Efficiently Solving Games with Expected Work Search

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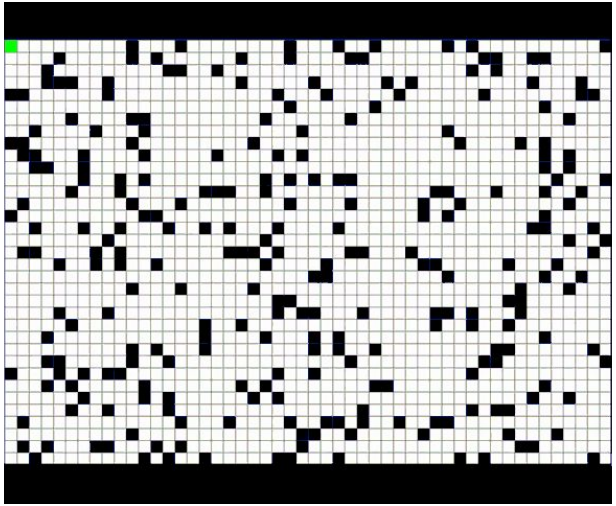
Introduction

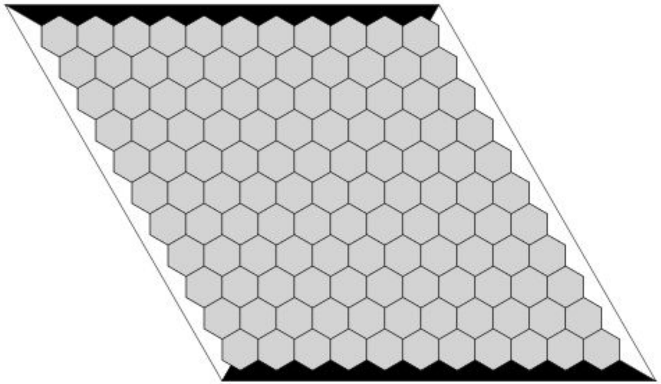
- Expected Work Search (**EWS**) is a new combinatorial search algorithm
    - Not a search engine like Google or Bing
    - Searches for solutions to problems with many possible positions
- Use **heuristics** to efficiently guide search
    - Can exponentially improve performance
- EWS can outperform existing algorithms
    - **3.8x** to **291x** faster when solving Go
    - **2.0x** to **3032x** faster when solving Hex

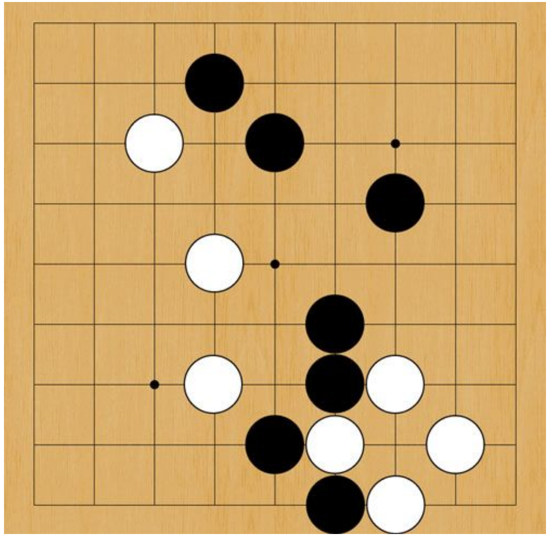


https://en.wikipedia.org/wiki/A*_search_algorithm#/media/File:Astarpathfinding.gif

Solving

- Evaluate EWS by **solving** two player perfect information games
  - Specifically Go and Hex
  - Well defined rules
  - Scalable difficulty
- Solving games prove who wins under **optimal play**
  - Any winning move = winning position
  - All losing moves = losing position

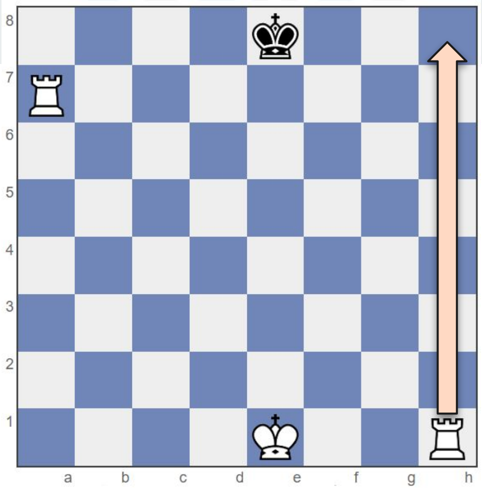


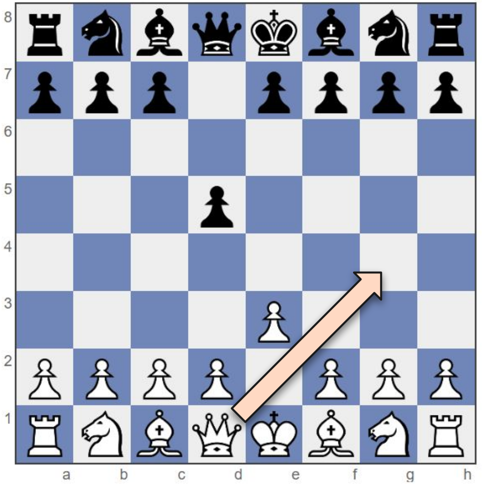


https://github.com/sotetsuk/pgx?tab=readme-ov-file https://www.irish-go.org/how-to-play-go/

Heuristics

- Win rate estimation
  - Monte Carlo Tree Search
  - Prioritizes strong moves
- Proof size estimation
  - Proof number search
  - Prioritizes quickly ending the game
- Both have strengths and weaknesses
- EWS combines both using Expected Work

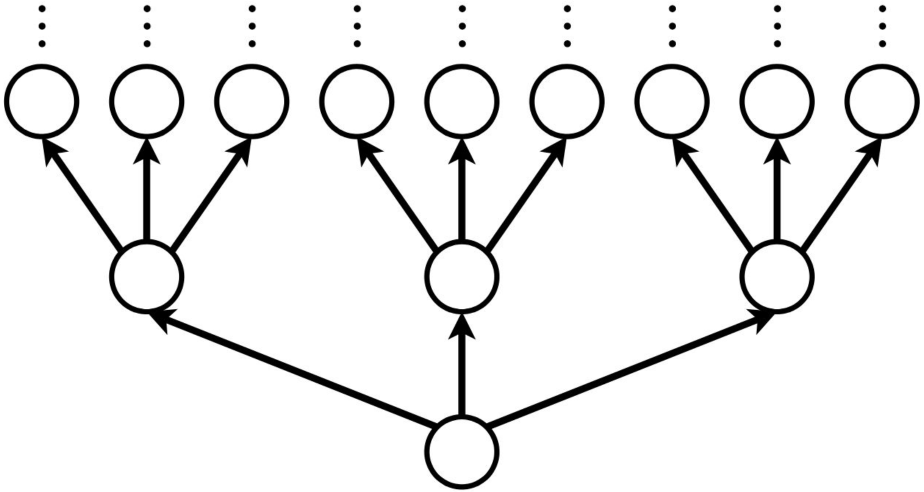




https://www.365chess.com/board_editor.php

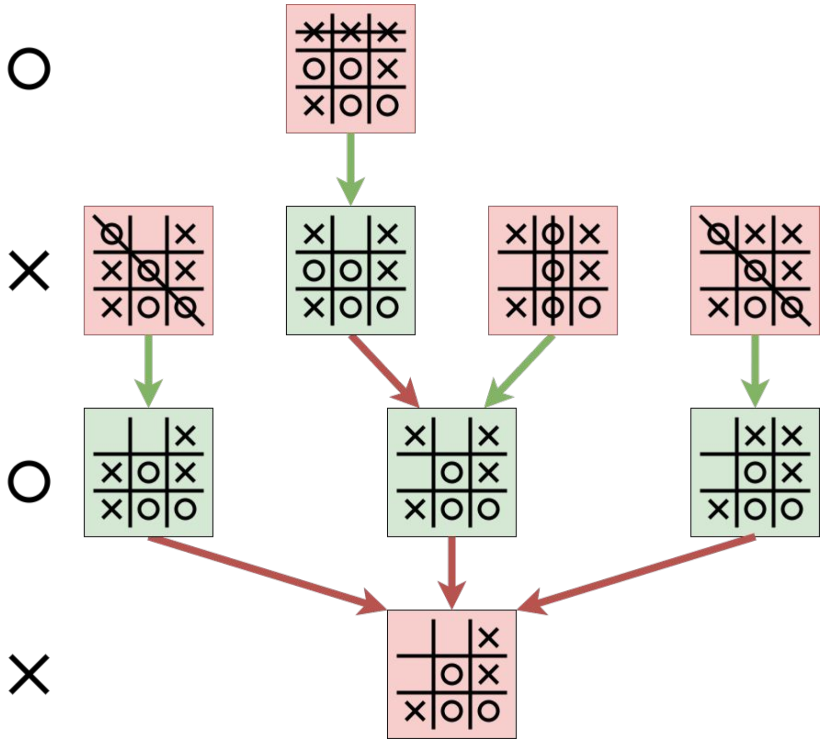
Expected Work Search

- Build a node tree in memory
- Nodes represent positions
- Each node has a win rate, and Expected Work estimate
- Each iteration of EWS builds the tree, refines node estimates, and backs up proofs



