

Computing Science (CMPUT) 657

Algorithms for Combinatorial Games

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5 × 6 Amazons Solution

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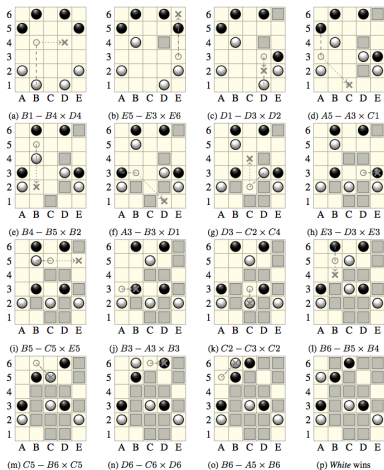


Figure 5.8: 5 × 6 board principal variation, White moves first and wins

Contributions

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- Paper: J. Song and M. Müller. An Enhanced Solver for The Game of Amazons. IEEE Transactions on Computational Intelligence and AI in Games (TCIAIG) 7(1), 16-27, 2015
- A strong evaluation function for computing correct bounds on the value of small full-board Amazons positions based on many types of local analysis.
- A df-pn based solver for Amazons which utilizes the evaluation components above and improves search efficiency, often by orders of magnitude.
- A technique for computing, storing and using databases of *blocker territories*.

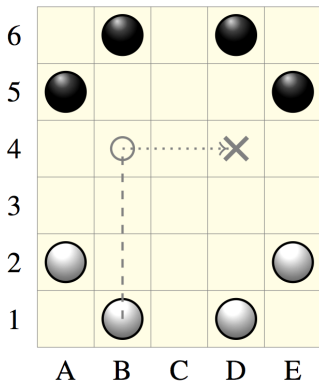
Contributions (2)

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- Two new static rules for improving the bounds on *active* local areas.
- A notation for expressing *tightened* and *relaxed* bounds on the value of a game.
- Techniques for exploiting knowledge about infinitesimals which often occur in Amazons, including an application of subzero thermography.
- An extensive empirical evaluation of the new solver.
- Solving the initial positions of Amazons on 4×5 , 5×4 , 4×6 , 5×6 and 4×7 boards to be first player wins, and solving 6×4 to be a second player win.

5 × 6 Amazons

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- 410 moves for first player
- Strong opening move shown
- Much harder than 5 × 5
- Much easier than 6 × 6
- Quite a bit easier than 6 × 5

Results on Rectangular Amazons Boards

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width height	4	5	6	7
4	2	①	①	①
5	①	1	①	?
6	②	?	?	?

- Winners for Amazons on small boards
- 1 = first player win, 2 = second player win
- New results are circled
- 5×4 much harder than 4×5
- 6×4 is second player win, much harder than 4×6 or even 4×7

Methods

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- Large number of databases
- Territories and blocker territories
- Active areas with both players
- Compute thermographs by retrograde analysis
- Several hundred databases
 - Many rectangle sizes, number of queens of each color
 - Covering all up to 2×6 and 3×5 , some 3×6 and 4×4

Bounds on Game Value

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- Use all fractions and some infinitesimals (5×5 proof used only integers)
- Three types of bounds from thermographs
- '[' and ']' - bound value is included
- '(' and ')' - bound value is excluded, “tightened bound”
- $\pm\epsilon$ - game is confused with bound, “relaxed bound”
- Example: $\text{bounds}(G) = [-3, -1)$ - $G \geq 3$ and $G < -1$
- Example: $\text{bounds}(G) = [-3, -1 + \epsilon]$ $G < -1 + \epsilon$ for all $\epsilon > 0$, but $G \leq -1$ is not proven
- Example: $-1 + *$, bounds $[-1 - \epsilon, -1 + \epsilon]$
- Addition follows the “obvious” rules

Bounds and Winners

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- Bounds b with lower bound $l(b)$, upper bound $u(b)$
- Black wins if:
 - $l(b) > 0$;
 - $l(b) = 0$ and the lower bound has been tightened;
 - $l(b) = 0$, the lower bound is not relaxed, and it is White to move.
- Similarly, White wins if:
 - $u(b) < 0$;
 - $u(b) = 0$ and the upper bound has been tightened;
 - $u(b) = 0$, the upper bound is not relaxed, and it is Black to move.

