# On NeST Graphs

by

Paul Kearney

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# Abstract

Relatively little is known about Neighborhood Subtree Tolerance (NeST) representations and the associated class of NeST graphs. Nevertheless, it can be shown that NeST representations and NeST graphs are of practical and theoretical interest: As a modeling tool, NeST representations generalize interval representations and tolerance representations while maintaining certain desirable properties of those representations. As a mathematical construct, NeST graphs fill a void within the literature and possess a theoretical richness of their own.

We will investigate NeST representations and NeST graphs by examining their historical context, presenting several applications, refining the definition of NeST representations and exploring various subclasses of NeST graphs. Along the way, several important questions and open problems from the literature are resolved, in particular, it is shown that: standard NeST representations are sufficient to represent all NeST graphs, fixed distance NeST graphs are exactly threshold tolerance graphs, proper NeST graphs are exactly unit NeST graphs and the class of NeST graphs is a proper subclass of weakly triangulated graphs.

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# List of Symbols

In the following $A$, $B$ and $W$ are sets, $R$ is a binary relation, $G$ is a graph, $\mathcal{T}$ is a tree, $T$ is an embedded tree, $F$ is a tolerance-free NeST representation, $S$ is a set of neighborhood subtrees and $\mathcal{T}$ is a set of tolerances. Entries without page references are not defined in this work.

| *Page* | *Symbol* | *Meaning* |
|---|---|---|
| | $A \subseteq B$ | $A$ is a *subset* of $B$. |
| | $A \subset B$ | $A$ is a *proper subset* of $B$. |
| | $A \cup B$ | The *set union* of $A$ and $B$. |
| | $A \cap B$ | The *set intersection* of $A$ and $B$. |
| | $A - B$ | The *set difference* of $A$ and $B$. |
| | $x \in A$ | *Set membership* ($x$ is an element of $A$). |
| | $\emptyset$ | The *empty set*. |
| | $\|A\|$ | The cardinality of $A$. |
| | $min A$ | The *minimum* value in $A$. |
| | $max A$ | The *maximum* value in $A$. |
| | $\|L\|$ | The euclidian *length* of the line segment $L$. |

| | | |
|---|---|---|
| 4 | $\omega(G)$ | The *clique number* of $G$. |
| 4 | $\alpha(G)$ | The *stability number* of $G$. |
| 4 | $\chi(G)$ | The *chromatic number* of $G$. |
| 4 | $\kappa(G)$ | The *clique cover number* of $G$. |
| 18 | $xRy$ | The assertion $(x, y) \in R$. |
| 19 | $V(G)$ | The *vertex set* of $G$. |
| 19 | $E(G)$ | The *edge set* of $G$. |
| 19 | $G_W$ | The subgraph of $G$ *induced* by $W \subseteq V(G)$. |
| 19 | $G(x)$ | The *connected component* of $G$ containing the vertex $x$. |
| 19 | $N_x$ | The *neighborhood* of the vertex $x$. |
| 19 | $M_x$ | The *non-neighborhood* of the vertex $x$. |
| 19 | $deg(x)$ | The *degree* of the vertex $x$. |
| 20 | $P(x, y)$ | The *path* in $T$ or $\mathcal{T}$ from $x$ to $y$. |
| 20 | $d(x, y)$ | The *distance* in $\mathcal{T}$ from $x$ to $y$. |
| 21 | $d(x, y)$ | The *distance* in $T$ from $x$ to $y$. |
| 20 | $(\mathcal{T}, w)$ | The *weighted tree* with tree $\mathcal{T}$ and *weight function* $w$. |
| 20 | $(f, \mathcal{T})$ | The *embedding* of $\mathcal{T}$ with *embedding function* $f$. |
| 22 | $T(c, r)$ | The *neighborhood subtree* of $T$ with center $c$ and radius $r$. |
| 22 | $\|T(c, r)\|$ | The *diameter* of the neighborhood subtree $T(c, r)$. |
| 24 | $(T, S, \mathcal{T})$ | A NeST representation. |
| 24 | $T_x$ | The neighborhood subtree of $T$ associated with the vertex $x$. |
| 24 | $c_x$ | The neighborhood subtree *center* associated with the vertex $x$. |
| 24 | $r_x$ | The neighborhood subtree *radius* associated with the vertex $x$. |
| 24 | $\tau_x$ | The *tolerance* associated with the vertex $x$. |

| | | |
|---|---|---|
| 24 | $T_{xy}$ | The *neighborhood subtree intersection* of $T_x$ and $T_y$. |
| 31 | $m(c,r)$ | The midpoint of a longest path in $T(c,r)$. |
| 31 | $l(c,r)$ | Half the length of a longest path in $T(c,r)$. |
| 36 | $B_x$ | Non-neighbors of $x$ which maximize neighborhood subtree intersection with $T_x$. |
| 37 | $\dot{z}$ | Any vertex of $B_z$. |
| 37 | $(T,S)$ | A tolerance-free NeST representation. |
| 37 | $\rho(F,G)$ | The *perturbation number* of $F$ and $G$. |
| 41 | $\mathcal{T}(v,w,ab,z)$ | A grafting of $\mathcal{T}$ with edges $vw$ and $ab$. |
| 48 | $k(T,S,\mathcal{T})$ | A *scaling* of the NeST representation $(T,S,\mathcal{T})$ by $k$. |
| 55 | $(T,X,c)$ | An $X$-tree with embedded tree $T$ and locator $c$. |
| 69 | $ax \bullet by$ | The assertion $P(a,x) \cap P(b,y) = \emptyset$. |

# Chapter 1

# Introduction

## 1.1 Historical Background

### 1.1.1 Interval Graphs

A graph[1] $G$ is an *intersection graph* if there exists a family $\mathcal{F}$ of nonempty sets and a function $S$ which identifies vertices of $G$ with sets of $\mathcal{F}$ such that two vertices form an edge in $G$ if and only if their associated sets in $\mathcal{F}$ have a nonempty intersection.

In 1957 Hajös asked which graphs can be represented as the intersection graphs of intervals on a line [12], that is, which graphs are *interval graphs*. Since then, interval graphs have become a widely studied class of graphs which have appeared in many diverse areas of study: archaeology (sequence dating), molecular biology (Benzer's problem), discrete mathematics (interval orders) and scheduling problems (room reservations), to name only a few. A detailed discussion of interval graph applications can be found in Golumbic [9] and Roberts [17].

[1]See section 1.3.1 for a formal definition of graph and other graph theoretic constructs.

Defined formally, a graph $G$ is an interval graph if there exists an indexed set, $I = \{L_v : v \in V(G)\}$, of closed intervals on the line such that the following edge condition holds:

$$xy \in E(G) \Leftrightarrow L_x \cap L_y \neq \emptyset .$$

$I$ is called an *interval representation* of $G$ and $G$ is called the graph *associated* with the interval representation $I$.

For an example of how interval representations can be used to model real problems, consider the problem of scheduling the sharing of a resource without preemption:

**Example 1** Let $D$ be a system resource and $S = \{p_1, p_2, \ldots, p_n\}$ a collection of clients requesting access to $D$. Let $R$ be an irreflexive and symmetric relation on $S$ defined such that $p_i R p_j$ if and only if $p_i$ and $p_j$ may access $D$ simultaneously. We may perceive the pair $(S, R)$ to be an undirected simple graph.

The relation $R$ is of relevance to a system scheduler trying to coordinate the access of clients to $D$. If $D$ were a file storage system then a system scheduler might define $p_i R p_j$ if and only if $p_i$ and $p_j$ are not requesting access to the same individual storage units within $D$. Similarly, if $D$ were a file system then a system scheduler might define $p_i R p_j$ if and only if $p_i$ and $p_j$ are not requesting access to the same file within $D$. In either case, $R$ characterizes when two processes can access $D$ simultaneously without causing an error.

The scheduling task is to produce a schedule which describes the access intervals of each $p_i$ to $D$ such that, if the access intervals of $p_i$ and $p_j$ overlap, then it must be that $p_i R p_j$. Restated, if $p_i$ and $p_j$ are not permitted to access $D$ simultaneously, as determined by $R$, then their access intervals in the schedule cannot overlap. We will refer to this as the *resource scheduling problem* $(D, S, R)$.

A solution schedule to the resource scheduling problem can be modeled by an interval representation $I = \{L_{p_i} : p_i \in S\}$ where $L$ is a time line and each interval $L_{p_i}$ represents the access interval of $p_i$ to $D$. If we do not allow preemption, an access interval can be modeled as a contiguous segment of time[2]. This is a plausible assumption since, for many resources, preemption is costly or impossible. If $G$ is the graph associated with the interval representation $I$ then $V(G) = S$ and $E(G)$ is defined by

$$p_i p_j \in E(G) \Leftrightarrow L_{p_i} \cap L_{p_j} \neq \emptyset\,.$$

Since $I$ represents a solution schedule to the resource scheduling problem $(D, S, R)$, we have that $L_{p_i} \cap L_{p_j} \neq \emptyset$ implies that $p_i R p_j$. Hence, $E(G)$ is a subset of $R$ which is equivalent to saying that $G$ is a subgraph of $(S, R)$. It follows that a solution schedule to the resource scheduling problem for $(D, S, R)$ is some subgraph of $(S, R)$ which is an interval graph.

A system scheduler may be satisfied with any interval subgraph of $(S, R)$ but most likely an economic solution will be desired. Suppose we define solution schedule $I$ to be better than solution schedule $J$ if the number of overlapping access intervals in $I$ is greater than the number of overlapping access intervals in $J$. That is, the interval graph associated with solution schedule $I$ will have more edges than the interval graph associated with solution schedule $J$. This economy criteria is plausible since a system scheduler might want to maximize the simultaneous use of resource $D$. Hence, an optimal solution is defined to be a largest (edgewise) subgraph of $(S, R)$ which is an interval graph.

[2]However, the model is powerful enough to model an access interval $L_{p_i}$ which is composed of multiple contiguous segments of time. For each segment we create a vertex in the associated graph. The associated graph is then a hypergraph with $p_i$ being the hypernode composed of all vertices associated with contiguous segments of $L_{p_i}$.

### 1.1.2 Interval Graphs are Perfect

This section introduces the concept of a *perfect graph*. Our main interest in perfect graphs is that there are efficient algorithms to solve several common optimization problems on perfect graphs, however, these problems are NP-complete for arbitrary graphs. We will not discuss perfect graph theory in depth but will present only the terminology and results necessary for our discussion.

Below are four (classic) parameters definable on a graph $G$ along with the four optimization problems commonly associated with these parameters:

1. The *clique number* of $G$, $\omega(G)$, is the size of the largest clique in $G$. Finding a largest clique in $G$ is called the *maximum clique problem.*

2. The *stability number* of $G$, $\alpha(G)$, is the size of the largest stable set in $G$. Finding a largest stable set of $G$ is called the *maximum stable set problem.*

3. The *chromatic number* of $G$, $\chi(G)$, is the least number of stable sets required to cover the vertices of $G$. Finding a smallest stable set cover of $G$ is called the *minimum coloring problem.*

4. The *clique cover number* of $G$, $\kappa(G)$, is the least number of cliques required to cover the vertices of $G$. Finding a smallest clique cover of $G$ is called the *minimum clique cover problem.*

Claude Berge [3] defined a graph $G$ to be perfect[3] if the following two properties,

[3]In fact, the *perfect graph theorem* states that $G$ satisfies $P1$ if and only if $G$ satisfies $P2$. This result is due to Lovász [15].

$P1$ and $P2$, are satisfied by all induced subgraphs $H$ of $G$:

$$\omega(H) = \chi(H) \tag{P1}$$

$$\alpha(H) = \kappa(H) \tag{P2}$$

The above four optimization problems occur frequently in graph theoretic applications. For arbitrary graphs, these problems are known to be NP-complete [6]. However, for the class of perfect graphs, these problems can be solved in polynomial time [11]. Hence, applications based upon perfect graphs benefit from the existence of algorithms which can solve these problems efficiently.

It has been shown that interval graphs are perfect [9]. Hence, applications based upon interval representations and interval graphs may utilize the fact that the above four optimization problems can be solved efficiently. To illustrate this we present an application modeled by an interval representation which requires a solution to the minimum coloring problem:

**Example 2** Suppose that $B$ is an important binary file that must be stored securely within some archiving system. To do this, a collection of contiguous subfiles of $B$, $B_1, B_2, \ldots, B_m$, are defined such that, for any bit $b$ in $B$, $b$ appears in at least two of the contiguous portions of $B$. The division of $B$ into contiguous subfiles can be modeled by an interval representation $I$ where $B$ is represented by a line and each $B_i$ is an interval on this line.

