Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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Tax vs Coupons

- Berlekamp's insight:
- Paying a tax of t...
- ...is like declining a coupon (or cheque) of value t
- In both cases, the opponent can get a benefit of t:
- Either directly through the tax, or from taking the coupon

Coupons and Enriched Environment

- The goal of temperature theory is to compare games with numbers
- Both approaches are equivalent
- Both extend down to temperature -1 and play of integers
- As with economic rules, the presence of an “enriched environment” with a large set of coupons makes a sum game much simpler
- We can simply play by “the book” of means and temperatures
- There is no more big advantage from getting “the last move at high temperature” - there is always a slightly less valuable coupon for the other player

Remember Sums of Switches?

- Switches are simple games of value $\{v | -v\} = \pm v$, where $v > 0$ is a number
- Such a switch has temperature v and mean 0
- Sums of switches are very easy to play
- Just play the switch with highest temperature
- Example: $G = \pm v_1 \pm v_2 \pm v_n$, where $v_1 > v_2 > ... > v_n > 0$
- Left stop $LS(G) = v_1 - v_2 + v_3 - +...$
- $RS(G) = -LS(G)$ by symmetry

Sums of Equal Switches

- Note: two switches of equal temperature cancel
- This is called *miai* in Go, important strategy in that game
- $\pm v + \pm v = 0$
- Example:
  - $G = \pm 5 \pm 5 \pm 4 \pm 3 \pm 3$
  - $LS(G) = 4$
  - Both moves in ± 5 and ± 4 are optimal (why?)
  - Playing in ± 3 first is a mistake:
  - Score would be $3 - 5 + 5 - 4 + 3 = 2$
- Sums of Equally Spaced Switches
  - $G = \pm x \pm 2x \pm 3x \pm ... \pm nx$
  - $LS(G) = \lceil n/2 \rceil x$

Coupon Stack (aka Enriched Environment)

- First in (Berlekamp 1996)
- Final version in (Berlekamp 2002)
- *Positively Enriched Environment* $\mathcal{E}_t$
- Fix $\delta > 0$
- Fix maximum temperature t, an integer multiple of δ
- $\mathcal{E}_t = \pm\delta + \pm 2\delta + ... + \pm t$
- In our TDS papers, we call it $C(t, \delta)$

Coupon Stack continued

- $\mathcal{E}_t = \pm\delta + \pm 2\delta + ... + \pm t$
- It is intuitive to think of all those switches as a *coupon stack*
- The only good move is to move in hottest switch $\pm t$
- “Take the largest coupon”
- For $t > \delta$, this leaves the stack $\mathcal{E}_{t-\delta}$ with one less coupon
- As a game:

$$\mathcal{E}_t = \mathcal{E}_{t-\delta} + \pm t = \{t + \mathcal{E}_{t-\delta} \mid -t + \mathcal{E}_{t-\delta}\}$$

Fully Enriched Environment

- After last positive coupon $\pm\delta$:
- Add coupons of values 0, $-\delta$, -2δ,...,-1
- Note: we cannot treat them as “switches” anymore
- E.g. $\{-\delta|\delta\} = 0$
- But we can simply define how to play a sum of any game *G* and a coupon stack:
- Either move in *G*, or take the top coupon
- In our TDS paper, we call these stacks $C_{-1}(t, \delta)$

Fully Enriched Environment at Negative Temperatures

- If G is a number with incentive less than t, players will prefer to move in $\mathcal{E}_t$
- Example: $G = 1/2, t = -1/8$
- Black would lose 1/2 by playing in 1/2, but only loses 1/8 by taking the top coupon in $\mathcal{E}_{-1/8}$
- Also, it will be White's turn next, and White will lose even more...

Bottom of the Stack

- We need a “large enough” supply of -1 coupons at the bottom of the stack
- Reason: similar as with economic rules
- If one player has an integer left, the opponent should be forced to take -1 coupons while the player finishes the integer game
- End of game: both players take a -1 coupon
- Remark: for loopy games we may need more than two such coupons
  - Taking a coupon lifts any repetition ban. Details later

Terminal Komi

- Last coupon, *terminal komi*: to avoid odd/even effects with taking -1 coupons at the end, we need a final coupon of value -1/2
- It should not matter which player begins taking the -1 coupons at the end
- Example: the following two sequences should have the same result

- Sequence 1
- Black takes -1
- Game over, White gets automatic -1/2
- Total: -1 - (-1/2) = -1/2 for Black