Bounds for Infinitesimals

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- 4 bounds correspond to the four subzero thermographs
- $[0, 0]$ for $G = 0$
- $(0, \epsilon]$ for $G > 0$
- $[-\epsilon, 0)$ for $G < 0$
- $[-\epsilon, +\epsilon]$ for G confused with 0.

Extra Rules for Infinitesimals

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- Collect $*$ separately, use $* + * = 0$.
- Black as the next player wins if:
 - no other infinitesimals exist and there is an odd number of $*s$;
 - no other infinitesimals exist and there is a single fuzzy infinitesimal;
 - all other infinitesimals are positive and there is either an odd number of $*$, or a single fuzzy infinitesimal, but not both.
- In all these cases Black wins by playing first in the fuzzy infinitesimal (which might be a $*$) and leaving a game $G^L \geq 0$ for White.

Local Search

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- Improve bounds of an active area
- Two local $\alpha\beta$ searches, one for each player going first
- Search score: difference in number of moves on the board
- Always allow Passes, to handle zugzwang positions
- Search values v_w and v_b for White and Black moving first:
- Bounds $[v_w - \epsilon, v_b + \epsilon]$
- Research question: can we get tighter bounds from these searches?
- Easier question: can we at least recognize 0? Other integers?

Example: A Win by *

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- Area A is recognized as a *
- Areas B , C and D are all 0
- Sum game is evaluated as *, first player can win
- Without databases: A still recognized as *, but B , C and D searched locally, with bounds $[-\epsilon, +\epsilon]$. No static winner

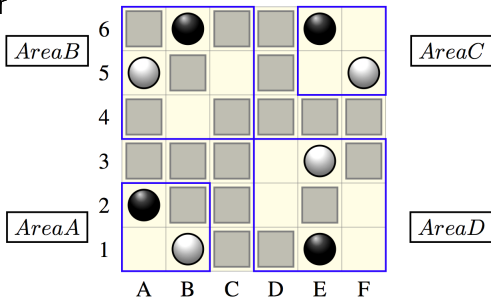


Figure: Black to move can win, value *.

Example: Local Search Finds a Win

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- B recognized as a $*$, $C = 1$ simple territory
- A not in DB, Local search bounds $[-\epsilon, 2 + \epsilon]$.
- $[-\epsilon, \epsilon] + 1 + [-\epsilon, 2 + \epsilon] = [1 - \epsilon, 3 + \epsilon]$, win for Black
- Without local search:
- A : bounds $[-2, 2]$, both have one safe move
- Global bounds $[-2, 2] + 1 + * = [-1, 3] + *$, not solved

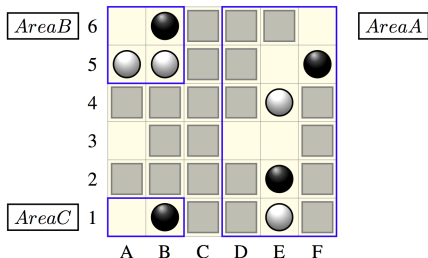


Figure: White to move, global bounds $[1 - \epsilon, 3 + \epsilon]$.

Experiment - old vs new Solver

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- Test cases from 5x5 paper
- Df-pn outperforms old $\alpha\beta$ solver, sometimes by more than an order of magnitude, in all large test cases and most small ones
- Using the databases sometimes reduces the number of nodes by another order of magnitude
- Territory databases yield only small improvements
- Most territories are very small and evaluated correctly even without DB

Solving 5×6 Amazons: A First Player Win

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- first move was fixed as White B1-B4xD4
- all 157 possible replies for Black were refuted
- Separate search tasks, with shared hash table did not clear table between searches
- 9 of 157 positions solved instantly
1 ply search and transposition table lookup
- 47 weakest Black moves took less than 1 second
- Hardest moves 2. E5-E3xE6 and E5-E3xE5 took 3-4 hours each
- Verification: “replay” solution trees without using hashing
Guards against possible errors from hash collisions