Given the division of $B$ into contiguous subfiles, an archiver needs to determine the least number of remote sites to store each $B_i$ such that if $B_i \cap B_j \neq \emptyset$ then $B_i$ and $B_j$ must be located at different sites. This is a security measure which ensures that no single bit $b$ is located at only one site.

The archiver's problem is equivalent to finding a smallest stable set cover of the interval graph $G$ associated with $I$, that is, the archiver must find a minimum coloring of $G$. This is seen by observing that we are attempting to assign remote sites (colors) to each subfile such that, if two subfiles intersect, i.e. they form an edge in the associated graph, then they are assigned different remote sites. The solution can be obtained in linear time [9].

### 1.1.3 Tolerance Graphs

The numerous applications modeled by interval representations is testimony to the power of this construct as a modeling tool. Furthermore, interval graphs are perfect which implies the existence of efficient algorithms to solve the four optimization problems of section 1.1.2. However, an interval representation is a simple model, insufficient for modeling many problems. *Tolerance representations*, introduced by Golumbic and Monma [8], generalize the interval representation model, allowing them to model a greater number of applications than interval representations. Furthermore, those graphs which have a tolerance representation, namely, *tolerance graphs*, are also perfect graphs.

Defined formally, a graph $G$ is a tolerance graph if there exists an indexed set, $I = \{L_v : v \in V(G)\}$, of closed intervals on the line and an indexed set of positive numbers, $t = \{t_v : v \in V(G)\}$, called *tolerances*, such that the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |L_x \cap L_y| \geq min\{t_x, t_y\}\,.$$

The pair $(I, t)$ is called a tolerance representation of $G$ and $G$ is called the graph *associated* with the tolerance representation $(I, t)$.

Tolerance representations generalize interval representations by introducing tolerances for overlap. The following intuitive descriptions of the edge conditions, as defined by interval and tolerance representations, demonstrate this generalization. Interval representations model the following edge condition:

*Two objects are associated if and only if their intervals on a line overlap.*

Tolerance representations model a similar, though more general, edge condition:

*Two objects are associated if and only if their intervals on a line overlap and the size of this overlap exceeds at least one of their tolerances for overlap.*

Observe that if a graph $G$ has an interval representation then it also has a tolerance representation; simply define all tolerances to be sufficiently small. However, there are graphs (applications) which are associated with (modeled by) tolerance representations but not by interval representations. For example, the cycle on four vertices, $C_4$, is a tolerance graph but not an interval graph. With respect to applications, tolerance representations are a more flexible model than interval graphs. Below is an example, reminiscent of Example 1 from section 1.1.1, of an application modeled by a tolerance representation but not by an interval representation.

**Example 3** Let $D$, $S$ and $R$ be defined as in Example 1. We call $p_i$ and $p_j$ *conflicting clients* if $\neg p_i R p_j$. In Example 1 there was the constraint that conflicting clients could not have overlapping access intervals in a solution schedule $I$ to the resource scheduling problem $(D, S, R)$. In this example, the system scheduler knows that $D$ can tolerate a certain amount, $t$, of overlap of the access intervals of conflicting clients. Such a situation arises naturally. For example, each access interval to $D$ may begin with an initialization period of length $t$ which does not interfere with other clients' access intervals to $D$.

A solution schedule $I$ must satisfy the condition

$$\neg p_i R p_j \Rightarrow |I_{p_i} \cap I_{p_j}| < t\,.$$

That is, if $p_i$ and $p_j$ are conflicting clients then their access intervals cannot overlap more than the allowable tolerance $t$. Hence, a solution schedule will be a tolerance representation whose associated graph is a subgraph of $(S, R)$.

See Golumbic, Monma and Trotter [10] for a brief discussion of other applications modeled by tolerance applications.

Interval graphs have the desirable property of being perfect graphs. It has been shown that the generalization of interval representations by tolerance representations preserves this property, that is, tolerance graphs are perfect [10]. This implies the existence of efficient algorithms to solve the four optimization problems of section 1.1.2 for tolerance graphs.

It is of interest that tolerance graphs are a superclass of many well-known classes of perfect graphs besides interval graphs. Among these are *permutation graphs*, *threshold graphs* and the complement of *threshold tolerance graphs*. See Golumbic [9] and Monma, Reed and Trotter [16] for more on these classes of graphs.

## 1.2 NeST Graphs

Interval representations are generalized to tolerance representations by the introduction of tolerances. Furthermore, tolerance graphs, despite being a superclass of interval graphs, are also a class of perfect graphs. The natural question to ask is whether there is a generalization of tolerance representations which is also associated with a class of perfect graphs. *Neighborhood subtree tolerance (NeST) representations*, recently introduced by Bibelnieks and Dearing [4], generalize tolerance representations and are associated with a class of perfect graphs, namely, *neighborhood subtree tolerance graphs*. NeST representations and NeST graphs are formally defined in section 1.3.2.

NeST representations generalize tolerance representations in the following way:

- the line is generalized to a tree embedded in the plane,
- intervals on the line are generalized to neighborhood subtrees of the embedded tree and
- interval size is generalized to neighborhood subtree diameter.

Intuitively, tolerance representations model the following edge condition:

> *Two objects are associated if and only if their intervals on a line overlap and the size of this overlap exceeds at least one of their tolerances for overlap,*

whereas NeST representations model the edge condition:

> *Two objects are associated if and only if their neighborhoods in an embedded tree overlap and the size of this overlap exceeds at least one of their tolerances for overlap.*

More formally, if a graph $G$ has a tolerance representation, then this tolerance representation is also a NeST representation of $G$ since the line can be considered to be an embedded tree and intervals on the line as neighborhood subtrees of this line. Hence, all tolerance graphs are NeST graphs and it follows that NeST representations are a generalization of tolerance representations. For examples of applications modeled by NeST representations see section 1.2.2.

NeST graphs are perfect graphs. Bibelnieks and Dearing demonstrated this by proving that NeST graphs are weakly triangulated [4]. A graph $G$ is *weakly triangulated* if neither itself nor its complement contains an induced cycle of five or more vertices. It is known that weakly triangulated graphs are perfect graphs [13]. Hence, NeST graphs are also perfect and, as a result, polynomial time algorithms exist to solve the four optimization problems of section 1.1.2 for NeST graphs..

### 1.2.1 Motivating NeST Graphs

In this section we attempt to motivate the study of NeST graphs. We have already seen that NeST representations generalize tolerance representations and that NeST graphs are perfect graphs. These properties alone make the class of NeST graphs interesting. However, there are other motivations for studying NeST graphs.

Trees are a naturally occurring structure and appear frequently in applications. Hence, generalizing the line to an embedded tree will allow NeST representations to model a greater number of applications than tolerance representations.

The generalization of the line to an embedded tree has been studied before. Interval graphs and interval representations have been generalized to the intersection graphs of subtrees in a tree. These graphs are called *subtree graphs*. Defined formally, a graph $G$ is a subtree graph if there is an embedded tree $T$ and an indexed set, $S = \{T_x : x \in V(G)\}$, of subtrees in $T$ such that the following edge condition holds:

$$xy \in E(G) \Leftrightarrow T_x \cap T_y \neq \emptyset \,.$$

$(T, S)$ is called a *subtree representation* of $G$ and $G$ is called the graph *associated* with $(T, S)$.

It has been shown that subtree graphs are exactly the triangulated graphs[4] [7]. This interesting characterization motivates the analogous generalization of tolerance representations by *subtree tolerance representations*.

A graph $G$ is a *subtree tolerance graph* if there is an embedded tree $T$, an indexed set, $S = \{T_x : x \in V(G)\}$, of subtrees in $T$ and an indexed set, $t = \{t_x : x \in V(G)\}$, of positive numbers called *tolerances*, such that the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |T_x \cap T_y| \geq min\{t_x, t_y\} \,.$$

$(T, S, t)$ is called a subtree tolerance representation of $G$ and $G$ is called the graph *associated* with $(T, S, t)$.

All graphs have a subtree tolerance representation [4]. Hence, subtree tolerance representations are too general. A problem with subtree tolerance representations is that any subtree of the embedded tree is permitted in the representation. To generalize tolerance representations, in an interesting way, the permissible subtrees in a subtree tolerance representation must be restricted. A natural solution is to restrict the subtrees so that they are neighborhood subtrees. This restriction to subtree tolerance representations yields NeST representations. Restriction to neighborhood subtrees is a natural solution in the following two ways:

1. NeST representations are intended to be a generalization of tolerance representations. We would like the intervals in a tolerance representation to have a

[4] A graph $G$ is called *triangulated* if it has no induced cycle on more than three vertices.

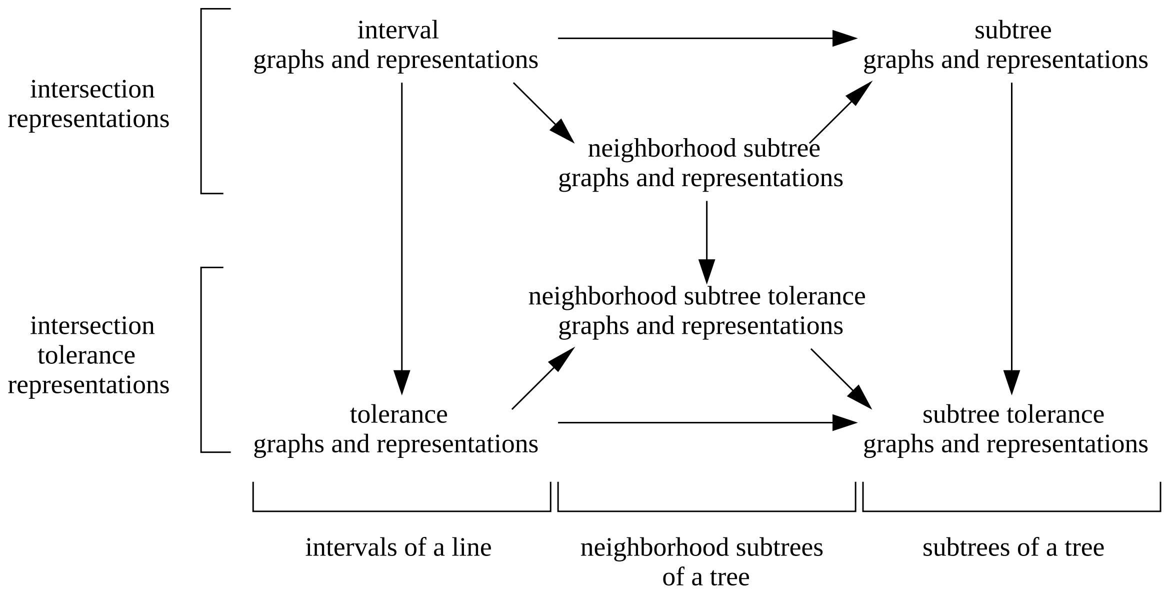



Figure 1.1: Generalizing interval graphs and representations.

natural analogue in a NeST representation. Since intervals on the line can be perceived as neighborhoods of the line, neighborhood subtrees of an embedded tree are a natural analogue.

2. With respect to applications, neighborhood subtrees occur naturally. That is, often it is the case that an object can be modeled by a neighborhood subtree of an embedded tree. For examples, see section 1.2.2.

The above discussion is summarized in Figure 1.1. The only undefined class of graphs in Figure 1.1 are *neighborhood subtree graphs*[5]. $G$ is a neighborhood subtree graph if it has a subtree representation $(T, S)$ where the subtrees in $S$ are neighborhood subtrees. $(T, S)$ is called a *neighborhood subtree representation* of $G$ and $G$ is called the graph *associated* with $(T, S)$.

[5]These will be discussed in section 3.5

The three classes of graphs appearing in the top half of Figure 1.1 are intersection graphs of, from left to right, intervals of a line, neighborhood subtrees of an embedded tree and subtrees of an embedded tree. The three classes of graphs appearing in the bottom half of Figure 1.1 are those classes produced when the three aforementioned representations are generalized by the introduction of tolerances. From left to right, these are tolerance representations, NeST representations and subtree tolerance representations.