Expected Work Calculation

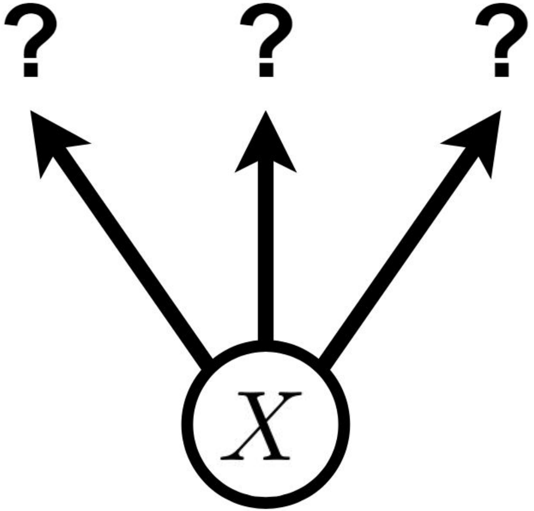
- How can we estimate the work it will take to solve a node?
- Inductively assume children have expected work values
- Separate cases for winning & losing
- Use MCTS style win rates to estimate winning probability
- Base case expected work values come from random simulations



Expected Work Calculation

- A position is either winning or losing

- Losing case:



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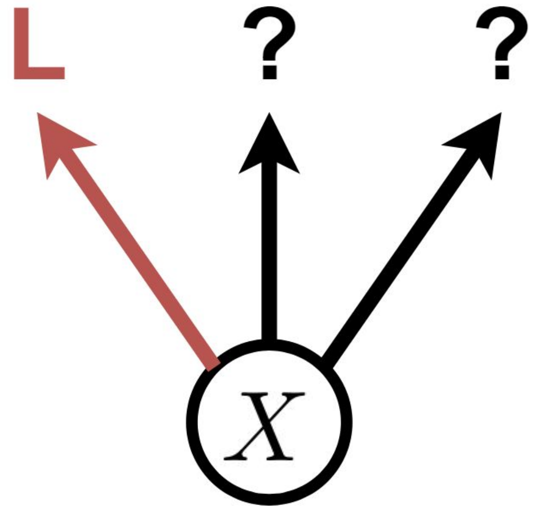

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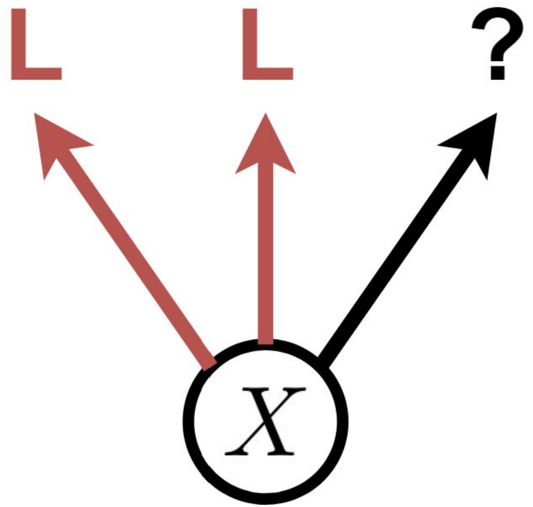

Expected Work Calculation

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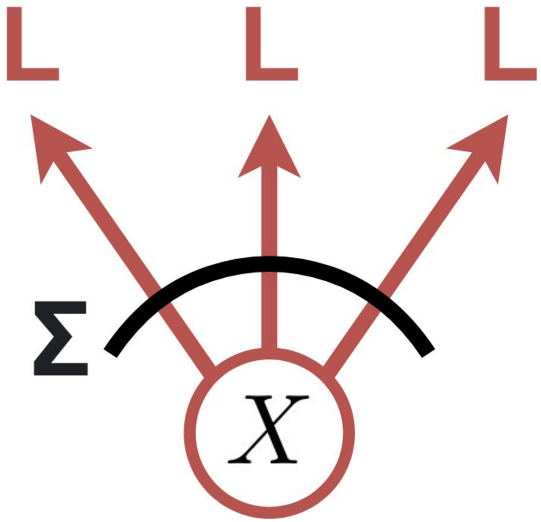
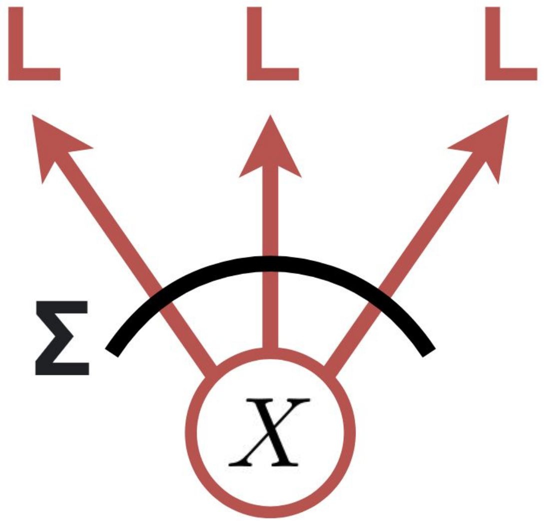


Expected Work Calculation

- A position is either winning or losing
- Losing case:
- Sum of winning children's EW

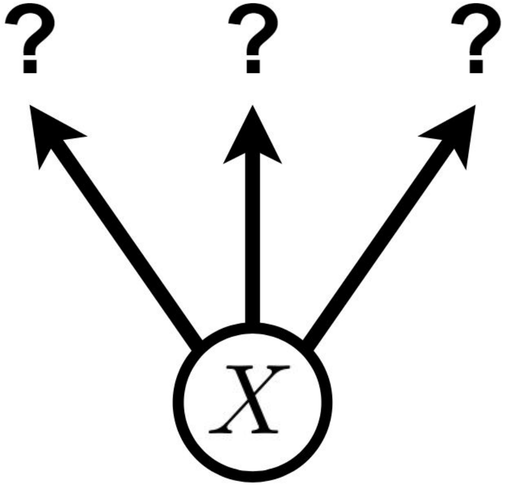


$$EW_{loss}(X) := \sum_{i=1}^{n} EW_{win}(C_i)$$

Expected Work Calculation

- A position is either winning or losing

- Winning case:



Expected Work Calculation

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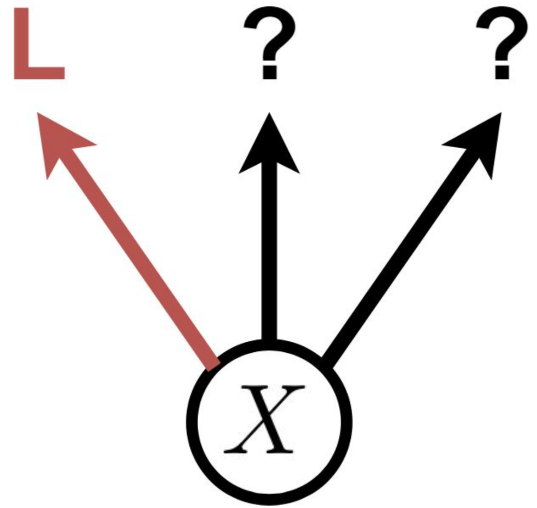
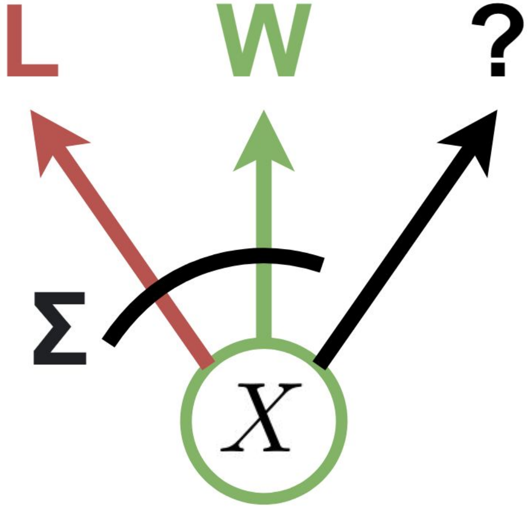
- Winning case:



Expected Work Calculation

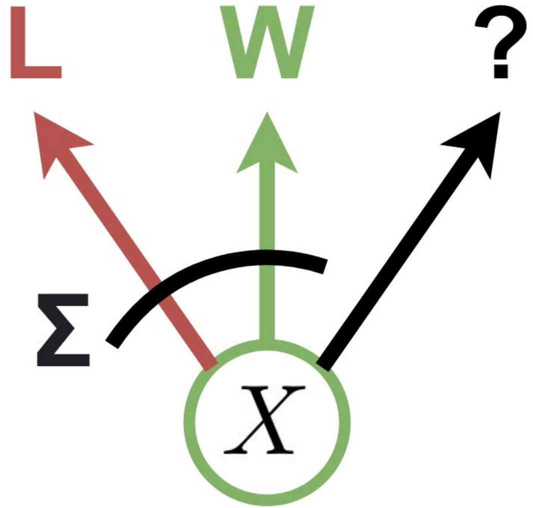
- A position is either winning or losing

- Winning case:



Expected Work Calculation

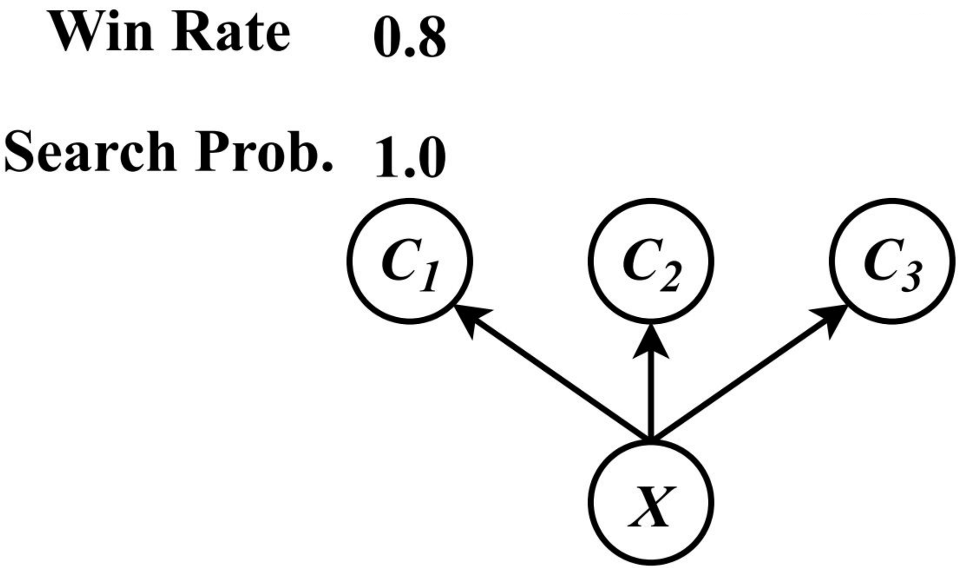
- A position is either winning or losing
- Winning case:
- **Weighted** sum of losing children's EW



$$EW_{win}(X) := \sum_{i=1}^{n} \left(EW_{loss}(C_i) \cdot \prod_{j=1}^{i-1} WR(C_j)\right)$$

$$EW_{win}(X) := \sum_{i=0}^{n} \left(EW_{loss}(C_i) \cdot \prod_{j=0}^{i-1} WR(C_j) \right) \qquad (2)$$

- Children are searched in order
- Search stops once a losing child is found
- The probability of searching a child = the probability none of the previous children are losing



$$EW_{win}(X) := \sum_{i=0}^{n} (EW_{loss}(C_i) \cdot \prod_{j=0}^{i-1} WR(C_j)) \quad (2)$$

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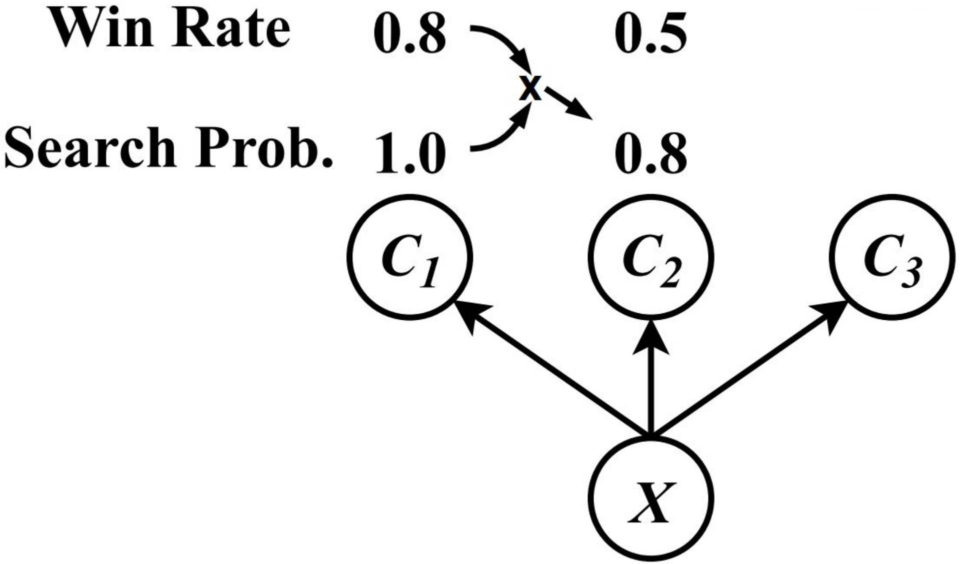

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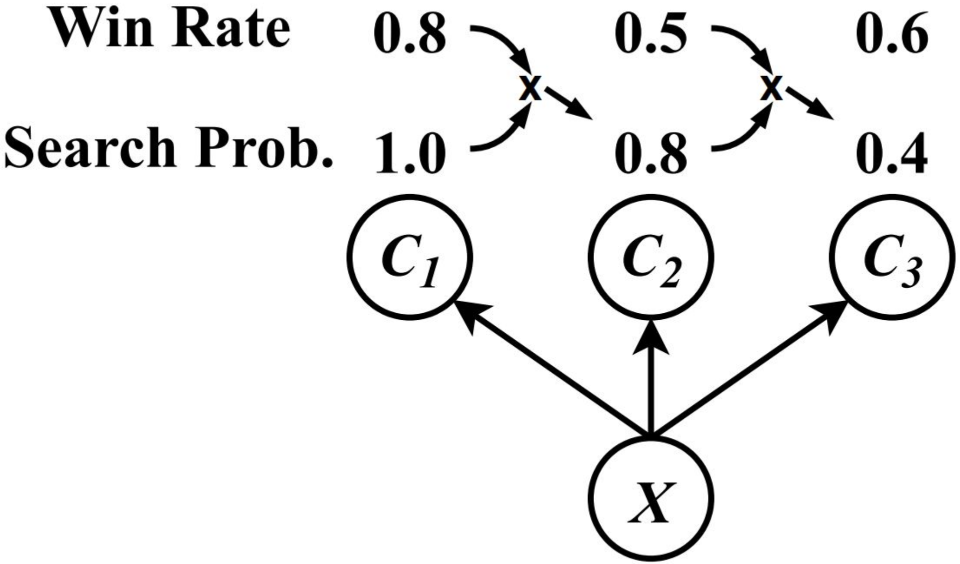

