorem
  - $G = \{G^L | G^R\}$
  - All options in G^L and G^R are numbers,
  - All options $G^L < G^R$.
  - Then G is the simplest number between G^L and G^R

What is the Simplest Number?

- Rule 1: any integer n is simpler than any nearby non-integer in open interval $(n-1, n+1)$
- Rule 2: among integers, smaller absolute value is simpler (0 is simplest of all)
- Rule 3: among dyadic fractions between two consecutive integers, one with smaller denominator is simpler
- Example: $3/8 \mid 13/16 = 1/2$
- $\{-3 \mid 5\} = 0$
  - A *zugzwang* (forced to move even you don't want in chess)
  - Both players would lose points by playing
- $\{-10, -8, -6 \mid -3, -2\} = -4$
  - Another zugzwang
  - -4 is simpler than -5
- Alternative definition uses *birthday* of games

Birthday

- We construct games recursively
- A game's *birthday* is the level of recursion which first generates it
- Simpler = has smaller birthday
- Birthday 0: only $0 = \{|\}$
- Birthday 1: only options with birthday 0
  - $1 = \{0|\}$, $-1 = \{|0\}$, $* = \{0|0\}$
- Birthday 2: only options with birthday 0 or 1
  - 18 total. Examples: $2 = \{1|\}$, $\{1|-1\}$, $\uparrow * = \{0, *|0\}$
- Birthday n: only options with birthday $< n$
  - Examples: integers n and $-n$ have birthday n
  - Switch $\pm n = \{n| - n\}$ has birthday $n + 1$

Summary

- A few extra facts about rational numbers
- Defined simplest number in an interval
- Defined Birthday of a game