- Sequence 2
- Black takes -1
- White takes -1
- Game over, Black gets automatic -1/2
- Total: -1 - (-1) + (-1/2) = -1/2 for Black

Economic Rules vs Conventional Play

- Mean and Temperature solve a sum game under economic rules
- *Orthodox* play: follow optimal economic strategy (sentestrat)
- What can they say about normal rules, where the last player wins?
- The key is to compare what happens in the game with orthodox play
- Account for wins and losses relative to orthodox play
- We have already done a little bit of that in past examples

Review - Goal of Playing a Sum Game vs Thermographs

- If we "just" want to win: (assume we are Left)
- Make the last move. This is possible if either $G > 0$, or $G \not\gtrless 0$ and we go first, or $G = 0$ and we go second
- If $LS(G) > 0$ and we go first, we can try to use analysis based on means and temperatures
- If $LS(G) = 0$ and we can win, we may need to know more about infinitesimals

Review - Goal of Playing a Sum Game vs Thermographs

- Basic relations between LS, RS, mean and temperature t
- Note: all the following is for $t \geq 0$ and stopping play at numbers
- $\mu(G) - t \leq RS(G) \leq \mu(G) \leq LS(G) \leq \mu(G) + t$
- Example for equality: simple switches $\{a|b\}$
- For coupon stacks, the bounds are about $t/2$
- When playing Sentestrat, we can achieve a score $v \geq \mu(G)$

Playing a Coupon Stack

- Consider playing a coupon stack $C(t, \delta)$
- The first player will win by about $t/2$ (precisely $t/2$ if δ is "small enough")
- Example 1: $C_{-1}(6, 1)$
- $LS(C(6, 1)) = 6 - 5 - 4 - 3 - 2 - 1 + 0 - (-1) + (-\frac{1}{2}) = 3.5$
- Problem: play below 0 does not add up to 0, but to 0.5. Not balanced.

Playing a Coupon Stack (2)

- Example 2: $C_{-1}(6, \frac{1}{4})$
- $6 - 5.75 + 5.5... = 3$ exactly
- Why?
- Play above 0 gives exactly 6/2 = 3:
- $(6-5.75)+(5.5-5.25)+...(0.5-0.25) = 12 \times 0.25 = 3$
- Play at 0 and below sums up to exactly 0:
- $0 - (-0.25) + (-0.5) - (-0.75) + (-1) - (-0.5) = 0$
- This is true for any $\delta = 2^{-n}$ and $n > 0$, so $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$,...

Playing a Coupon Stack Plus a Game

- Example: $G + C$, switch $G = 4| - 1$, stack $C = C_{-1}(4, \frac{1}{4})$
- Left goes first: $t(G) = 2.5 < 4$, so first, both take coupons
- Left: $4 - 3\frac{3}{4} + 3\frac{1}{2} - 3\frac{1}{4} + 3 - 2\frac{3}{4} = \frac{3}{4}$ for Left
- Now, Left can take either $2\frac{1}{2}$ coupon, or play in G. Assume Left plays in G
- Left: $G \to G^L = 4$
- Only a stack $C = C_{-1}(5/2, \frac{1}{4})$ is now left, and it is Right's turn to go first. Score half of $5/2$ for Right, so $\frac{-5}{4}$
- Minimax score: $\frac{3}{4} + 4 - \frac{5}{4} = \mathbf{3\frac{1}{2}}$
- Compare with just playing the stack without G, $LS(C) = 4/2 = 2$
- Left has gained 3/2 **which is *exactly* the mean of G!**

Playing a Coupon Stack Plus a Game - Comment

- In general, play will start with a series of coupons
- When the temperature of C drops to “around” G, the first move in G will occur
- If that move increases the temperature, we will see a series of moves in G
- When the first stable follower is reached, play will switch back to taking more coupons
-

Berlekamp's Main Theorem About Coupon Stacks

- Berlekamp (1996), but notation from our TDS paper (2004)
- $V(G, p)$ = stop value of game G for player p going first
- $V(G, Left) = LS(G)$
- $V(G, Right) = RS(G)$
- Main Theorem:
- If δ is "small enough" and t is "high enough", then for coupon stack $C = C_{-1}(t, \delta)$:
- $V(G + C, p) = \mu(G) + V(C, p)$

What is “small enough” and “high enough”?