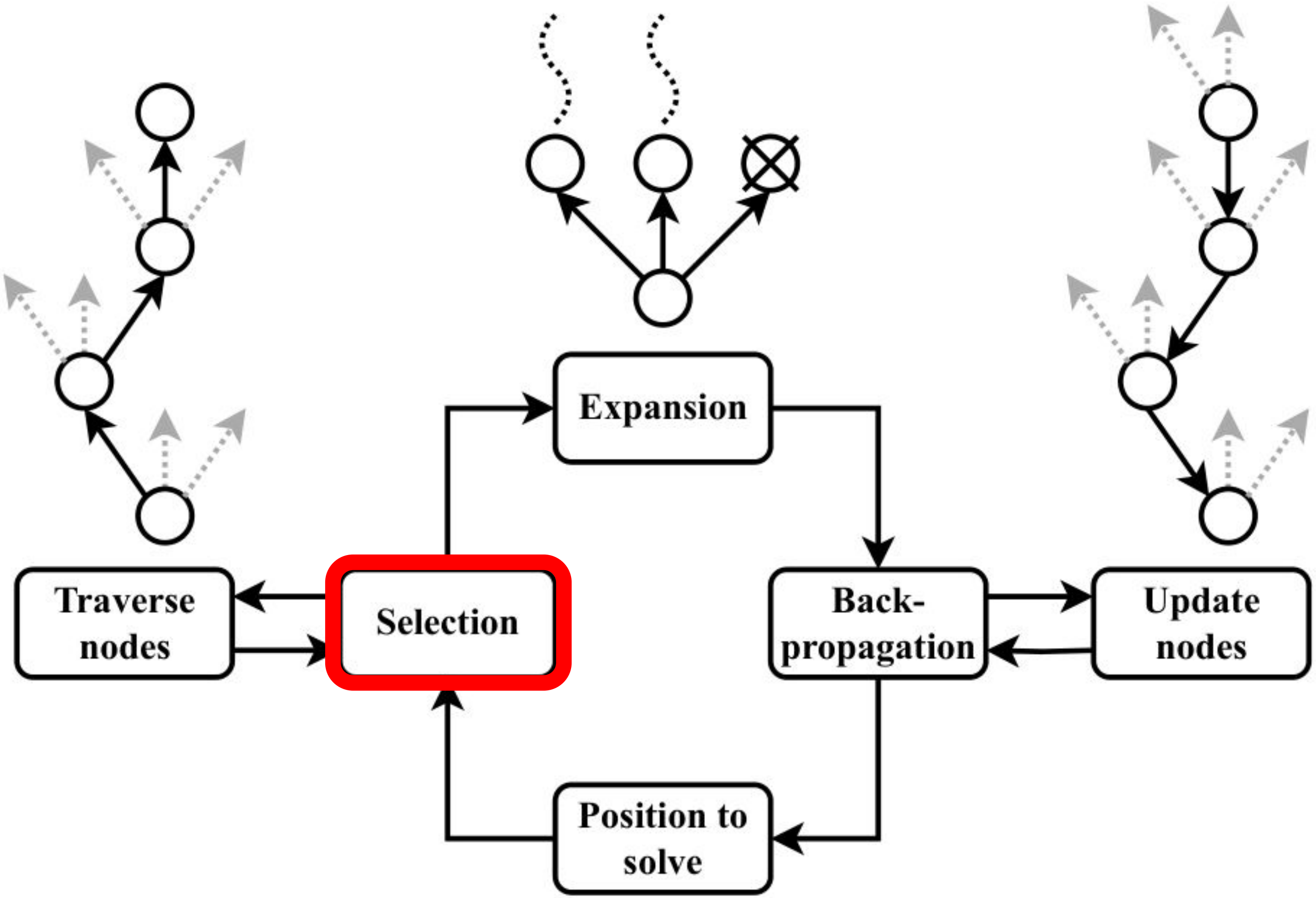
Expansion
Traverse nodes
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Back-propagation
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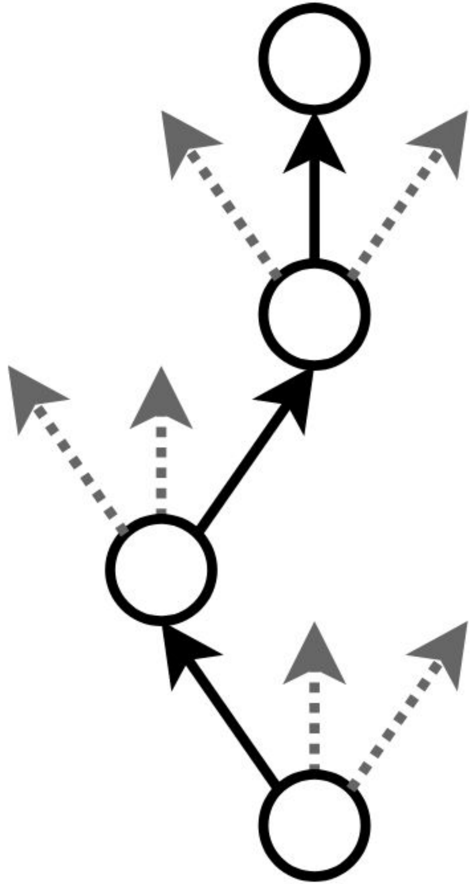
Position to solve

Selection

- Traverse the search tree
- Select the best child according to the move ordering
- If the selected child has been expanded, continue selection
- Otherwise, expand the leaf node



Child Ordering

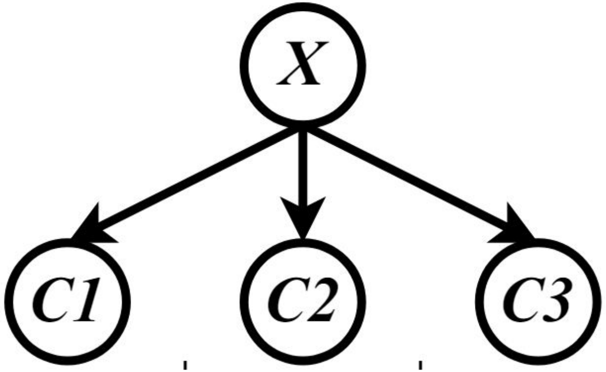
- Sort children with:

$$EW_{loss}(A)/(1-WR(A)) < EW_{loss}(B)/(1-WR(B))$$
$$\iff$$
$$A < B$$

- Best first move ordering
- Move ordering of children determines EW_{win}
    - **This ordering minimizes EW_{win}**

Child Ordering

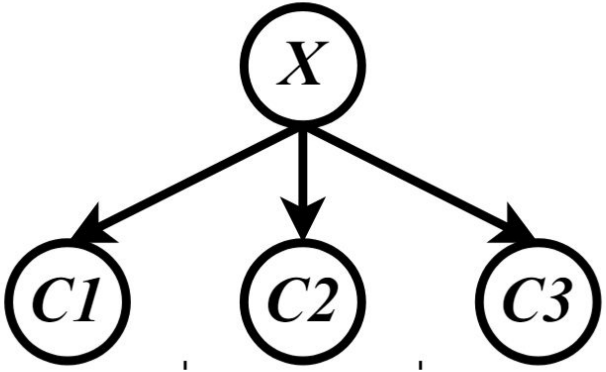
$$EW_{loss}(A)/(1-WR(A)) < EW_{loss}(B)/(1-WR(B))$$



	C1	C2	C3
WR	0.6	0.2	0.15
EW_{loss}	900	1000	1500

Child Ordering

$$EW_{loss}(A)/(1-WR(A)) < EW_{loss}(B)/(1-WR(B))$$



WR	0.6	0.2	0.15
EW_{loss}	900	1000	1500
EW/(1-WR)	**2250**	**1250**	**1765**

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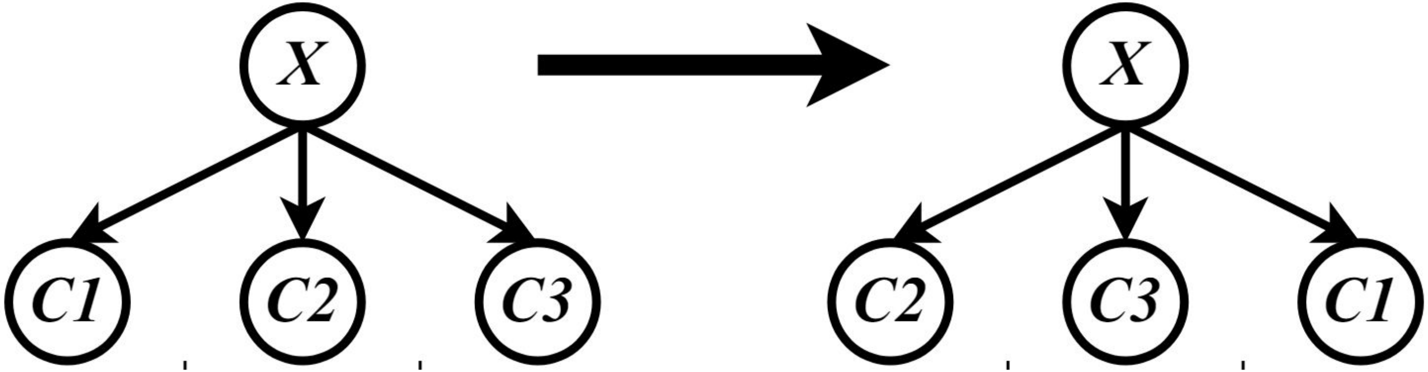