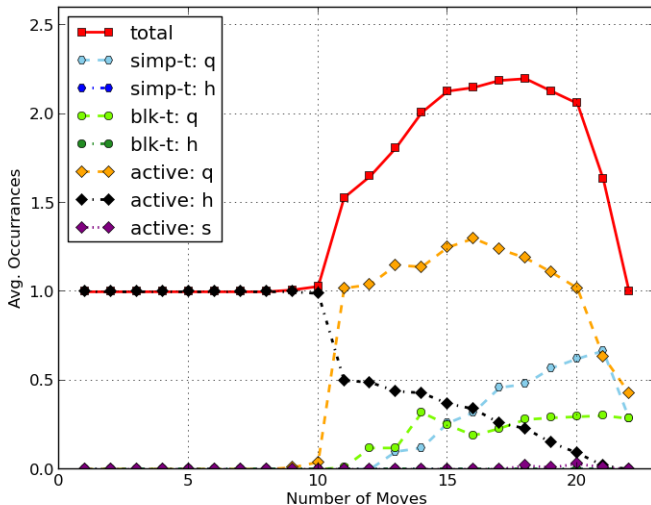
Solving 5×6 - Details

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- 9 of 157 solved instantly by a 1 ply search through a transposition table lookup
- Example
 - First, Black move 2. E5-C5xC4 refuted by a line starting with White 3. D1-D3xC2
 - The position after move 3 is stored in table as proven win
 - Next, Black move 2. E5-C5xC2 can be immediately refuted by White 3. D1-D3xC4
 - Both lines lead to the same position after three moves, swapping who plays the arrow shots to C2 and C4
 - (show details on whiteboard)

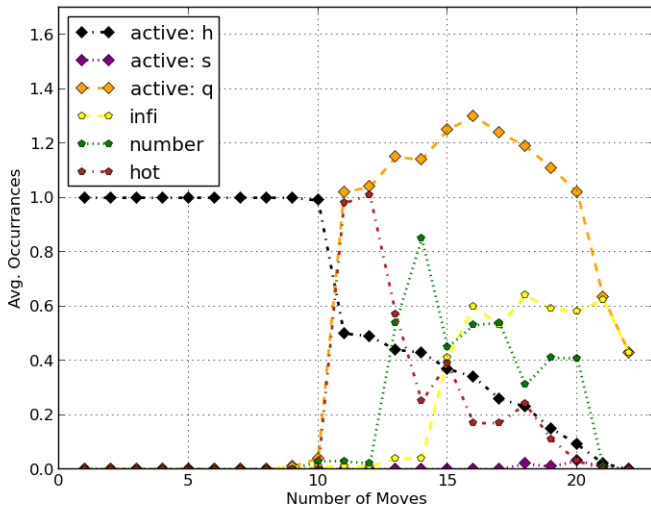
Types of Areas in Search - 5×6 Self-Play Samples

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Types of Areas in Search - Active Only

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Legend

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Legend	Meaning
total	the sum of all types of areas
simp-t: q	simple territories queried from the databases
simp-t: h	simple territories computed by heuristics
blk-t: q	blocker territories queried from the databases
blk-t: h	blocker territories computed by heuristics
active: q	active areas queried from the databases
active: h	active areas computed by heuristics
active: s	active areas computed by local searches
infi	infinitesimals in active: q
number	numbers in active: q
hot	hot games in active: q

Summary

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- Solved 5×6 and a number of other boards
- Strong use of CGT for pruning
- 6×5 seems within reach but still a bit large
- See <https://webdocs.cs.ualberta.ca/~mmueller/amazons/6x5-solution.html>
- Potential for more improvements, e.g. make local searches better (use TDS+ ???)
- 6×6 still needs better algorithms (or a really big computer...)