We may also perceive Figure 1.1 as three columns composed of two classes of graphs each. The leftmost column are those representations based on intervals of a line. The rightmost column are those representations based on subtrees of an embedded tree. The middle column are those representations based on neighborhood subtrees of an embedded tree.

In closing, we summarize our motivation for studying NeST graphs:

1. NeST representations generalize tolerance representations.

2. NeST graphs are perfect graphs.

3. The generalization of the line by an embedded tree will allow NeST representations to model a greater number of applications.

4. The generalization of interval representations by subtree representations resulted in an interesting characterization of triangulated graphs. The analogous generalization of tolerance representations by subtree tolerance representations, however, is not interesting. By restricting subtrees to be neighborhood subtrees it is likely that the resulting generalization of tolerance representations, and the class of associated graphs, will be of practical and theoretical interest.

### 1.2.2 NeST Graph Applications

Here we present a few applications which can be modeled by NeST representations (but not by tolerance representations). These applications will also utilize the efficient algorithms which solve the four optimization problems of section 1.1.2.

**Application 1** Let $R$ be an acyclic, connected road system on which centers for various emergency response vehicles, say ambulances, are located. Let $A$ be the set of ambulance centers. An administrator wishes to decide the *service radius*, $r_p$, for each ambulance center $p$ in $A$, where distances are measured along the road system. The service radius, together with the location of an ambulance center, defines an *area of responsibility*, $R_p$, for each ambulance center. Observe that each area of responsibility is a neighborhood subtree of $R$. The three conditions that must be met when assigning service radii are:

- Each point in $R$ is within the responsibility of at least one ambulance center.
- The service radius of any ambulance center does not exceed an upper bound $r$. The upper bound reflects the policy that no person should wait longer than a prespecified amount of time for ambulance response.
- The size of the overlap of the area of responsibility of any two ambulance centers does not exceed some upper bound $t$. This tolerance for overlap is introduced to enforce economy in the assignment of service radii to the ambulance centers. That is, it would be inefficient to allow two ambulance centers to have areas of responsibility which overlap more than $t$.

Let $G$ be a graph where $V(G) = A$ and $E(G) = \emptyset$. The administrator must find a NeST representation $(R, S, \mathcal{T})$ of $G$, where $S$ is the set of ambulance center areas of responsibility and $\tau_p = t$, for each $p$ in $A$, such that

- $S$ covers $R$ and
- for all $p \in A$, $r_p \leq r$.

Observe that the third condition above is satisfied if $(R, S, \mathcal{T})$ is a NeST representation of $G$. An approximate solution to the problem is one in which $|E(G)|$ is minimized. That is, the first two conditions take priority over the third.

**Application 2** Let $R$ be a road system as in Application 1; assume here that the emergency response service is the fire department. Let $f_1, \ldots, f_m$ be the location of $m$ fire stations on $R$ and let $s_1, \ldots, s_n$ be the location of $n$ suburbs on $R$. We assume that each fire station $f_i$ has a service radius $r_{f_i}$ and each suburb $s_i$ can be represented as a neighborhood subtree of $R$ with radius $r_{s_i}$.

A fire insurance company wishes to sell a new policy to the home owners of the $n$ suburbs. The policy offers better rates to those home owners who live in a suburb which falls completely within the area of responsibility of more than one fire station. Furthermore, the greater the number of fire stations which are responsible for a suburb the better the policy rates for the home owners in that suburb. The following model will aid the insurance company in determining which policy a particular home owner is eligible for. Assign an infinite (that is, very large) tolerance to each fire station. Let the tolerance of any suburb be exactly twice the radius of that suburb. Define a graph $G$ where the vertices of $G$ are the fire stations along with the suburbs by

$$xy \in E(G) \;\Leftrightarrow\; |R_x \cap R_y| \geq min\{t_x, t_y\},$$

where $R_x$, $t_x$, $R_y$ and $t_y$ are the neighborhood subtrees and tolerances associated with $x$ and $y$. We observe that

- $f_i f_j \notin E(G)$ for all $f_i$ and $f_j$,
- $s_i s_j \notin E(G)$, since $|R_{s_i} \cap R_{s_j}| \geq min\{t_{s_i}, t_{s_j}\}$ implies that $R_{s_i} \subseteq R_{s_j}$ or $R_{s_j} \subseteq R_{s_i}$, which implies that $s_i \equiv s_j$ (we do not allow suburbs within suburbs) and
- $f_i s_j \in E(G) \Leftrightarrow |R_{f_i} \cap R_{s_j}| \geq min\{t_{f_i}, t_{s_j}\} = 2r_j \Leftrightarrow R_{s_j} \subseteq R_{f_i} \Leftrightarrow s_j$ is within the area of responsibility of station $f_i$.

We can see that the model is a NeST representation of the graph $G$ in which the degree of vertex $s_i$ is the number of fire stations which share responsibility for $s_i$ and the neighbors of vertex $f_j$ are the suburbs $f_j$ is responsible for.

The insurance company would like to determine the best policy (from the home owner's point of view) for which a customer is eligible given the above representation, regardless of whether the customer lives in a suburb or in a rural district along $R$. To model this we alter the above representation. Remove all suburbs from the representation and allow all fire station tolerances to be arbitrarily small (but still positive). Let $H$ be the graph where $V(H)$ is the set of fire stations and $E(H)$ is defined by

$$xy \in E(H) \Leftrightarrow |R_x \cap R_y| \geq min\{t_x, t_y\}.$$

That is, the vertices $f_i$ and $f_j$ form an edge in $H$ if and only if their areas of responsibility overlap. Observe that if some home owner is within the area of responsibility of $k$ fire stations, then those $k$ fire stations form a complete subgraph of $H$. Furthermore, a collection of neighborhood subtrees satisfies the Helly property: A collection of sets $S$ satisfies the *Helly property* if, for any $S' \subseteq S$, if every pairwise intersection between sets in $S'$ is nonempty then the intersection of all sets in $S'$ is nonempty. Hence, if a set $M$ of fire stations form a complete subgraph of $G$ then there is a location on $R$ which is in the area of responsibility of all fire stations in $M$. It follows that determining the best possible policy is equivalent to solving the maximum clique problem on $H$. Since NeST graphs are perfect, this can be solved in polynomial time.

**Application 3** Let $T$ be an embedded tree, $X$ a set of locations in $T$ and $k$ some positive number. Define the relation $E$ on $X$ by

$$xy \in E \Leftrightarrow d(x,y) \leq k\,.$$

That is, we associate $x$ and $y$ if and only if they are "$k$-close" in $T$. We call $(T, X, k)$ a *proximity representation* of the graph $(X, E)$ and $(X, E)$ the graph *associated* with $(T, X, k)$.

We can generalize the above model by defining a set of positive numbers $K = \{k_x : x \in X\}$ and the relation $E$ by

$$xy \in E \Leftrightarrow d(x,y) \leq min\{k_x, k_y\}\,.$$

That is, we associate $x$ and $y$ if and only if they are either "$k_x$-close" or "$k_y$-close". We call $(T, X, K)$ a *generalized proximity representation* of the graph $(X, E)$ and $(X, E)$ the graph *associated* with $(T, X, K)$.

In future sections we will see that the class of graphs associated with proximity representations are exactly fixed tolerance NeST graphs (see section 3.5) and that the class of graphs associated with generalized proximity representations are exactly proper NeST graphs (see section 3.3).

There are numerous applications which might utilize one of the above representations:

1. Suppose $T$ is a tree network of computer sites where $V(T)$ denotes the computer sites within the network and $E(T)$ denotes the communication links between

sites within the network. Furthermore, suppose the network is designed, for security measures, such that a site $v \in V(T)$ can only communicate, directly or indirectly via other sites, with sites which are within $r$ communication links.

We can represent the above network by the proximity representation $(T', V(T), r)$ where $T'$ is an embedding of $T$ where each link in $E(T)$ is mapped to a line segment of unit length. The edge set of the graph $G$ associated with $(T', V(T), r)$ satisfies the following condition:

$$xy \in E(G) \Leftrightarrow x \text{ and } y \text{ can communicate}.$$

Suppose we need to distribute data to all sites in the network. To minimize the number of sites we need to "seed" with the data, we must find a minimum dominating set of $G$. That is, we must find a minimal subset $W$ of $V(G)$ such that, for all $v \in V(G)$, either $v \in W$ or $v$ is adjacent to a vertex in $W$. This problem can be solved in polynomial time.

2. Let $T$ be a weighted *phylogenetic tree* over some species set $X$ [1]. We wish to investigate the relationship $E$ between species which are within $k$ evolutionary units of each other. Hence, $(T, X, k)$ is a proximity representation of the graph $(X, E)$ which characterizes this evolutionary proximity relation.

## 1.3 Definitions

### 1.3.1 General Definitions

A *binary relation* $R$ on the set $W$ is a subset of $W \times W$. We will abbreviate $(x, y) \in R$ to $xy \in R$ and also to $xRy$. $R$ is *reflexive* if, for all $w \in W$, $wRw$. $R$ is called *irreflexive* if not $wRw$, for all $w \in W$. $R$ is called *symmetric* if, for all $x, y \in W$, $xRy$ implies $yRx$. $R$ is called *transitive* if, for all $x, y, z \in W$, $xRy$ and $yRz$ implies $xRz$.

A *graph* $G$ is specified by its *vertex set* $V(G)$ and its *edge set* $E(G)$ where $E(G)$ is an irreflexive ($G$ is a *simple* graph) and symmetric ($G$ is an undirected graph) binary relation on $V(G)$. If $xy \in E(G)$ we say "$x$ and $y$ are neighbors" or "$x$ sees $y$". If $xy \notin E(G)$ we say "$x$ and $y$ are non-neighbors" or "$x$ misses $y$". For any $x \in V(G)$ we define the *neighborhood* of $x$, denoted $N_x$, to be the set $\{z \in V(G) : xz \in E(G)\}$. Similarly, the *non-neighborhood* of $x$, denoted $M_x$, is the set $\{z \in V(G) - \{x\} : xz \notin E(G)\}$. The *degree* of a vertex $v$ in $G$, denoted $deg(v)$, is the number of neighbors of $v$. A *hypergraph* is a graph $G$ except that $E(G)$ is a subset of the power set of $V(G)$ (i.e. the set of all subsets of $V(G)$).

$H$ is a *subgraph* of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that, if $xy \in E(H)$ then $x, y \in V(H)$. The subgraph of $G$ *induced* by $W \subseteq V(G)$, denoted $G_W$, is the graph with vertex set $W$ and edge set $\{xy \in E(G) : x, y \in W\}$.

We say graphs $G_1$ and $G_2$ are *isomorphic* if there exists a bijection $f$ from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$.

The graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_k\}$, $k \geq 3$ and edge set $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$ is called a *cycle* on $k$ vertices. A graph $G$ is called *acyclic* if it contains no induced cycles. The graph $G$ where $V(G) = \{v_1, v_2, \ldots, v_k\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ is called a *path* on $k$ vertices.

Two vertices, $x$ and $y$, in $G$ are *connected* if there exists a path in $G$ from $x$ to $y$. $G_W$ is called a *connected component* of $G$ if all pairs of vertices in $W$ are connected and, for all $x \in V(G) - W$, $x$ is not connected to a vertex in $W$. The connected component of $x$ in $G$, denoted $G(x)$, is the connected component of $G$ which contains

$x$. A graph $G$ is called *connected* if it has exactly one connected component.

A graph $\mathcal{T}$ is called a *tree* if it is connected and acyclic. $P(x, y)$ denotes the unique induced path in $\mathcal{T}$ from $x$ to $y$. A *leaf* of $\mathcal{T}$ is any vertex of degree 1. If $w$ is a real-valued function on $E(\mathcal{T})$ then $(\mathcal{T}, w)$ is called a *weighted tree* and $w$ is called a *weight function*. In a weighted tree, $d(x, y)$ denotes the sum of the weights of the edges in $P(x, y)$. In an unweighted tree, $d(x, y)$ denotes the number of edges in $P(x, y)$.

A real-valued function $d$ on some space $S$ is a *metric* if, for all $x, y, z \in S$, $d(x, x) = 0$, $d(x, y) = d(y, x)$, $d(x, y) \geq 0$ and $d(x, y) + d(y, z) \geq d(x, z)$.