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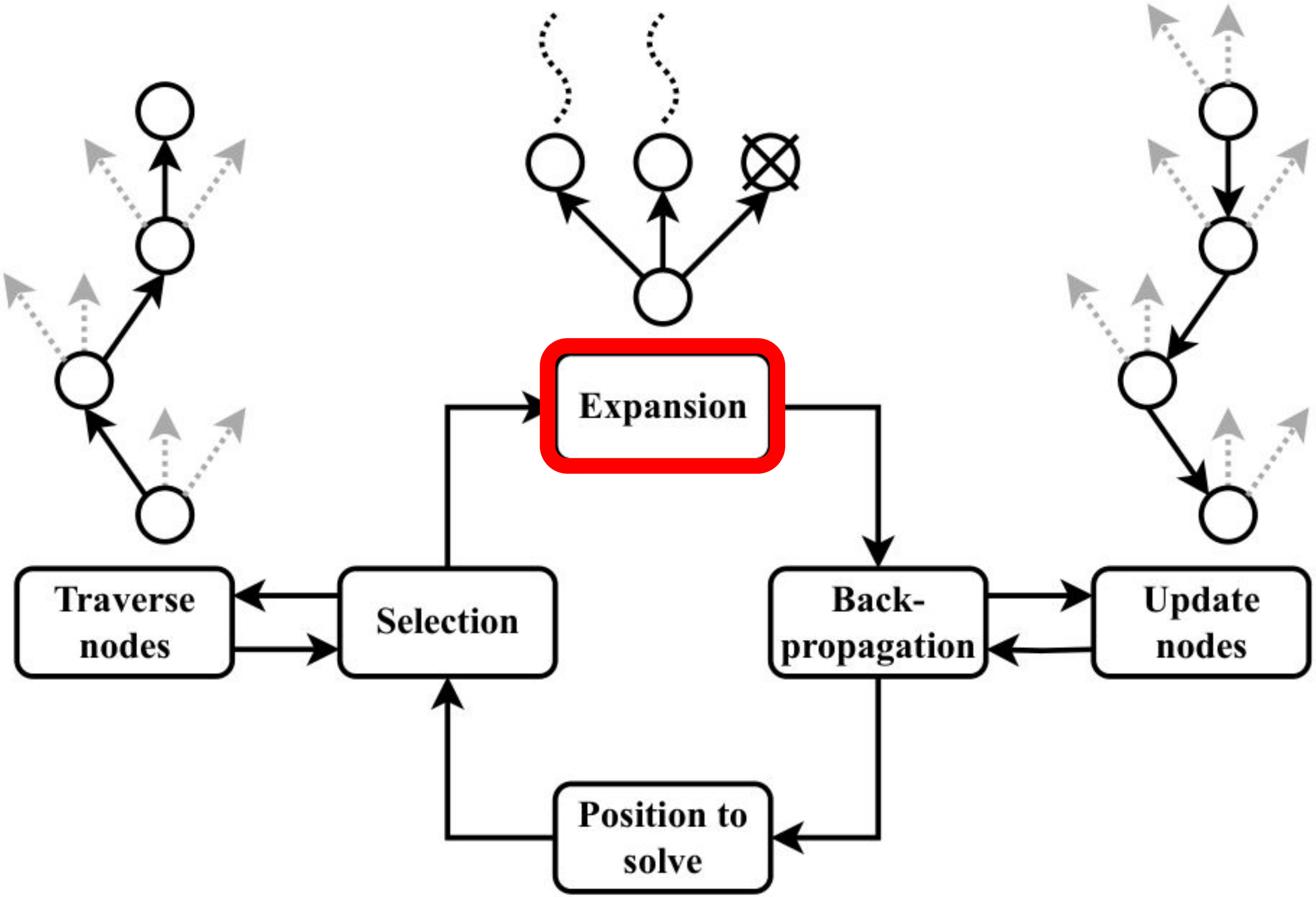
0.2	0.15	0.6
1000	1500	900
1250	**1765**	**2250**

Selection

- Traverse the search tree
- Select the best child according to the move ordering
- If the selected child has been expanded, continue selection
- Otherwise, expand the leaf node

Algorithm 1 SelectBackpropagate X: returns whether X is solved and if so whether X is winning

```
C := X.children[0]                                        ▷ Selection
Make move C.move
if C.expanded then
    isSolved, isWinning = SelectBackpropagate(C)
else
    isSolved, isWinning = Expand(C)
Undo move C.move                                          ▷ Backpropagation
if isSolved and isWinning then
    Remove C from X.children
    if X.children is empty then
        return true, false                                ▷ Solved loss
else if isSolved and not isWinning then
    return true, true                                     ▷ Solved win
Sort X.children with (3)
Update X.winRate
Update X.expectedWorkLoss with (1)
Update X.expectedWorkWin with (2)
return false, false                                       ▷ Unsolved
```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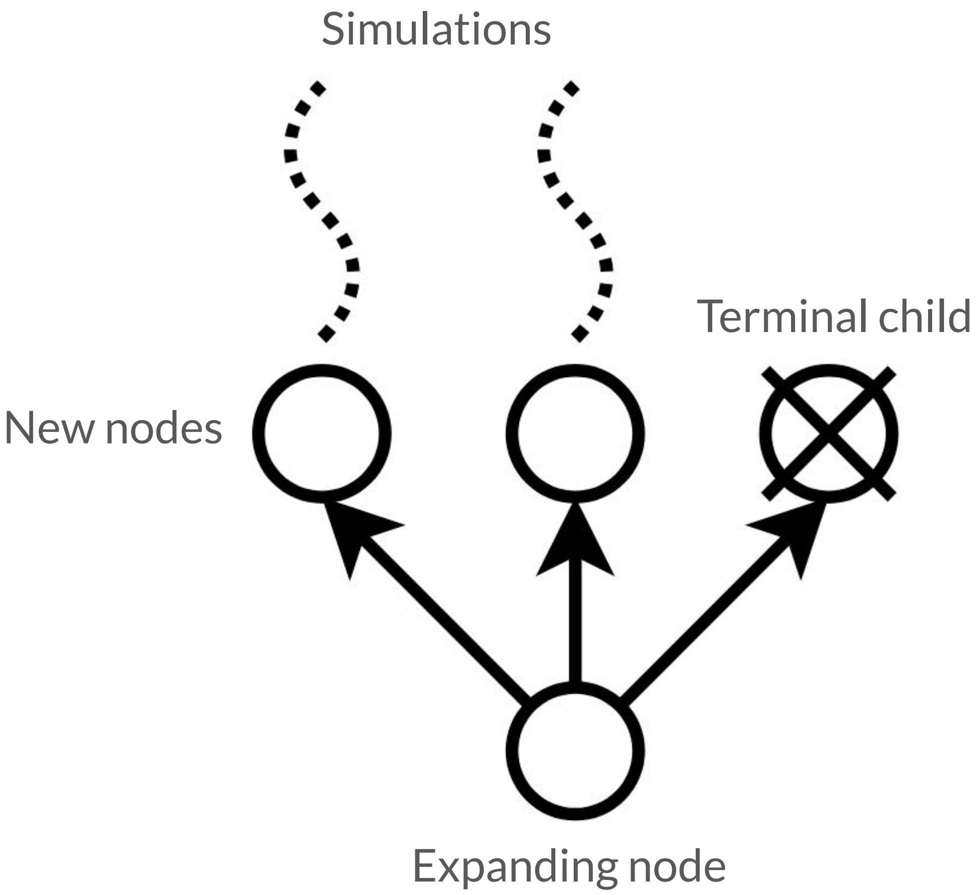
Expansion

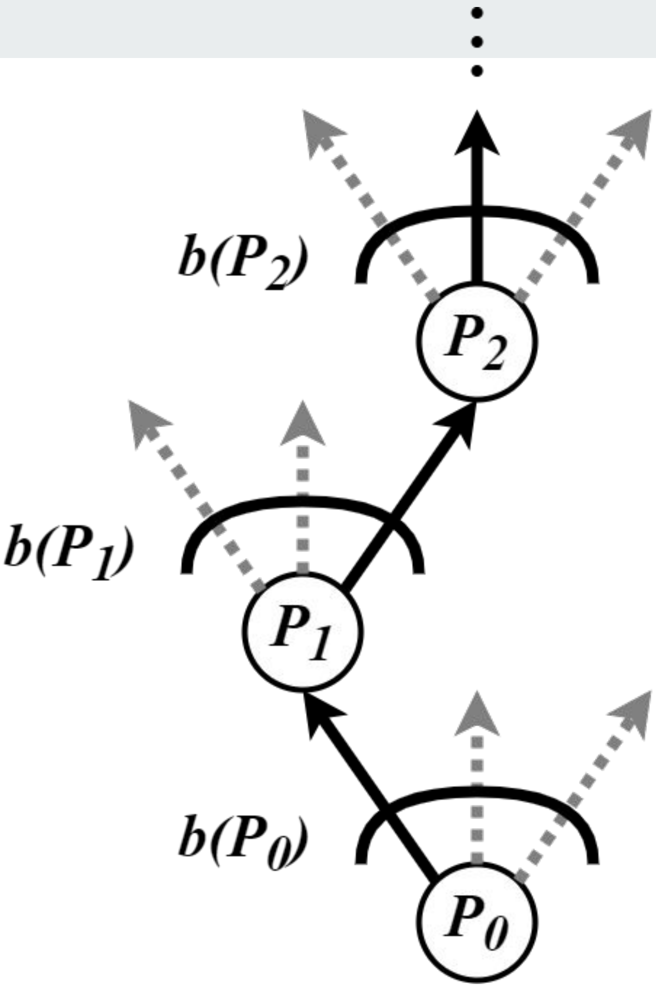
- Create new nodes
- Check for terminal children
- Initialize WR and EW with random simulation results
- Return if the expanded node is proven