- Small enough δ:
  Choose δ such that all temperatures of all positions in all subgames are multiples of 2δ
- All temperatures are dyadic fractions of form $k/2^n$
- Let n' be the largest such n
- Then $\delta = 1/2^{n'+1}$ works
- High enough t:
  Any t higher than the max. temperature of all subgames

Consequence of Main Theorem: How to Compute $\mu(G)$

- Stack $C = C_{-1}(t, \delta)$, t is multiple of 2δ
- Theorem: $V(G + C, p) = \mu(G) + V(C, p)$
- Stack alone: $V(C, p) = t/2$
- Find $v = V(G + C, p)$ by minimax search of $G + C$
- We can now compute $\mu(G) = v - t/2$ (!)

- Remark: what if t is *odd* multiple of δ?
- First player gets one extra coupon, we must adjust for that in search

How to Compute $t(G)$

- We can also compute $t(G)$ from TDS
- "$t(G)$ is the lowest temperature when optimal play needs to first switch from C to G"
- What does it mean? To explain the algorithm, we need three ingredients
  - How to play sums of switches (we know)
  - Understand the *principal variation* (PV) in minimax search
  - How to deal with one-sided sente

Principal Variation (PV)

- *Principal variation* (PV) in minimax search:
- Sequence of optimal moves by both sides
- Achieves minimax score
- With equally good moves, we can have several PV
- Earlier example: $G + C$,
  switch $G = \{4| - 1\}$, stack $C = C_{-1}(4, \frac{1}{4})$
- Start of PV, if Left goes first: both take coupons
- 1. Left $C(4)$, 2. Right $C(3\frac{3}{4})$, 3. Left $C(3\frac{1}{2})$,
  4. Right $C(3\frac{1}{4})$,...
- Remark: if one player makes a mistake, and the other player plays perfectly, then the score will differ from minimax in favor of the other player

Problems with Finding Temperature from PV

- Minor problem: game and coupon may have exactly same value in minimax
- Example PV continued:
  ... 5. Left $C(3)$, 6. Right $C(2\frac{3}{4})$
- Now, two equally valuable moves:
- Play in $G = 4|-1$, or take $C(2\frac{1}{2})$
- We have two PV:
- 7. Left $G \rightarrow G^L = 4$ 8. Right $C(2\frac{1}{2})$ 9. Left $C(2\frac{1}{4})$...
- 7. Left $C(2\frac{1}{2})$ 8. Right $G \rightarrow G^R = -1$ 9. Left $C(2\frac{1}{4})$...
- Both have the same minimax result
- Only different moves in both lines are 7 and 8
- Their difference in value is same for both PV:
  $4 - 2.5 = 2.5 - 1$

Problems with Finding Temperature from PV (2)

- In general, it is hard to control the search to always prefer say a move in C over one in G
- We do not know which of the two PV will be returned
- PV1: first play in G between $C(2\frac{3}{4})$ and $C(2\frac{1}{2})$
- PV2: first play in G between $C(2\frac{1}{2})$ and $C(2\frac{1}{4})$
- How to get exact t from that?

Problems with Finding Temperature from PV (3)

- Major problem: one-sided sente
- Consider earlier example $G = \{\{20|5\}|0\}$ plus stack $C = C_{-1}(7\frac{1}{2}, \frac{1}{4})$
- Left can play $G \to G^L$ at any temperature between $7\frac{1}{2}$ and 5
- PV_1: 1. Left $G \to G^L$, 2. Right $G^L \to G^{LR} = 5$, 3. Left $C(7\frac{1}{2})$ ...
  - Minimax score $5 + \frac{15}{4} = \frac{35}{4}$
- PV_2: 1. Left $G \to G^L$, 2. Right $C(7\frac{1}{2})$, 3. Left $G^L \to G^{LL} = 20$, 4. Right $C(7\frac{1}{4})$, 5. Left $C(7)$ ...
  - Minimax score $20 - 7\frac{1}{2} - 7\frac{1}{4} + \frac{7}{2} = \frac{35}{4}$

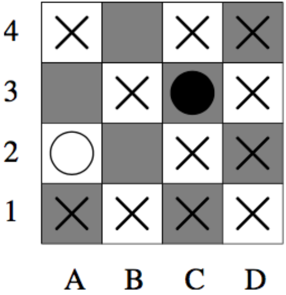
Problems with Finding Temperature from PV (4)