If $C$ and $S$ are graph classes such that every graph in $S$ is also in $C$ then $S$ is called a *subclass* of $C$ and $C$ is called a *superclass* of $S$.

### 1.3.2 Defining NeST Graphs

#### Embedded Trees

Basic to the definition of NeST graphs is the concept of an *embedded tree*.

Let $\mathcal{T}$ be a tree. We call $T = (f, \mathcal{T})$ an *embedding of* $\mathcal{T}$ or simply an *embedded tree* if

- $f$ is a one-to-one function, called the *embedding function*, with domain $V(\mathcal{T}) \cup E(\mathcal{T})$,
- for all $v \in V(\mathcal{T})$, $f(v)$ is a point in the plane,
- for all $uv \in E(\mathcal{T})$, $f(uv)$ is the line segment from $f(u)$ to $f(v)$ and

- for $e_1 \neq e_2 \in E(\mathcal{T})$,

$$f(e_1) \cap f(e_2) = \begin{cases} f(v) & \text{if } e_1 \cap e_2 = \{v\}, \\ \emptyset & \text{if } e_1 \cap e_2 = \emptyset. \end{cases}$$

The embedded tree $T$ will also be associated with the following set of points in the plane

$$\Big(\bigcup_{v \in V(\mathcal{T})} \{f(v)\}\Big) \cup \Big(\bigcup_{e \in E(\mathcal{T})} f(e)\Big).$$

This definition is consistent with that found in Tamir [19].

Having defined an embedded tree $T = (f, \mathcal{T})$, there are many subsidiary definitions that will be useful. For $v \in V(\mathcal{T})$, $f(v)$ is called a *node* of $T$. If $v$ is a leaf of $\mathcal{T}$ then $f(v)$ is called an *endpoint* of $T$. If $v$ is not a leaf of $\mathcal{T}$ then $f(v)$ is called an *internal node* of $T$. For $uv \in E(\mathcal{T})$ we call $f(uv)$ an *edge* of $T$. A *halfline* is any point set of the form $\{b + tv : t \geq 0\}$ where $b$ and $v$ are fixed, two dimensional vectors.

An example of a tree $\mathcal{T}$ and an embedding $T = (f, \mathcal{T})$ appears in Figure 1.2. $\mathcal{T}$ is the tree $(\{1,2,3,4,5,6\}, \{13, 23, 34, 45, 46\})$. The nodes of $T$ are $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(6)$. The edges of $T$ are $f(13)$, $f(23)$, $f(34)$, $f(45)$ and $f(46)$. The endnodes of $T$ are $f(1)$, $f(2)$, $f(5)$ and $f(6)$. The endedges of $T$ are $f(13)$, $f(23)$, $f(45)$ and $f(46)$. $f(3)$ and $f(4)$ are internal nodes and $f(34)$ is an internal edge.

With respect to an embedded tree $T$, $P(x, y)$ denotes the unique path in $T$ connecting the points $x$ and $y$; $d(x, y)$ denotes the Euclidean length of the path $P(x, y)$. The *connected components*, or more simply, the *components* of $T - \{p\}$, where $p$ is some point in $T$, is the collection $T_1, T_2, \ldots, T_m$ of embedded trees resulting from the removal of $p$ from $T$.

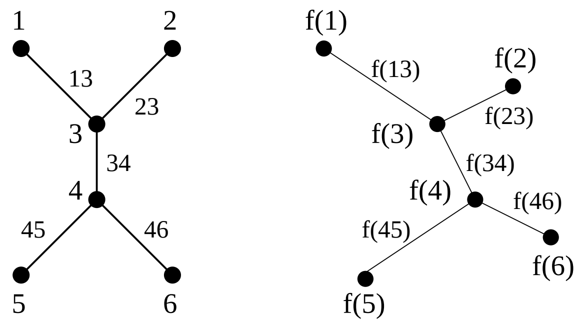



Figure 1.2: A tree $\mathcal{T}$ and its embedding $T = (f, \mathcal{T})$.

### Neighborhood Subtrees

If $T$ is an embedded tree then a *neighborhood subtree* of $T$, with *center* $c \in T$ and *radius* $r \geq 0$, denoted $T(c, r)$, is the set of points $\{x \in T : d(x, c) \leq r\}$.

We define our notion of neighborhood subtree size. Let $S$ be any connected subset of points of an embedded tree $T$. Define the function $|\cdot|$ as follows:

$$|S| = \begin{cases} max\{d(p_1, p_2) : p_1, p_2 \in S\} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

For any neighborhood subtree $T(c, r)$, we call $|T(c, r)|$ the *diameter* of $T(c, r)$. Observe that $|T(c, r)|$ is the length of a longest path in $T(c, r)$.

Our definition of diameter differs from that found in Bibelnieks and Dearing [4], where $|T(c, r)|$ is defined to be $2r$. Our reasons for this slight change are as follows.

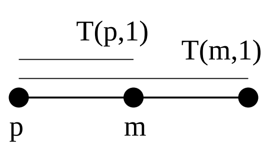



Figure 1.3: $T(p, 1)$ and $T(m, 1)$.

First, consider the embedded tree $T$ consisting of a line segment two units long, with endpoint $p$ and midpoint $m$. Let $T(p, 1)$ and $T(m, 1)$ be two neighborhood subtrees defined on $T$ (see Figure 1.3). Bibelnieks' and Dearing's [4] definition of diameter gives us $|T(p,1)| = 2 = |T(m,1)|$, however, $T(m,1)$ is obviously twice as large as $T(p,1)$. Defining the diameter of a neighborhood subtree to be the length of its longest path gives the more satisfactory $|T(p,1)| = 1$ and $2 = |T(m,1)|$.

In general, when a neighborhood subtree center is located near an endpoint of the embedded tree it may happen that the length of a longest path in the neighborhood subtree is less than twice the neighborhood subtree's radius. In this case the diameter of the neighborhood subtree is not accurately measured by twice its radius.

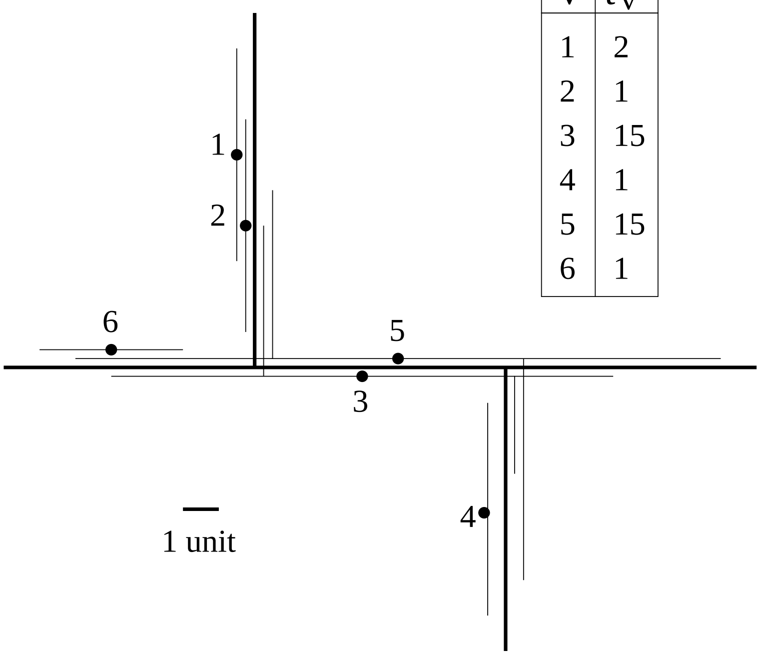
Second, defining diameter only for neighborhood subtrees, as do Bibelnieks and Dearing [4], causes an awkward special case: the diameter of the empty set is not defined. This results from the fact that a neighborhood subtree can never be empty. Since diameter is used to measure neighborhood subtree intersections it is necessary that it be defined for empty intersections. Our definition of diameter extends to cover this special case.

In section 2.1 we will see that our definition of diameter is, in practice, the same as that defined by Bibelnieks and Dearing when restricted to measuring the size of *standard neighborhood subtrees.*

### NeST Graphs

**Definition 1.3.1** *A graph $G$ is a* neighborhood subtree tolerance (NeST) graph *if there exists an embedded tree $T$, an indexed set, $S = \{T(c_v, r_v) : v \in V(G)\}$, of neighborhood subtrees of $T$ and an indexed set, $\mathcal{T} = \{\tau_v : v \in V(G)\}$, of positive numbers, called* tolerances*, such that the following edge condition holds:*

$$xy \in E(G) \Leftrightarrow |T(c_x, r_x) \cap T(c_y, r_y)| \geq min\{\tau_x, \tau_y\}.$$




| v | $t_v$ |
|---|---|
| 1 | 2 |
| 2 | 1 |
| 3 | 15 |
| 4 | 1 |
| 5 | 15 |
| 6 | 1 |

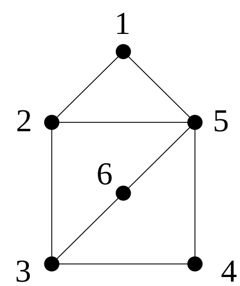



Figure 1.4: A NeST representation and associated graph.

*The triple* $(T, S, \mathcal{T})$ *is called a* neighborhood subtree tolerance (NeST) representation *of* $G$ *and* $G$ *is called the graph* associated *with the NeST representation* $(T, S, \mathcal{T})$.

We shall use $T_x$ as shortened notation for the neighborhood subtree $T(c_x, r_x)$. We shall also denote the intersection of $T_x$ and $T_y$ by $T_{xy}$. Whenever a NeST representation $(T, S, \mathcal{T})$ is referred to, it may be assumed that $T_x, c_x, r_x$ and $\tau_x$ denote the representation's neighborhood subtree, center, radius and tolerance, respectively, for some vertex $x$ in the NeST graph associated with $(T, S, \mathcal{T})$. An example of a NeST representation and its associated graph appears in Figure 1.4.

## 1.4 Overview

We give a brief overview of the chapters and sections which follow.

**Chapter 1:** The intent of this chapter is to examine the background of, motivate, illustrate and precisely define NeST representations and NeST graphs.

Section 1.1 gives a historical perspective for NeST representations. The ancestors of NeST representations, namely, interval representations and tolerance representations are presented and illustrated by examples. Various properties of these representations and their associated graphs are explored.

With the historical background of NeST representations in hand, section 1.2 motivates the study of NeST representations. In particular, it is shown that NeST representations are a natural and practical generalization of interval representations and tolerance representations while NeST graphs preserve key algorithmic properties possessed by interval graphs and tolerance graphs. Furthermore, NeST representations fill a void in the current literature. Finally, NeST representations are illustrated with an assortment of applications.

Section 1.3 defines the terminology we will be using throughout this work. A precise definition of NeST representations is given. This definition corrects an error found in the original paper on NeST representations [4].

Section 1.4 is this overview section.

**Chapter 2:** A NeST representation is a complicated mathematical structure with many parameters. This chapter filters from the definition of NeST representations redundant information. This strategy is applied to the three components of a NeST representation, namely, the set of neighborhood subtrees, the set of tolerances and the embedded tree.

Section 2.1 introduces the concept of maximal and truncated neighborhood subtrees. It is shown that maximal neighborhood subtrees have a distinct theoretical and practical advantage over their truncated counterparts. We introduce standard NeST representations which sidestep the problem of truncated neighborhood subtrees, without loss of generality. We discuss how truncated neighborhood subtrees arise in the generalization of interval representations and tolerance representations by NeST representations. Through this analysis we derive an alternative generalization of interval representations and tolerance representations which is more natural than NeST representations yet equivalent.

In section 2.2 we discover that tolerances in a NeST representation do not need to be specified explicitly, but instead, can be defined implicitly in a "tolerance-free" NeST representation. That is, tolerances in a NeST representation of a graph are redundant.

Section 2.3 restricts the domain of embedded trees necessary for a NeST representation. The tree operation of grafting is introduced and utilized to derive a sufficient condition on embedded trees.

**Chapter 3:** A better understanding of NeST representations and NeST graphs can be achieved by investigating the subclasses of NeST graphs associated with NeST representations which have been restricted in obvious ways. Two of the classes we investigate are introduced here, namely, fixed diameter and fixed tolerance NeST graphs, whereas proper and fixed tolerance NeST graphs were introduced by Bibelnieks and Dearing [4].