Initial Evaluation

- Random simulations
- Win rates = wins / visits
- Initialize EW as the sum of the branching factors of positions in random simulations

$$EW_0(X) := \sum_{i=0}^{m} b(P_i)$$

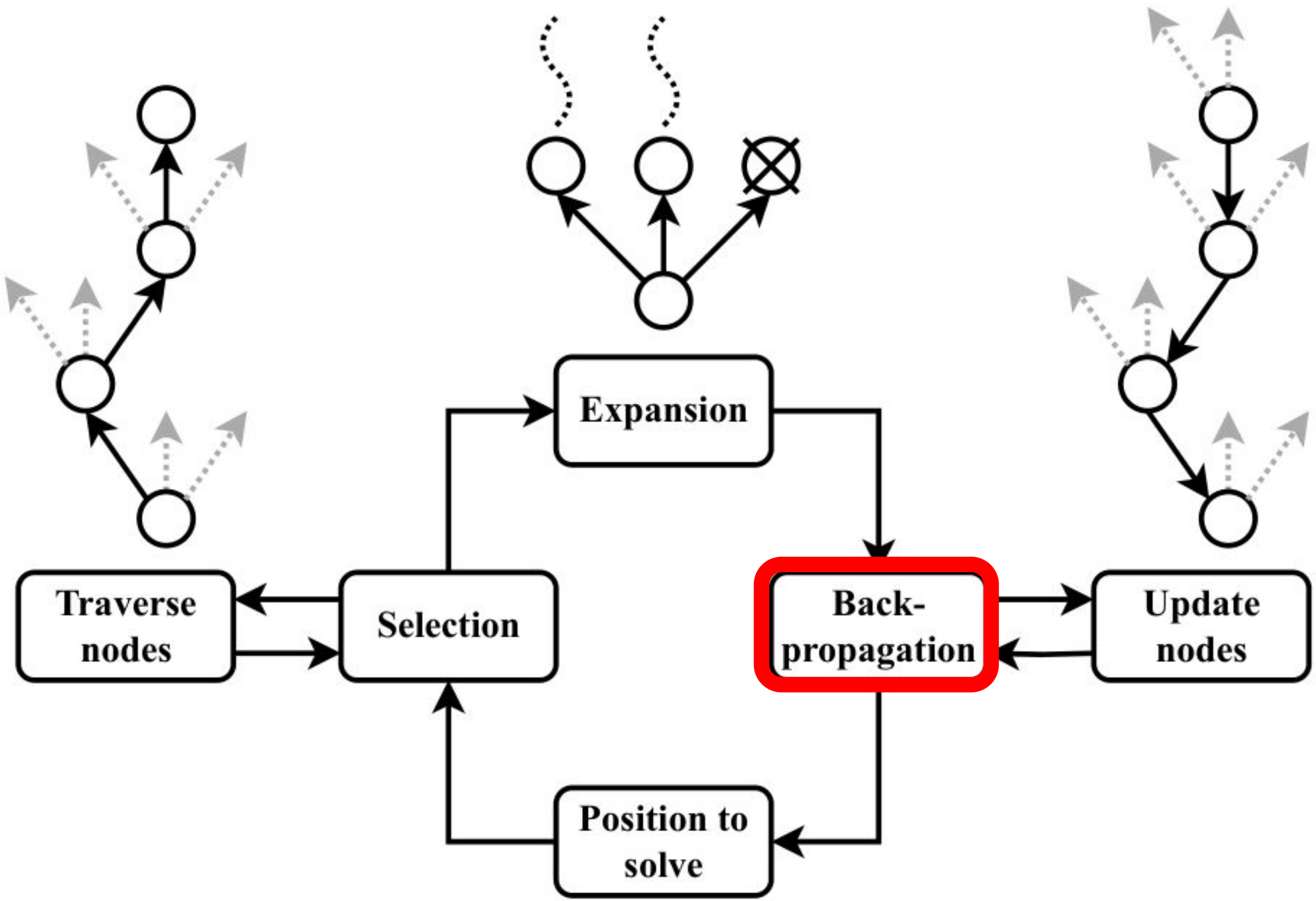


Expansion

- Create new nodes
- Check for terminal children
- Initialize WR and EW with random simulation results
- Return if the expanded node is proven

Algorithm 2 Expand X: returns whether X is solved and if so whether X is winning

```
X.expanded := true
for all legal moves m do
    if m is a terminal winning move for X then
        return true, true                          ▷ Solved win
    else if m is not a terminal losing move for X then
        Create node C
        C.expanded = false
        C.move := m
        Evaluate C.winRate
        Evaluate C.expectedWorkLoss
        Evaluate C.expectedWorkWin
        Add C to X.children
if X.children is empty then
    return true, false                             ▷ Solved loss
else
    return false, false                            ▷ Unsolved
```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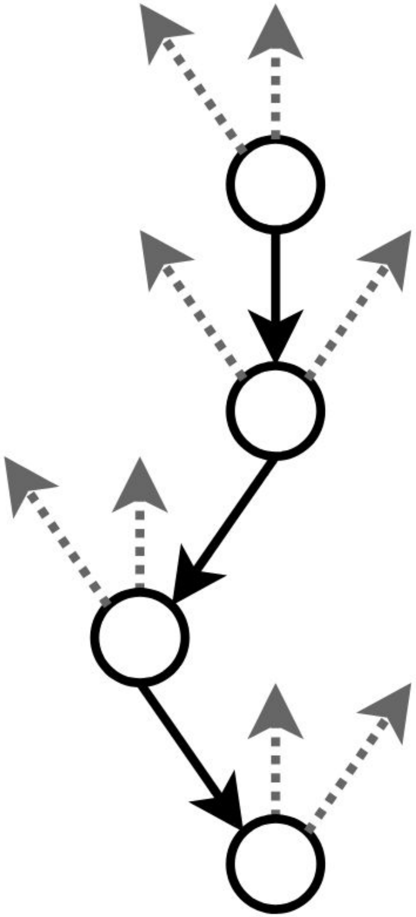
Backpropagation

- Back up new information along the path chosen by selection:
- Win rate
- Expected Work
- Proofs
- Proven nodes are pruned