- Left can play first in G at $t = 7\frac{1}{2}$, or $t = 7$, or ... $t = 5$:
- 1. Left $C(7\frac{1}{2})$ 2. Right $C(7\frac{1}{4})$ 3. Left $C(7)$ ... 10. Right $C(5\frac{1}{4})$ 11. Left $G \rightarrow G^L$ 12. Right $G^L \rightarrow G^{LR} = 5$ 13. Left $C(5)$ ...
  - Minimax score $5 \times \frac{1}{4} + 5 + \frac{5}{2} = \frac{35}{4}$
- Left can even take $C(5)$ and let White play in G:
- 1. Left $C(7\frac{1}{2})$ 2. Right $C(7\frac{1}{4})$ 3. Left $C(7)$ ... 10. Right $C(5\frac{1}{4})$ 11. Left $C(5)$ 12. Right $G \rightarrow G^R = 0$ 13. Left $C(4\frac{3}{4})$ 14. Right $C(4\frac{1}{2})$ ...
  - Minimax score $5 \times \frac{1}{4} + 5 + 0 + 4\frac{3}{4} - \frac{9}{4} = \frac{35}{4}$

Finding Temperature from PV

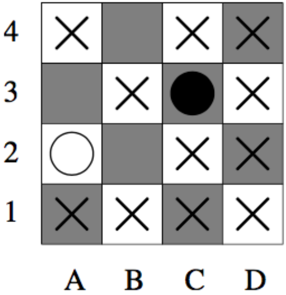
- We do not know from the PV whether a player was forced to play in G, or could have taken another coupon
- Solution:
  - Do another search
  - Force the player to take one more coupon
  - If minimax score changes: coupon was worth less, player needed to play in G
  - If minimax score is the same, try again, force player to take even more coupons
- $t(G)$ is the lowest temperature when optimal play needs to first switch from C to G

Example from Paper - Mean



- Search $G + C$, with stack $C = C_{-1}(t_{max} = \frac{17}{8}, \delta = \frac{1}{8})$
- Minimax score of $G + C$ $V(G + C, Left) = \frac{15}{8}$
- Minimax score of stack: $V(C, \text{Left}) = \lceil \frac{17}{2} \rceil \cdot \frac{1}{8} = \frac{9}{8}$
- $\mu(G) = \frac{15}{8} - \frac{9}{8} = \frac{3}{4}$

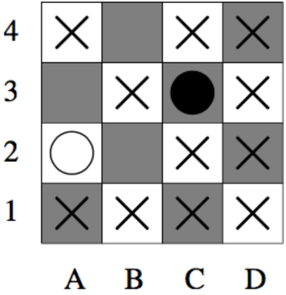
Example from Paper - CV



1. $C(\frac{17}{8})$ **2.** $C(\frac{16}{8})$ **3.** $C(\frac{15}{8})$
4. $C(\frac{14}{8})$ **5.** $C(\frac{13}{8})$ **6.** $C(\frac{12}{8})$
7. $C(\frac{11}{8})$ **8. A2–A3**$\times$**B2** **9.** $C(\frac{10}{8})$
10. $C(\frac{9}{8})$ **11.** $C(\frac{8}{8})$ **12.** $C(\frac{7}{8})$
13. $C(\frac{6}{8})$ **14.** $C(\frac{5}{8})$ **15.** $C(\frac{4}{8})$
16. $C(\frac{3}{8})$ **17.** $C(\frac{2}{8})$ **18.** $C(\frac{1}{8})$
19. $C(0)$ **20.** $C(-\frac{1}{8})$ **21.** $C(-\frac{2}{8})$
22. $C(-\frac{3}{8})$ **23. C3–B4**$\times$**C3**.

- After White move 8, $G^R = -1/2$, and $t(G^R) = -1/2$
- After Black move 23, $G^{RL} = -1$
- The program recognizes this as integer -1, stops search
- The rest of the stack is divided up starting with $C(-\frac{4}{8})$ for White, total value $\frac{2}{8}$ for Black

Example from Paper - Temperature



- First switch from C to G in PV:
- **7.** $C(\frac{11}{8})$ **8. A2–A3$\times$B2 9.** $C(\frac{10}{8})$
- Set initial estimate to $t = \frac{11}{8}$
- What if we force players to play $C(\frac{10}{8})$ instead of G?
- Answer: minimax score stays the same, so we can lower estimate to $t = \frac{10}{8}$
- What if we force players to play $C(\frac{9}{8})$ instead of G?
- Answer: minimax score changes. Playing in G is better than $C(\frac{9}{8})$
- Stop, with $t(G) = \frac{10}{8} = \frac{5}{4}$

Summary

- Coupon stack is an alternative model for computing means and temperature
- Same result as with taxes, but simpler to compute with
- Main theorem relates minimax search results to mean value of game
- TDS exploits this to find exact means and temperatures
- Next time: approximate TDS and playing sums of complex games