In section 3.2 we investigate fixed diameter NeST graphs. We show that these graphs are exactly unit NeST graphs which are analogous to unit interval and unit tolerance graphs. More importantly, we show that fixed diameter NeST graphs are proper NeST graphs. We also discuss the relationship between fixed diameter and fixed radius NeST representations.

Proper NeST representations and proper NeST graphs are the topic of section 3.3. Proper NeST graphs are analogous to proper interval and proper tolerance graphs. The question of whether unit interval and proper interval graphs are the same, as well as the question of whether unit tolerance and proper tolerance graphs are the same, have been asked and answered (yes to the former, no to the latter) in the literature. We ask and answer the question of whether unit NeST and proper NeST graphs are the same (yes). We close the section with a tolerance and radius-free characterization of proper NeST graphs.

Section 3.4 introduces fixed distance NeST graphs. We prove that a fixed distance NeST graph has a NeST representation where the embedded tree is an embedded star. We also demonstrate that fixed distance NeST graphs are a subclass of proper NeST graphs. Our main result is in response to a question posed by Monma, Reed and Trotter [16]: the class of fixed distance NeST graphs is exactly the class of threshold tolerance graphs. This equivalence implies polynomial recognition of fixed distance NeST graphs.

The final subclass of NeST graphs we explore is the class of fixed tolerance NeST graphs. The main result of section 3.5 is a characterization of fixed tolerance NeST graphs.

**Chapter 4:** In this chapter we address some unanswered questions regarding class inclusions and NeST graphs. These questions are both posed by ourselves and posed in the literature.

In section 4.1 we answer the question posed by Bibelnieks and Dearing [4]: is the class of NeST graphs a proper subclass of weakly triangulated graphs? To answer this question we utilize results obtained in earlier chapters of the thesis.

In section 4.2 we examine class inclusion relationships among proper, fixed distance and fixed tolerance NeST graphs.

**Chapter 5:** This chapter presents a summary of our results and conclusions. Open problems in the area of NeST representations and NeST graphs are discussed as well as possible directions for future research.

# Chapter 2

# Refining NeST Representations

The first hurdle that one encounters when working with NeST representations is their complex structure. In this chapter we will filter from the definition of a NeST representation redundant information in an attempt to make this hurdle less obstructive. We will do this by examining each of the main components of a NeST representation $(T, S, \mathcal{T})$, namely, the embedded tree $T$, the set of neighborhood subtrees $S$ and the set of tolerances $\mathcal{T}$.

## 2.1 Standard Neighborhood Subtrees

In this section we motivate, define and explore what we call standard neighborhood subtrees. It will be shown that standard neighborhood subtrees have practical and theoretical advantages over arbitrary neighborhood subtrees.

A neighborhood subtree $T(c, r)$ is called *maximal* if $|T(c, r)| = 2r$. The following lemma illustrates why the adjective "maximal" is appropriate:

**Lemma 2.1.1** *For any neighborhood subtree $T(c, r)$, $|T(c, r)| \leq 2r$.*

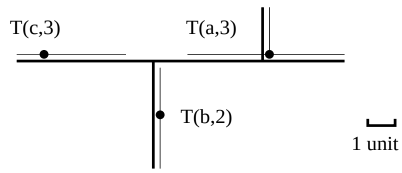



Figure 2.1: Maximal and truncated neighborhood subtrees.

*Proof:* Since $d$ is a metric on the embedded tree $T$, the triangle inequality holds, that is, $d(i,j) \leq d(i,k) + d(j,k)$ for all points $i, j, k \in T$. If $|T(c,r)| > 2r$ then there is a path $P(a,b)$ in $T(c,r)$ such that $d(a,b) > 2r$. However, $r \geq d(a,c)$ and $r \geq d(b,c)$ imply $2r \geq d(a,c) + d(b,c)$. By the triangle inequality we have $2r \geq d(a,b) > 2r$; a contradiction. $\square$

A neighborhood subtree which is not maximal will be called *truncated.* $T(c,3)$ in Figure 2.1 is a truncated neighborhood subtree whereas $T(a,3)$ and $T(b,2)$ are not. The existence of truncated neighborhood subtrees in a NeST representation is undesirable for the following reasons:

- The diameter of a maximal neighborhood subtree $T(c,r)$ depends only upon the value of $r$ and can be computed in constant time. The diameter of a truncated neighborhood subtree depends upon $r$, $c$ and the embedded tree $T$. Hence, determining the diameter of a truncated neighborhood subtree cannot, in general, be computed in constant time and requires more information to do so.

- The analysis of NeST representations where all neighborhood subtrees are maximal is simpler than analyzing those with truncated neighborhood subtrees because the diameter of a maximal neighborhood subtree is independent of the embedded tree.

The complications incurred by the presence of truncated neighborhood subtrees can be avoided by the use of standard neighborhood subtrees. We shall see that every NeST graph has a NeST representation in which each neighborhood subtree is a standard neighborhood subtree. Furthermore, standard neighborhood subtrees are always maximal.

**Definition 2.1.2** *We call* $T(m(c,r), l(c,r))$ *the* standard neighborhood subtree *of* $T(c,r)$ *where*

1. $m(c,r)$ *is the midpoint of a longest path in* $T(c,r)$ *and*

2. $l(c,r)$ *is half the length of a longest path in* $T(c,r)$.

**Lemma 2.1.3** *The standard neighborhood subtree of a neighborhood subtree is uniquely defined.*

*Proof:* Let $T(c,r)$ be a neighborhood subtree. To show that $T(m(c,r), l(c,r))$ is uniquely defined it is sufficient to show that $m(c,r)$ and $l(c,r)$ are unique. Obviously, $l(c,r)$ is unique since it is the length of a longest path in $T(c,r)$.

To see that $m(c,r)$ is also unique, suppose that $P(a,b)$ and $P(x,y)$ are two longest paths in $T(c,r)$ with midpoints $c_1$ and $c_2$ where $d(c_1,c_2) > 0$. We have that $d(a,b) = 2l(c,r) = d(x,y)$. Either $P(a,c_1) \cap P(c_1,c_2) = \{c_1\}$ or $P(b,c_1) \cap P(c_1,c_2) = \{c_1\}$, similarly, either $P(x,c_2) \cap P(c_1,c_2) = \{c_2\}$ or $P(y,c_2) \cap P(c_1,c_2) = \{c_2\}$. Without loss of generality, assume that $P(a,c_1) \cap P(c_1,c_2) = \{c_1\}$ and $P(x,c_2) \cap P(c_1,c_2) = \{c_2\}$. It follows that $P(a,x) = P(a,c_1) \cup P(c_1,c_2) \cup P(x,c_2)$ which implies that $d(a,x) = 2l(c,r) + d(c_1,c_2) > 2l(c,r)$. This contradicts the maximality of $d(a,b)$ and $d(x,y)$. By contradiction, $d(c_1,c_2) = 0$. Hence, the midpoints of all longest paths in $T(c,r)$ are identical which implies that $m(c,r)$ is unique. □

The following lemma demonstrates that a neighborhood subtree and its standard neighborhood subtree are identical point sets.

**Lemma 2.1.4** *$T(c,r) = T(m(c,r), l(c,r))$, for any neighborhood subtree $T(c,r)$.*

*Proof:* Let $P(a,b)$ be a longest path in $T(c,r)$. First we show that $T(m(c,r), l(c,r)) \subseteq T(c,r)$. We have that $m(c,r)$ is on $P(a,c)$ or $P(b,c)$, hence, $r \geq d(a,c) = d(c, m(c,r)) + d(a, m(c,r))$ or $r \geq d(b,c) = d(c, m(c,r)) + d(b, m(c,r))$. In either case, $r \geq d(c, m(c,r)) + l(c,r)$. If $x \in T(m(c,r), l(c,r))$ then $l(c,r) \geq d(x, m(c,r))$ and so $r \geq d(c, m(c,r)) + d(x, m(c,r)) \geq d(c,x)$. This proves $x \in T(c,r)$.

To see that $T(c,r) \subseteq T(m(c,r), l(c,r))$ first assume that $x$ is an element of $T(c,r) - T(m(c,r), l(c,r))$. This assumptions gives us $d(x, m(c,r)) > l(c,r)$. Now, $d(a,x) = d(a, m(c,r)) + d(x, m(c,r))$ or $d(b,x) = d(b, m(c,r)) + d(x, m(c,r))$. Thus, either $d(a,x) = l(c,r) + d(x, m(c,r)) > 2l(c,r)$ or $d(b,x) = l(c,r) + d(x, m(c,r)) > 2l(c,r)$. In either case, the maximality of $d(a,b)$ in $T(c,r)$ is contradicted. This proves that $T(c,r) \subseteq T(m(c,r), l(c,r))$. $\square$

**Lemma 2.1.5** *A neighborhood subtree is maximal if and only if it is a standard neighborhood subtree.*

*Proof:* ($\Rightarrow$) Let $T(c,r)$ be a maximal neighborhood subtree in some NeST representation. Let $P(a,b)$ be a maximal path in $T(c,r)$. By Lemma 2.1.4, $P(a,b)$ is maximal in $T(m(c,r), l(c,r))$ as well. Hence, $l(c,r) = r$. Since $d(a,c) = d(b,c) = r = |P(a,b)|/2$, $c$ is the midpoint of $P(a,b)$, hence, $c = m(c,r)$.

($\Leftarrow$) If $T(c,r)$ is a standard neighborhood subtree then, by definition, $|T(c,r)| = 2r$, hence, $T(c,r)$ is maximal. $\square$

**Lemma 2.1.6** *If $G$ is a NeST graph then $G$ has a NeST representation in which $c = m(c,r)$ and $r = l(c,r)$, for each neighborhood subtree $T(c,r)$.*

*Proof:* Let $G$ have a NeST representation $(T, S, \mathcal{T})$. Replace all neighborhood subtrees in $S$ with their standard neighborhood subtrees. $(T, S, \mathcal{T})$ remains a NeST representation of $G$ since, by Lemma 2.1.4, $T(c,r) = T(m(c,r), l(c,r))$, for all neighborhood subtrees $T(c,r)$. $\square$

**Corollary 2.1.7** *If $G$ is a NeST graph then $G$ has a NeST representation where all neighborhood subtrees are maximal.*

*Proof:* By Lemma 2.1.6, $G$ has a NeST representation in which all neighborhood subtrees are standard neighborhood subtrees. Hence, by Lemma 2.1.5, $G$ has a NeST representation where all neighborhood subtrees are maximal. $\square$

A consequence of Corollary 2.1.7 is that we may restrict our analysis of NeST representations, without loss of generality, to those NeST representations whose neighborhood subtrees are maximal. This fact motivates the following definition:

**Definition 2.1.8** *A NeST representation is called a* standard NeST representation *if all neighborhood subtrees are standard neighborhood subtrees.*

Observe that when measuring neighborhood subtree size in a standard NeST representation there is no practical difference between the definition of diameter found in Bibelnieks and Dearing [4] and the definition of diameter found here.

Consider the relationship between tolerance representations, NeST representations and neighborhood maximality. Intervals on the line can be perceived as neighborhoods

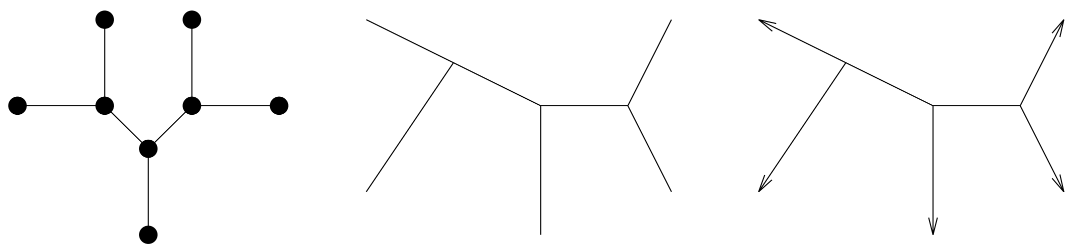


Figure 2.2: A tree $\mathcal{T}$, an embedding of $\mathcal{T}$ and an extended embedding of $\mathcal{T}$.

of the line. Since the line in a tolerance representation is unbounded, intervals in a tolerance representation are always maximal neighborhoods. NeST representations are a generalization of tolerance representations yet neighborhoods in an embedded tree aren't necessarily maximal. The reason for this is that the embedded tree has endpoints, hence, neighborhood subtrees can be truncated.