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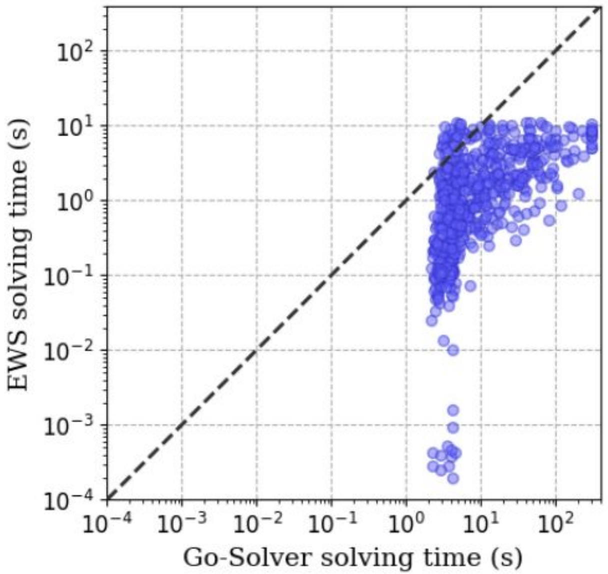
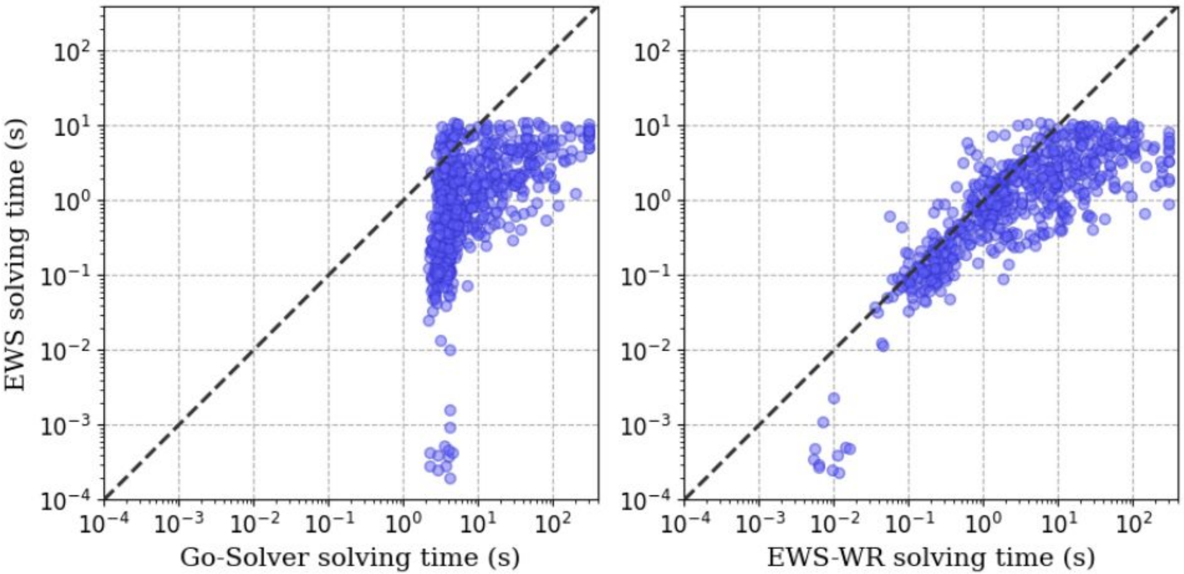
Sort X.children with (3)
Update X.winRate
Update X.expectedWorkLoss with (1)
Update X.expectedWorkWin with (2)
return false, false                                  ▷ Unsolved
```

Results

- Compared against different independent solving implementations
    - GoSolver, MIGOS, Enhanced AB, Morat PNS, Morat MCTS
- Ablation study
    - Removed proof-size / win-rate information
    - Performance drops
- Evaluated a variety of Go positions
    - Strong general performance
    - New results on empty boards
- Evaluated empty nxn Hex
    - Strong results with and without knowledge



Program	Av. solve time (s)	# Faster than EWS	# Timeouts
EWS	2.32	-	-
Go-Solver	22.63	31 (5.2%)	11 (1.8%)



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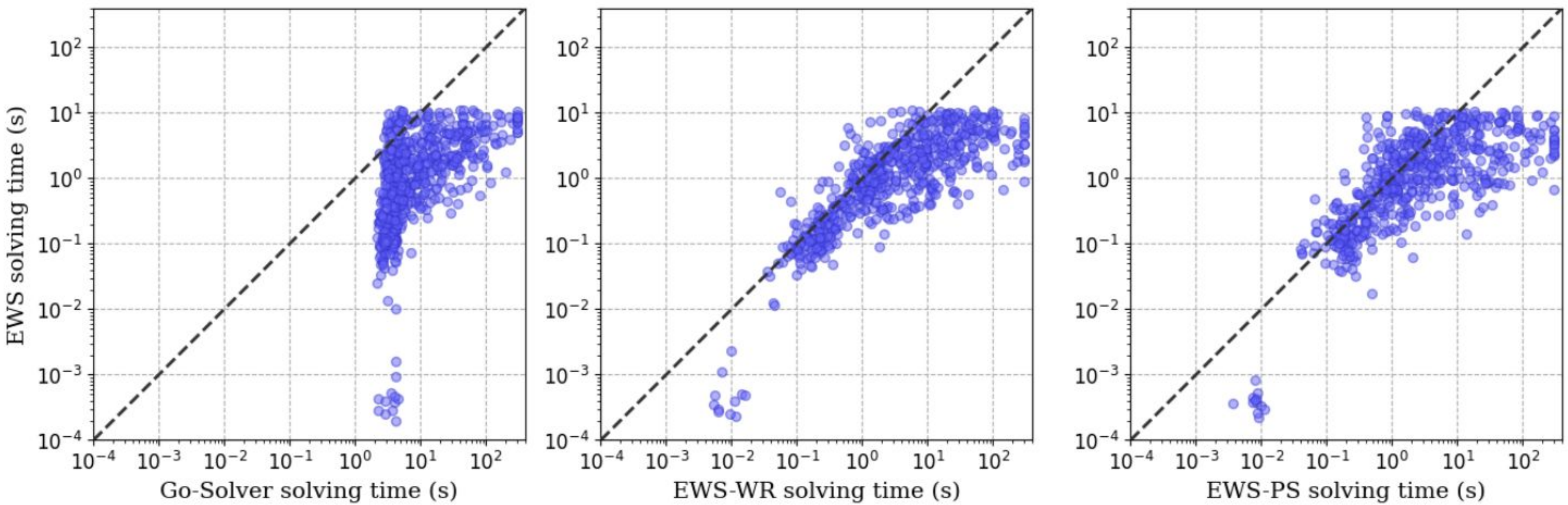
EWS-WR	19.70	96 (16.0%)	11 (1.8%)



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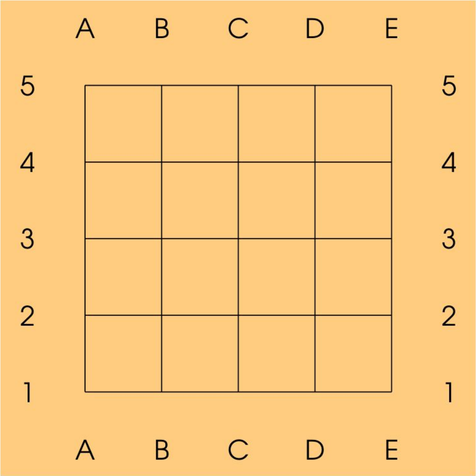
EWS-WR	19.70	96 (16.0%)	11 (1.8%)
EWS-PS	19.58	171 (28.5%)	14 (2.3%)

Solving Square Go Boards

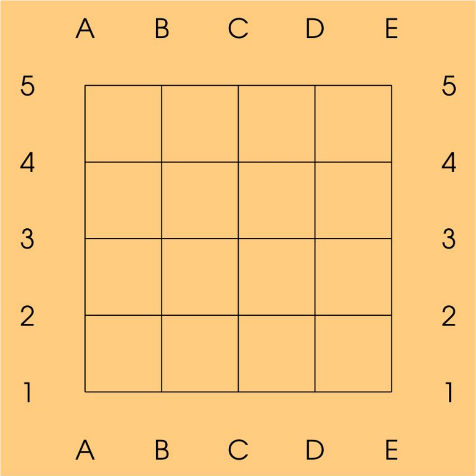
- Compare against:
    - EWS-WR (without win rate estimation)
    - EWS-PS (without proof size estimation)



Program	3×3 Time (s)	3×3 Nodes	4×4 Time (s)	4×4 Nodes	5×5 Time (h)	5×5 Nodes
EWS	0.053	161	13.626	495,494	5.252	2,605,781,360
EWS-WR	0.074	796	40.396	1,562,718	> 24	-
EWS-PS	0.070	232	18.320	744,169	> 24	-

Solving Square Go Boards

- Compare against:
  - EWS-WR (without win rate estimation)
  - EWS-PS (without proof size estimation)
  - Go-Solver (iterative deepening alpha beta)
  - MIGOS (state of the art using Japanese rules)



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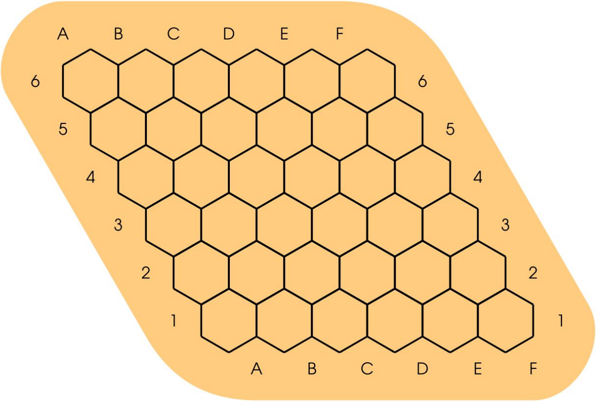
EWS-WR	0.074	796	40.396	1,562,718	> 24	-
EWS-PS	0.070	232	18.320	744,169	> 24	-
Go-Solver	1.299	1,628	51.522	799,607	> 24	-
MIGOS SSK	< 3.3	$\sim$ 25,118	$\sim$3,960	$\sim$3,162,277,660	-	-