Although standard NeST representations avoid the problem of truncated neighborhood subtrees, they do not avoid the fact that tolerance representations have an inherent property, namely, neighborhood maximality, that NeST representations do not possess. In a generalization of tolerance representations we would like to maintain such nice properties. This problem can be solved by an alternative generalization of tolerance representations which is more elegant, yet equivalent to, NeST representations. We call this generalization an *extended NeST representation.*

An extended embedded tree $T$ is an embedding of a tree $\mathcal{T}$ except that leaves of $\mathcal{T}$ are mapped to nonintersecting halflines. See Figure 2.2 for an example.

All neighborhoods of an extended embedding tree are maximal for the same reason that neighborhoods of a line are maximal: there are no endpoints. We define an extended NeST representation to be a NeST representation $(T, S, \mathcal{T})$ except that $T$ is an extended embedded tree. It is easily seen that every extended NeST representation $(T, S, \mathcal{T})$ yields a NeST representation $(T', S', \mathcal{T}')$ where $T'$ is an (unextended) embedded tree: let $S' = S$, $\mathcal{T}' = \mathcal{T}$ and $T' = T(p,r)$ where $T(p,r)$ is a neighborhood subtree of $T$ with arbitrary center $p$ and radius $r$ sufficiently large so that, for all $T_x \in S$, $T_x \subset T(p,r)$.

The following theorem proves the converse to be true, that is, any graph which has a NeST representation also has an extended NeST representation. This result is not obvious since, in general, $T_x \neq T'_x$, where $T_x \in S$ and $T'_x \in S'$ are defined as above.

**Theorem 2.1.9** *If a graph has a NeST representation then it has an extended NeST representation.*

*Proof:* Let $G$ have a standard NeST representation $R = (T, S, \mathcal{T})$. Define an extended NeST representation $R' = (T', S', \mathcal{T}')$ where $T'$ is the extended embedded tree obtained by attaching a halfline $L_z$ to each endpoint $z$ of $T$. Let $S' = S$ and $\mathcal{T}' = \mathcal{T}$, hence, for all $x \in V(G)$, $c'_x = c_x$, $r'_x = r_x$ and $\tau'_x = \tau_x$.

We will show that $R'$ is an extended NeST representation of $G$. It is sufficient to show that $|T'_{xy}| = |T_{xy}|$, for all pairs $x, y \in V(G)$. We proceed with a case by case analysis.

**Case 1:** $T_{xy} = \emptyset$.

If $T_x$ and $T_y$ do not intersect then we have that $d(c_x, c_y) > r_x + r_y$. Since $d(c'_x, c'_y) = d(c_x, c_y)$, $r'_x = r_x$ and $r'_y = r_y$ it follows that $T'_x$ and $T'_y$ do not intersect either. Hence, $|T_{xy}| = 0 = |T'_{xy}|$.

**Case 2:** $T_x \not\subset T_y$, $T_y \not\subset T_x$ and $T_{xy} \neq \emptyset$.

In this case we have $|T_{xy}| = r_x + r_y - d(c_x, c_y)$ (referencing forward to Lemma 3.3.3). Since $T'_x \not\subset T'_y$, $T'_y \not\subset T'_x$ and $T'_{xy} \neq \emptyset$ it follows that $|T'_{xy}| = r'_x + r'_y - d(c'_x, c'_y) = |T_{xy}|$.

**Case 3:** $T_x \subset T_y$

This case is symmetric with $T_y \subset T_x$. $T_x \subset T_y$ implies that $|T_{xy}| = |T_x| = 2r_x = 2r'_x = |T'_x|$. Since $|T'_{uv}| \geq |T_{uv}|$, for all $u, v \in V(G)$, and $|T'_x| \geq |T'_{xy}|$ it must be that $|T'_{xy}| = |T_{xy}|$.

Having examined all cases we conclude that $|T'_{xy}| = |T_{xy}|$, for all $x, y \in V$. $\square$

It is an interesting fact that the graphs associated with NeST representations and those associated with extended NeST representations are identical. Nothing is gained or lost by restricting embedded trees to be bounded.

## 2.2 The Role of Tolerances

Our aim in this section is to simplify NeST representations by eliminating tolerances. Note that the removal of tolerances is done in an effort to better understand the class of NeST graphs; we are not suggesting a redefinition of NeST representations without tolerances as tolerances are an important modeling component of NeST representations. The "tolerance-free" characterization of NeST graphs we achieve suggests a simpler, yet sufficient, description of NeST graphs and will be exploited in the study of various subclasses of NeST graphs discussed later.

Given a graph $G$ with NeST representation $(T, S, \mathcal{T})$ we define the set $B_x$, for every vertex $x \in V(G)$, by

$$B_x = \{z \in M_x : |T_{xz}| \geq |T_{xy}|, \text{for all } y \in M_x\}.$$

$B_x$ is the set of non-neighbors of $x$ that maximize the size of their neighborhood subtree intersections with the neighborhood subtree of $x$. Observe that $B_x$ may be empty if $M_x = \emptyset$.

**Definition 2.2.1** *The pair* $(T, S)$*, where* $T$ *is an embedded tree and* $S$ *is a set of neighborhood subtrees in* $T$*, is a* tolerance-free NeST representation *of the graph* $G$ *if the following edge condition holds:*

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\},$$

*where, for a vertex* $z$*,* $\dot{z}$ *is any vertex in* $B_z$*, and* $|T_{z\dot{z}}| = 0$ *whenever* $B(z)$ *is empty.*

Before stating and proving the main result of this section, we define the *perturbation number* of a tolerance-free NeST representation of a graph $G$. The perturbation number measures the amount of "flexibility" a tolerance-free NeST representation and graph possess.

**Definition 2.2.2** *The perturbation number,* $\rho(R, G)$*, of a tolerance-free NeST representation* $R = (T, S)$ *of a graph* $G$ *is defined by*

$$\rho(R, G) = \begin{cases} min(D) & \text{if } D \neq \emptyset, \\ 0 & \text{if } D = \emptyset, \end{cases}$$

*where*

$$D = \{|T_{xy}| - |T_{x\dot{x}}| : |T_{xy}| > |T_{x\dot{x}}|, \;\; xy \in E(G)\}.$$

Stating that $G$ is a NeST graph is equivalent to stating that there is a NeST representation $(T, S, \mathcal{T})$ such that $G$ and $(T, S, \mathcal{T})$ satisfy the following relation:

$$xy \in E(G) \;\;\Leftrightarrow\;\; |T_{xy}| \geq min\{\tau_x, \tau_y\}.$$

There is redundancy in this characterization. In particular, the set of tolerances $\mathcal{T}$ is redundant. The following theorem proves that this redundancy exists and can be extracted systematically.

**Theorem 2.2.3** *$(T, S, \mathcal{T})$ is a NeST representation of $G$ if and only if $(T, S)$ is a tolerance-free NeST representation of $G$.*

*Proof:* ($\Leftarrow$) Let $G$ have a tolerance-free NeST representation $(T, S)$. Define a NeST representation $(T, S, \mathcal{T})$ where each tolerance $\tau_x \in \mathcal{T}$ is defined by

$$\tau_x = |T_{x\dot{x}}| + \epsilon$$

and $\epsilon$ is given by

$$\epsilon = \begin{cases} \rho((T,S),G)/2 & \text{if } \rho((T,S),G) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

To prove that $(T, S, \mathcal{T})$ is a NeST representation of $G$ we must verify that

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\},$$

for all $x, y \in V(G)$.

Suppose that $xy \in E(G)$. From the definition of a tolerance-free NeST representation it follows that $|T_{xy}| > |T_{x\dot{x}}|$ or $|T_{xy}| > |T_{y\dot{y}}|$; assume the former without loss of generality. Since $|T_{xy}| > |T_{x\dot{x}}|$ it follows that $\rho((T,S),G) > 0$ and $|T_{xy}| - |T_{x\dot{x}}| \geq \rho((T,S),G) > \epsilon$. Hence, $|T_{xy}| > |T_{x\dot{x}}| + \epsilon = \tau_x$.

Suppose that $xy \notin E(G)$. From the definition of a tolerance-free NeST representation it follows that $|T_{xy}| \leq |T_{x\dot{x}}|$ and $|T_{xy}| \leq |T_{y\dot{y}}|$. Since $\epsilon > 0$ it follows that $|T_{xy}| < |T_{x\dot{x}}| + \epsilon = \tau_x$ and $|T_{xy}| < |T_{y\dot{y}}| + \epsilon = \tau_y$.

($\Rightarrow$) Let $G$ have a NeST representation $(T, S, \mathcal{T})$. We claim that $(T, S)$ is a tolerance-free NeST representation of $G$. To prove this we must verify that

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\},$$

for all $x, y \in V(G)$.

Suppose that $xy \in E(G)$. It follows that either $|T_{xy}| \geq \tau_x$ or $|T_{xy}| \geq \tau_y$; assume the former without loss of generality. Observe that for all $z \in M_x$, $\tau_x > |T_{xz}|$. In particular, $\tau_x > |T_{x\dot{x}}|$. It follows that $|T_{xy}| > |T_{x\dot{x}}|$.

Suppose that $xy \notin E(G)$. By definition of $\dot{x}$ and $\dot{y}$, $xy \notin E(G)$ implies that $|T_{xy}| \leq |T_{x\dot{x}}|$ and $|T_{xy}| \leq |T_{y\dot{y}}|$. □

In closing, we make three observations.

1. Theorem 2.2.3 and its proof characterize the role of tolerances in a NeST representation as that of placeholders. It is not necessary that these placeholders be specified explicitly as in a NeST representation: tolerances can be implicitly defined by a tolerance-free NeST representation. This simplification of NeST representations, though not surprising, will prove to be useful.

2. The above result proves that a tolerance-free NeST representation $(T, S)$ of a graph $G$ is obtained from a NeST representation $(T, S, \mathcal{T})$ of $G$ simply by dropping $\mathcal{T}$ from the triple $(T, S, \mathcal{T})$. Conversely, the proof of Theorem 2.2.3 describes a method by which we can reconstruct a set of tolerances $\mathcal{T}$ from a tolerance-free NeST representation $(T, S)$ of $G$ to obtain a NeST representation $(T, S, \mathcal{T})$ of $G$. It is important to realize that the embedded tree $T$ and the set of neighborhood subtrees $S$ defined in the tolerance-free NeST representation and the NeST representation are identical.

3. For every NeST representation there is a unique associated graph. That is, the triple $(T, S, \mathcal{T})$ and the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\}$$

completely specify a unique graph $G$. This is not true for tolerance-free NeST representations. That is, the pair $(T, S)$ and the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

may be satisfied by several different graphs $G$.

## 2.3 Sufficient Embedded Trees

For every embedded tree $T$ there is a unique weighted tree $(\mathcal{T}, w)$ such that $T = (f, \mathcal{T})$, $|f(e)| = w(e)$, for all $e \in E(\mathcal{T})$ and $\mathcal{T}$ has no degree 2 vertices. Embedded trees are a complex component of NeST representations and it is to our advantage to establish sufficient structural properties upon these trees and their unique weighted counterparts. In this section we introduce the tree operation of *grafting* which will be used in proving a sufficiency condition for embedded trees in NeST representations.

### 2.3.1 Embedding Weighted Trees

It is common knowledge that any tree is embeddable in the plane. In fact, any weighted tree $(\mathcal{T}, w)$ has an embedding $(f, \mathcal{T})$ which preserves the weight function $w$. Several algorithms which produce planar drawings of weighted trees appear in Barthélemy and Guénoche [1].

**Definition 2.3.1** *An embedding $(f, \mathcal{T})$ of the weighted tree $(\mathcal{T}, w)$ is* weight preserving *if $|f(e)| = w(e)$, for all $e \in E(\mathcal{T})$.*

**Corollary 2.3.2** *Any weighted tree $(\mathcal{T}, w)$ has a weight preserving embedding.*

*Proof:* We omit proof though the reader is directed to Barthélemy and Guénoche [1]. □

Given an embedded tree $T$ we can also define a weighted tree for which $T$ is a weight preserving embedding.

**Definition 2.3.3** *If $T$ is an embedded tree then the* discrete tree *associated with $T$, $(\mathcal{T}, w)$, is defined by:*

- *For each $p \in T$ such that $p$ is an endpoint of $T$ or the intersection of line segments in $T$, add a vertex $v_p$ to $V(\mathcal{T})$.*
- *Let $v_p v_q \in E(\mathcal{T})$ iff $p$ and $q$ share a line segment in $T$.*

Discrete trees will be utilized in the following section.

### 2.3.2 Grafting

In this section we define the tree operation of *grafting.* Grafting will be used to show that if a graph $G$ has a NeST representation with embedded tree $(f, \mathcal{T})$ then it is sufficient that $\mathcal{T}$ have only degree one and degree three vertices. This sufficiency condition restricts the number of viable trees $\mathcal{T}$ for a NeST representation.

Informally, grafting is the operation of severing a subtree of a tree $\mathcal{T}$ and splicing it back onto an edge of $\mathcal{T}$.

**Definition 2.3.4** *If $\mathcal{T}$ is a tree with edges $vw$ and $ab$ then $\mathcal{T}' = \mathcal{T}(v, w, ab, z)$ is called a* grafting of $\mathcal{T}$ *where*

- *$z \notin V(\mathcal{T})$,*
- *$ab \in \mathcal{T}_{E(\mathcal{T})-\{vw\}}(v)$,*

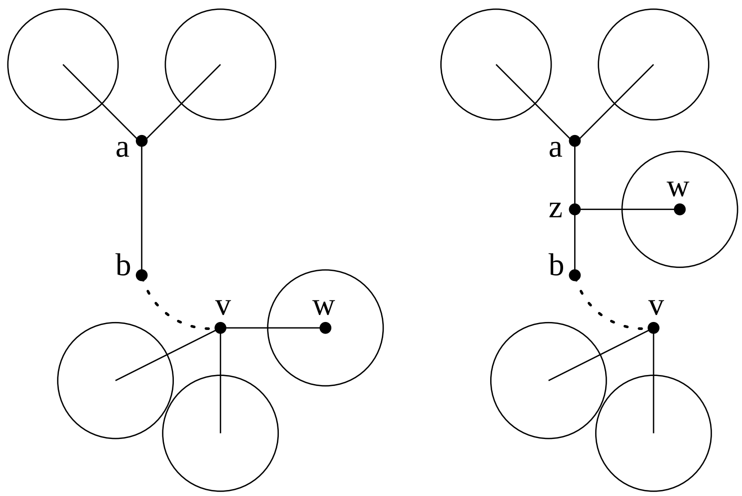



Figure 2.3: $\mathcal{T}$ and the grafting $\mathcal{T}(v, w, ab, z)$.

- $V(\mathcal{T}') = V(\mathcal{T}) \cup \{z\}$ *and*

- $E(\mathcal{T}') = (E(\mathcal{T}) \cup \{az, bz, wz\}) - \{ab, vw\}$.

Hence, $\mathcal{T}(v, w, ab, z)$ is the tree obtained by severing the edge $vw$, inserting the new vertex $z$ onto the edge $ab$ and then adding an edge between $w$ and $z$. This operation is depicted in Figure 2.3.

We can extend the notion of grafting trees to grafting embedded trees:

**Definition 2.3.5** *The embedded tree $T' = (f', \mathcal{T}')$ is a* grafting *of the embedded tree $T = (f, \mathcal{T})$ if $\mathcal{T}'$ is a grafting of $\mathcal{T}$.*

Grafting of trees and embedded trees will be employed in the proof of the following structural lemma:

**Lemma 2.3.6** *If $G$ is a NeST graph then $G$ has a tolerance-free NeST representation $(T, S)$ where the discrete tree associated with $T$ has only degree 1 or degree 3 vertices.*

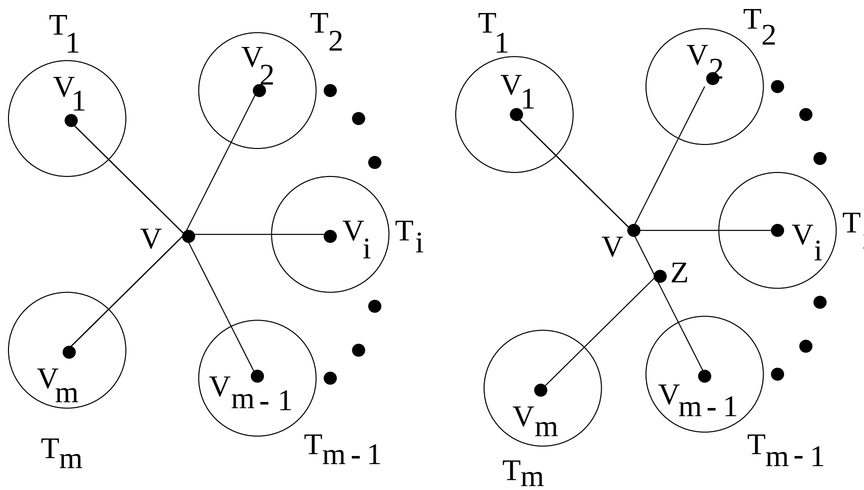



Figure 2.4: Reducing the degree of $v$ by grafting.

*Proof:* Let $(T, S)$ be a tolerance-free NeST representation of $G$ and $(\mathcal{T}, w)$ the discrete tree associated with $T$. Suppose $v \in V(\mathcal{T})$ and $deg(v) = 2$ where $N_v = \{x, y\}$. Define $\mathcal{T}'$ by

$$V(\mathcal{T}') = V(\mathcal{T}) - \{v\} \text{ and}$$

$$E(\mathcal{T}') = (E(\mathcal{T}) \cup \{xy\}) - \{vx, vy\}.$$

Now define $w'$ by:

$$w'(e) = \begin{cases} w(e) & \text{if } e \neq xy, \\ w(vx) + w(vy) & \text{if } e = xy. \end{cases}$$

Let $T'$ be a weight preserving embedding of $(\mathcal{T}', w')$. Define a tolerance-free NeST representation $(T', S')$ for $G$ where neighborhood subtree centers are placed in $T'$ such that $d(c'_x, c'_y) = d(c_x, c_y)$, for all $x, y \in V(G)$. Allow the neighborhood subtree radii to be the same as in $(T, S)$. It follows that $(T', S')$ is a tolerance-free NeST representation of $G$. We can repeat the above for all vertices in the discrete tree of degree 2.

We assume that $\mathcal{T}$ has no degree two vertices. Suppose that $v \in V(\mathcal{T})$ and $deg(v) = m > 3$. Let $v_1, v_2, \ldots, v_m$ be the $m$ neighbors of $v$ in $\mathcal{T}$. Let $\mathcal{T}_i = \mathcal{T}_{V(\mathcal{T})-\{v\}}(v_i)$, $1 \leq i \leq m$, be the $m$ connected components of $\mathcal{T}_{V(\mathcal{T})-\{v\}}$.

Let $\mathcal{T}'$ be the grafting $\mathcal{T}(v, v_m, vv_{m-1}, z)$. This grafting appears in Figure 2.4. Observe that the degree of $v$ in $\mathcal{T}'$ is one less than the degree of $v$ in $\mathcal{T}$. Define a weight function $w'$ for $\mathcal{T}'$ as follows:

$$w'(e) = \begin{cases} \delta & e = vz, \\ w(vv_{m-1}) & e = v_{m-1}z, \\ w(vv_m) & e = v_m z, \\ w(e) & \text{otherwise}, \end{cases}$$

where $0 < \delta < p((T,S),G)/2$. In the case that $p((T,S),G) = 0$, $G$ has no edges and thus has a tolerance-free NeST representation where the embedded tree is a single vertex. Assume $p((T,S),G) \neq 0$.

Let $T'$ be a weight preserving embedding of $(\mathcal{T}', w')$. Define a tolerance-free NeST representation $(T', S')$ for $G$ where neighborhood subtree centers are placed in $T'$ such that $d(c'_x, c'_y) = d(c_x, c_y)$, for all $x, y \in V(G)$. Allow the neighborhood subtree radii be the same as in $(T, S)$.

We claim that $(T', S')$ is a tolerance-free NeST representation of $G$. We must show that the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |T'_{xy}| > min\{|T'_{x\dot{x}}|, |T'_{y\dot{y}}|\}\,.$$

First observe that $||T_{xy}| - |T'_{xy}|| \leq \delta$. Suppose $xy \in E(G)$. Since $(T,S)$ is a tolerance-free NeST representation of $G$ it follows that $|T_{xy}| > |T_{x\dot{x}}|$, without loss of generality. In the worst case, $|T'_{xy}| = |T_{xy}| - \delta$ and $|T'_{x\dot{x}}| = |T_{x\dot{x}}| + \delta$. Hence, $|T'_{xy}| - |T'_{x\dot{x}}| =$

$|T_{xy}| - \delta - |T_{x\dot{x}}| - \delta = |T_{xy}| - |T_{x\dot{x}}| - 2\delta > |T_{xy}| - |T_{x\dot{x}}| - p((T,S),G)$. But by definition, $p((T,S),G) \leq |T_{xy}| - |T_{x\dot{x}}|$, so $|T_{xy}| - |T_{x\dot{x}}| - p((T,S),G) \geq 0$ and it follows that $|T'_{xy}| > |T'_{x\dot{x}}|$.

Suppose $xy \notin E(G)$. By definition, $|T'_{xy}| \leq |T'_{x\dot{x}}|$. Hence, we have shown that the edge condition holds so $(T', S')$ is a tolerance-free NeST representation of $G$. We can repeat the above process until the degree of $v$ is three. Furthermore, the process does not increase the degree of any other vertices. Hence, applying the process to all vertices of degree greater than three results in a tolerance-free NeST representation of $G$ where the discrete tree has only degree one and degree three vertices. $\square$

# Chapter 3

# Subclasses of NeST Graphs

## 3.1 Purpose

The strategy behind this chapter is to achieve a better understanding of NeST representations and NeST graphs by examining important subclasses of NeST graphs. Each of the subclasses we study are obtained by restricting NeST representations in some obvious and natural way.

The subclasses we will be studying are *fixed diameter NeST graphs*, *proper NeST graphs*, *fixed distance NeST graphs* and *fixed tolerance NeST graphs*. Fixed diameter NeST graphs are obtained by restricting all neighborhood subtrees in a NeST representation to be the same size. Proper NeST graphs are obtained by requiring that no neighborhood subtree in a NeST representation be properly contained within another neighborhood subtree. Fixed distance NeST graphs are obtained by requiring all neighborhood subtree centers to be equidistant in the embedded tree of a NeST representation. Finally, fixed tolerance NeST graphs are obtained by requiring all tolerances in a NeST representation to be equal.

The subclasses we examine have a theoretical motivation which will be discussed in their respective sections. Proper and fixed tolerance NeST graphs already appear in the literature [4], whereas fixed diameter and fixed distance NeST graphs are introduced here.

## 3.2 Fixed Diameter NeST Graphs

We begin our study of subclasses of NeST graphs with the class of fixed diameter NeST graphs. Fixed diameter NeST graphs are those graphs which have a NeST representation in which all neighborhood subtrees have the same diameter. The theoretical motivation for studying fixed diameter NeST graphs is that they are analogous to fixed interval graphs and fixed tolerance graphs. We shall show that fixed diameter NeST graphs are exactly *unit NeST graphs* (fixed diameter NeST graphs where all neighborhood subtree diameters have unit size). More importantly, we shall show that fixed diameter NeST graphs are proper NeST graphs. The analogues of these problems have been studied with respect to interval graphs and tolerance graphs.

**Definition 3.2.1** *A graph $G$ is a* fixed diameter NeST graph *if there exists a NeST representation of $G$ in which all neighborhood subtree diameters are equal. Such a NeST representation is called a* fixed diameter NeST representation*. $G$ is a* unit NeST graph *if it has a NeST representation where all neighborhood subtree diameters have unit size. Such a NeST representation is called a* unit NeST representation*.*

Before we continue our discussion on fixed diameter NeST graphs, we introduce the concept of *scaling* a NeST representation. If $(T, S, \mathcal{T})$ is a NeST representation then we scale $(T, S, \mathcal{T})$ by a positive constant $k$, denoted $k(T, S, \mathcal{T})$, if

- the embedded tree $T$ is expanded (or contracted) uniformly by a factor $k$ such that the distance between any two points, $a$ and $b$, in the expanded tree is equal to $kd(a,b)$ where $d(a,b)$ is the distance from $a$ to $b$ in $T$,
- all neighborhood subtree radii are multiplied by a factor $k$ and
- all neighborhood subtree tolerances are multiplied by a factor $k$.