Solving Square Hex Boards

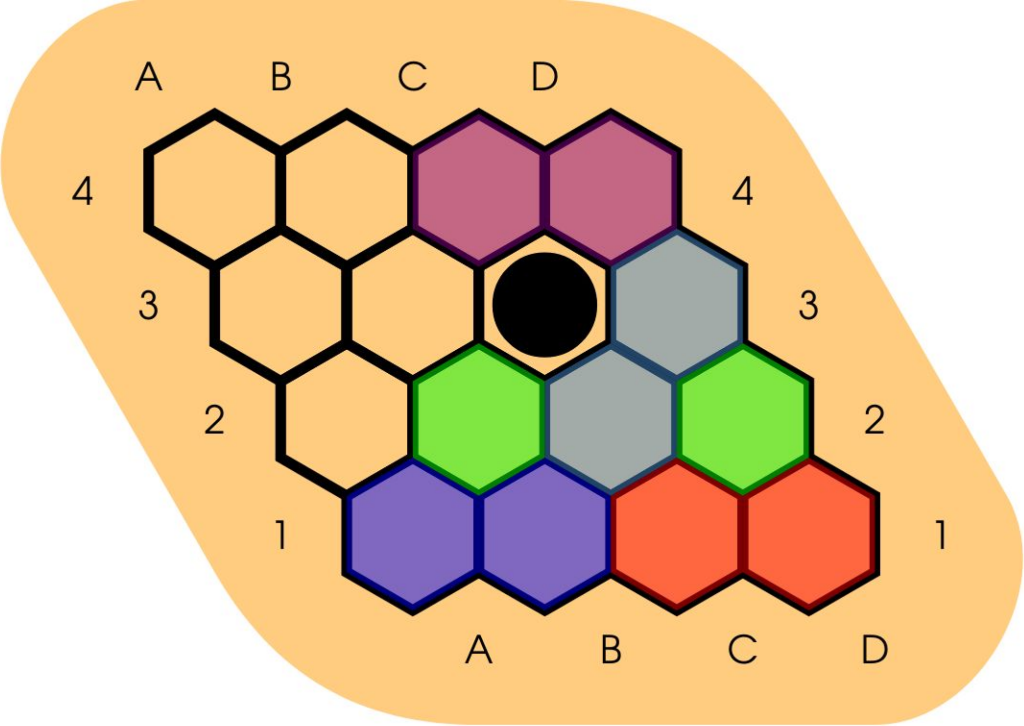
- Comparing against:
  - An enhanced alpha beta solver we developed
  - Morat Monte Carlo Tree Search
  - Morat Proof Number Search



Program	4×4 Time (s)	4×4 Nodes	5×5 Time (s)	5×5 Nodes	6×6 Time (h)	6×6 Nodes
EWS	0.002	283	0.253	37,034	0.422	93,963,192
Enhanced AB	0.004	1,673	3.935	698,402	> 24	-
Morat MCTS	0.035	4,644	29.669	3,554,546	> 24	-
Morat PNS	0.054	8,871	758.102	154,539,591	> 24	-

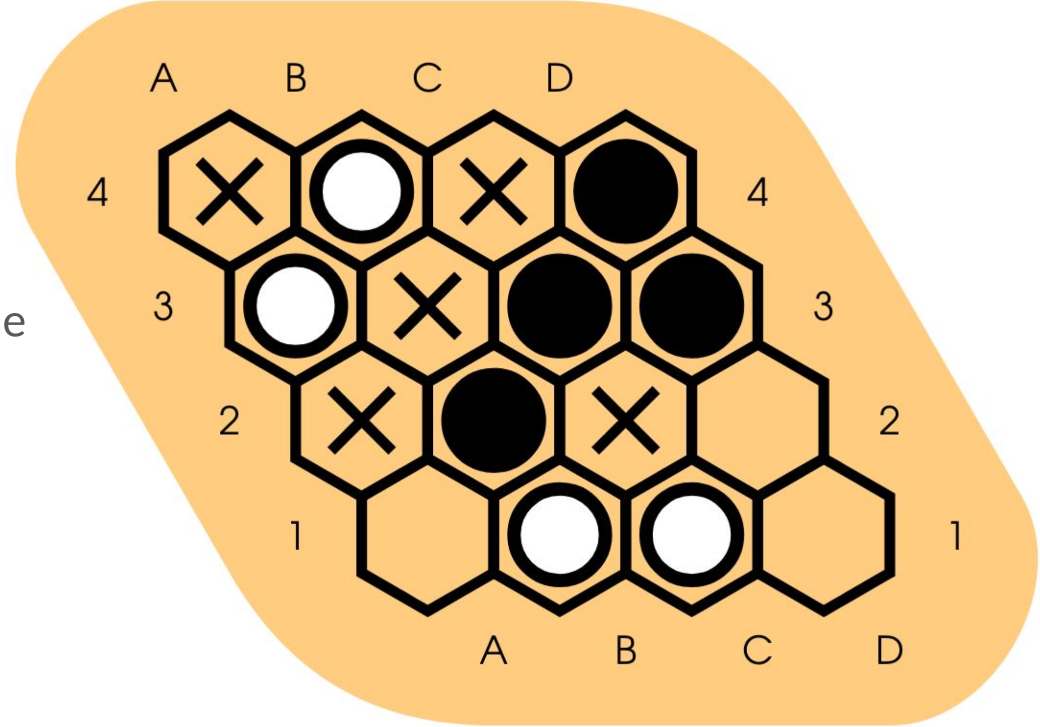
Solving with Hex Knowledge

- Virtual Connections
- Strategies to guarantee connections
- Full board connections determine the winner
- Intersections of first player VCs form a must-play zone



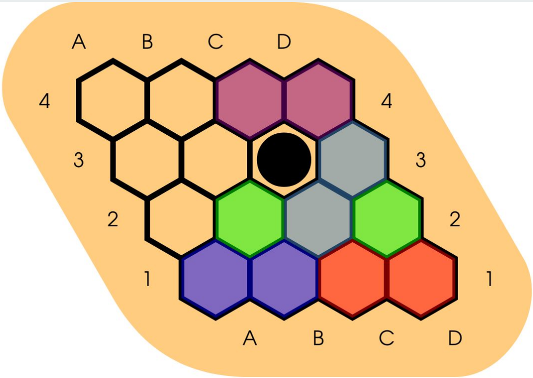
Solving with Hex Knowledge

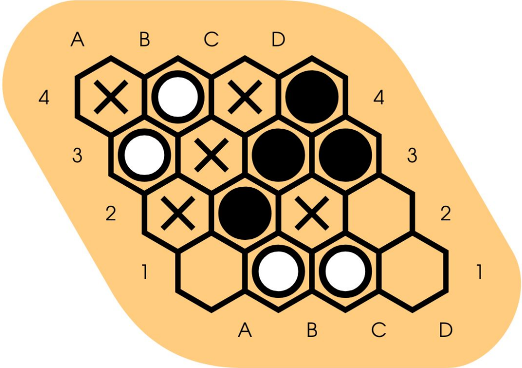
- Dead cells / Inferior cell analysis
- Some moves are irrelevant to the game's theoretical outcome
- Dead cells can be filled in, inferior moves can be ignored
- Found with pattern matching



Solving with Hex Knowledge

- Significantly reduces search space
- For 6x6 Hex: 93,963,192 nodes -> 26 nodes
- 8x8 can be solved
- Search is much slower, but more node efficient





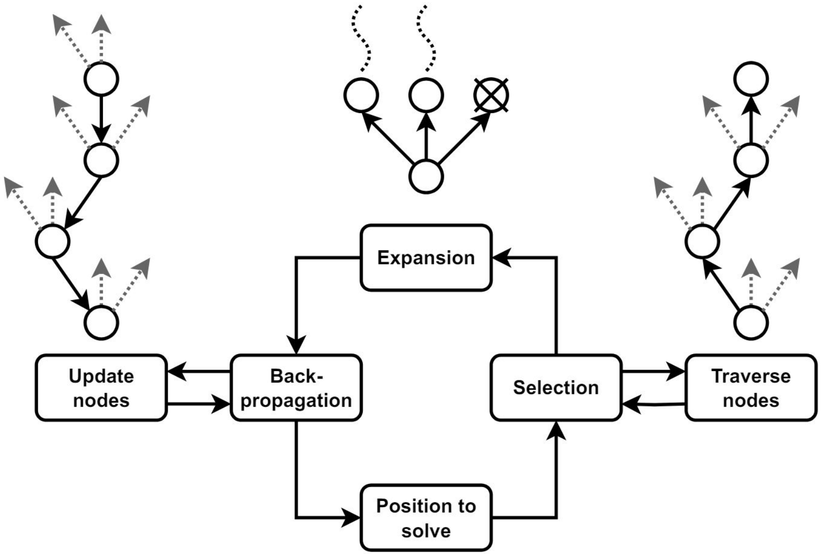
Program	6×6 Time (s)	6×6 Nodes	7×7 Time (s)	7×7 Nodes	8×8 Time (s)	8×8 Nodes
EWS with knowledge	0.005	26	0.588	2,318	234.449	555,158

Conclusion

- Expected Work Search combines proof size and win rate heuristics
  - Balances these heuristics
  - Covers previous weaknesses

- Strong results on Go and Hex
  - New results solving 5x5 Go with pos. superko
  - More Hex knowledge is needed for S.O.T.A results

- Plenty of future work for EWS



Thank you for your time!

Questions?

References

[1] Allis, L., Herik, H., and Huntjens, M. Go-Moku and Threat-Space Search. *Computational Intelligence 12* (1994), pp. 7–23.

[2] Allis, L., van der Meulen, M., and van den Herik, H. Proof-Number Search. *Artificial Intelligence 66*, 1 (1994), pp. 91–124.

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