The following lemma confirms the intuition that the scaled NeST representation $k(T,S,\mathcal{T})$ is a NeST representation of $G$ if and only if $(T,S,\mathcal{T})$ is a NeST representation of $G$.

**Lemma 3.2.2** *$(T,S,\mathcal{T})$ and $k(T,S,\mathcal{T})$ are NeST representations of the same graph.*

*Proof:* The equality of $(T,S,\mathcal{T})$ and $k(T,S,\mathcal{T})$ is easily exhibited by scaling the unit of measure[1]. □

**Corollary 3.2.3** *$G$ is a fixed diameter NeST graph if and only if it is a unit NeST graph.*

*Proof:* Let $(T,S,\mathcal{T})$ be a fixed diameter NeST representation of $G$ where all neighborhood subtree diameters are $D$. Since scaling preserves the ratio of neighborhood subtree diameters, $(1/D)(T,S,\mathcal{T})$ is a fixed diameter NeST representation of $G$ where all diameters have unit size.

The converse is true by definition. □

Proper NeST graphs are those graphs which have a NeST representation where no neighborhood subtree is properly contained within another. This class of graphs is formally introduced in section 3.3. Proper interval and proper tolerance graphs

[1] Thanks to David Gregory for this observation.

have analogous definitions. It is known that unit interval and unit tolerance graphs are proper interval [18] and proper tolerance graphs [10], respectively. Below we show that the same is true for NeST graphs. That is, fixed diameter NeST graphs (i.e. unit NeST graphs) are proper NeST graphs.

**Theorem 3.2.4** *A fixed diameter NeST representation is a proper NeST representation.*

*Proof:* Let $(T, S, \mathcal{T})$ be a fixed diameter NeST representation. Suppose that $T_x \subset T_y$ for some $x, y \in V(G)$ and let $D = |T_x| = |T_y|$. Consider the standard neighborhood subtrees $T(m(c_x, r_x), l(c_x, r_x))$ and $T(m(c_y, r_y), l(c_y, r_y))$ of $T_x$ and $T_y$. For brevity, let $c'_x = m(c_x, r_x)$, $r'_x = l(c_x, r_x)$, $c'_y = m(c_y, r_y)$ and $r'_y = l(c_y, r_y)$. Recall from section 2.1 that $T_x = T(c'_x, r'_x)$ and $T_y = T(c'_y, r'_y)$, hence, $T(c'_x, r'_x) \subset T(c'_y, r'_y)$.

Now, $c'_x$ is the midpoint of the path $P(a, b)$ where $a$ and $b$ are maximally distant points in $T(c'_x, r'_x)$. However, since $|T(c'_x, r'_x)| = D = |T(c'_y, r'_y)|$ it follows that $a$ and $b$ are maximally distant in $T(c'_y, r'_y)$ as well. Since $T(c'_y, r'_y)$ is a standard neighborhood subtree, it follows from Lemma 2.1.3 that $c'_y = c'_x$ and $r'_y = r'_x$. However, this implies that $T(c'_x, r'_x) = T(c'_y, r'_y)$. We conclude that $(T, S, \mathcal{T})$ is a proper NeST representation. $\square$

A closing comment on fixed diameter NeST representations. Consider those NeST representations obtained by the restriction that all neighborhood subtree radii be equal. Call these NeST representations *fixed radius NeST representations.* Intuitively, we may equate fixed radius NeST representations with fixed diameter NeST representations. However, this intuition is incorrect: two neighborhood subtrees in a NeST representation may have equal radii yet have different diameters. This occurs when neighborhood subtrees are truncated. However, if truncated neighborhood subtrees do not occur, fixed radius and fixed diameter NeST representations are identical. In other words, fixed radius standard NeST representations can be equated with fixed diameter NeST representations. The divergence between fixed radius and fixed diameter NeST representations does not occur in the analogous interval and tolerance representations. That is, when intervals on the line are perceived as neighborhoods, fixed radius and fixed diameter interval representations, as well as fixed radius and fixed diameter tolerance representations, are identical. As discussed in section 2.1, this divergence is a side effect of generalizing the unbounded line by an embedded tree with endpoints.

In the next section on proper NeST graphs, we will again encounter fixed diameter NeST graphs as they play a central role in the understanding of proper NeST graphs.

## 3.3 Proper NeST Graphs

In this section we explore proper NeST graphs which were informally introduced in section 3.2. Proper NeST graphs are analogous to proper interval and proper tolerance graphs which are well-studied subclasses of interval and tolerance graphs, respectively.

**Definition 3.3.1** *A graph $G$ is called a* proper NeST graph *if it has a NeST representation $(T, S, \mathcal{T})$ where no neighborhood subtree is properly contained within another neighborhood subtree. That is, for all $x, y \in V(G)$, $T_x \not\subset T_y$. We call a NeST representation with this property a* proper NeST representation.

Roberts [18] raised and resolved the question of whether unit interval graphs are the same as proper interval graphs (yes); Bogart, Isaak, Langley and Fishburn [5] resolved the analogous question, posed by Golumbic, Monma and Trotter, of whether unit tolerance graphs are the same as proper tolerance graphs (no). As the major result of this section, we raise and resolve the obvious question of whether fixed diameter NeST graphs (equivalently, unit NeST graphs) are the same as proper NeST graphs (yes). We close the section with a tolerance-free and radius-free characterization of proper NeST graphs.

**Lemma 3.3.2** *If $G$ is a proper NeST graph then there exists a proper NeST representation of $G$ in which all neighborhood subtree intersections are nonempty.*

*Proof:* We know that the result is true for NeST graphs in general [4]. However, we must show that it holds for proper NeST graphs as well.

Suppose that $G$ has a proper NeST representation $(T, S, \mathcal{T})$. The following procedure is used by Bibelnieks and Dearing [4] to modify the NeST representation $(T, S, \mathcal{T})$ of a graph $G$ so that all neighborhood subtree intersections are non-empty:

- Let $M = max\{d_{xy} - r_x - r_y : x, y \in V(G), x \neq y\}$.
- If $M \leq 0$ then stop,
- else add $M/2$ to all radii, add $M$ to all tolerances and attach line segments of length $M/2$ to all endpoints of $T$.

The above procedure enlarges all of the neighborhood subtrees, and the embedded tree $T$, such that a proper NeST representation remains proper after being transformed by the procedure. $\square$

**Lemma 3.3.3** *If $T_x \not\subset T_y$, $T_y \not\subset T_x$ and $T_{xy} \neq \emptyset$ then $|T_{xy}| = r_x + r_y - d(c_x, c_y)$.*

*Proof:* Bibelnieks and Dearing have shown that, if $T_x \not\subset T_y$, $T_y \not\subset T_x$ and $T_{xy} \neq \emptyset$ then $T_{xy}$ is a neighborhood subtree with radius $1/2(r_x + r_y - d(c_x, c_y))$ [4]. It follows from their result that $|T_{xy}| = r_x + r_y - d(c_x, c_y)$. □

The main result of this section follows:

**Theorem 3.3.4** *If $G$ has a proper NeST representation then $G$ has a fixed diameter, standard NeST representation.*

*Proof:* Let $R = (T, S, \mathcal{T})$ be a proper standard NeST representation of $G$. Let $r = max\{r_x : x \in V(G)\}$. We define a new NeST representation $R' = (T', S', \mathcal{T}')$ as follows:

1. $T' = (\bigcup_{x \in V(G)} L_x) \cup T$ where, for each $x \in V(G)$, $L_x$ is a line segment of length $2r$ attached to $T$ at $c_x$.

2. For all $x \in V(G)$, $c'_x$ is located on $L_x$ such that $d(c'_x, c_x) = r - r_x$.

3. For all $x \in V(G)$, $r'_x = r$ and $\tau'_x = \tau_x$.

**Claim 1:** $R'$ is a fixed diameter NeST representation.

Since $r'_x = r$ for all $x \in V(G)$, we need only show that all neighborhood subtrees in $R'$ are maximal.

Consider $x \in V(G)$. Let $p \in L_x$ such that $d(c'_x, p) = r$. Let $P(a, b)$ be a maximal path in $T_x$. Since $T_x$ is maximal, $d(a, c_x) = r_x$. Consider the path $P(a, p)$ in $T'_x$. $P(a, p) = P(a, c_x) \cup P(c_x, c'_x) \cup P(c'_x, p)$, hence, $d(a, p) = r_x + (r - r_x) + r = 2r$. It follows that $T'_x$ is maximal.

**Claim 2:** $R'$ is a standard NeST representation.

In Claim 1 we proved that all neighborhood subtrees were maximal. By Lemma 2.1.5, all neighborhood subtrees are standard neighborhood subtrees.

**Claim 3:** $R'$ is a NeST representation of $G$.

It must be shown that, for all $x, y \in V(G)$, the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |T'_{xy}| \geq min\{\tau'_x, \tau'_y\}.$$

By Lemma 3.3.2, we may assume that $|T'_{xy}| \neq \emptyset$. Since $R'$ is a fixed diameter NeST representation, $R'$ is a proper NeST representation by Theorem 3.2.4. By Lemma 3.3.3, and the fact that $R'$ is proper, it follows that $|T'_{xy}| = r'_x + r'_y - d(c'_x, c'_y) = 2r - d(c'_x, c'_y)$. Observe that $d(c'_x, c'_y) = d(c_x, c'_x) + d(c_x, c_y) + d(c_y, c'_y) = (r - r_x) + d(c_x, c_y) + (r - r_y)$, thus, $|T'_{xy}| = 2r - (2r - r_x - r_y + d(c_x, c_y)) = r_x + r_y - d(c_x, c_y)$. By Lemmas 3.3.2 and 3.3.3, we may assume that $|T_{xy}| = r_x + r_y - d(c_x, c_y)$ and so $|T'_{xy}| = |T_{xy}|$.

Since $\tau'_x = \tau_x$ for all $x \in V(G)$, it follows from

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\}$$

that

$$xy \in E(G) \Leftrightarrow |T'_{xy}| \geq min\{\tau'_x, \tau'_y\}.$$

□

**Corollary 3.3.5** *$G$ is a proper NeST graph if and only if $G$ is a fixed diameter NeST graph (equivalently, unit NeST graph).*

*Proof:* Theorem 3.3.4 and Theorem 3.2.4. □

As mentioned earlier, interval graphs possess the property that unit interval graphs are identical to proper interval graphs. The generalization of interval graphs to tolerance graphs, by the introduction of tolerances, results in the loss of this property for tolerance graphs. That is, unit tolerance graphs are not equal to proper tolerance graphs. It is an interesting fact that this property is regained in the generalization of tolerance graphs to NeST graphs by generalizing the line to a tree.

The following definition will be utilized in our characterization of proper NeST graphs:

**Definition 3.3.6** *$(T, X, c)$ is called an* $X$-tree *if $T$ is an embedded tree, $X$ is a finite set and $c : X \to T$; $c$ is called the* locator *of the $X$-tree.*

Informally, an $X$-tree is an embedded tree in which the elements of $X$ have been located. We now state and prove our tolerance-free and radius-free characterization of proper NeST graphs.

**Theorem 3.3.7 (Proper NeST Graph Characterization)** *$G$ is a proper NeST graph if and only if there exists a $V(G)$-tree $(T, V(G), c)$ such that*

$$xy \in E(G) \Leftrightarrow \begin{cases} d(c_x, c_y) < d(c_x, c_p), \text{ for all } p \in M_x \\ \qquad\qquad\qquad or \\ d(c_x, c_y) < d(c_y, c_q), \text{ for all } q \in M_y \end{cases} .$$

*Proof:* ($\Rightarrow$) Suppose $G$ is a proper NeST graph. By Theorem 3.3.4, let $(T, S, \mathcal{T})$ be a fixed diameter, standard NeST representation of $G$. We may assume that all neighborhood subtree intersections are nonempty by Lemma 3.3.2. Let $(T, S)$ be the corresponding tolerance-free NeST representation of $G$.

$(T, S)$ satisfies the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

which, by application of Lemma 3.3.3, can be expanded to

$$xy \in E(G) \Leftrightarrow r_x + r_y - d(c_x, c_y) > min\{r_x + r_{\dot{x}} - d(c_x, c_{\dot{x}}), r_y + r_{\dot{y}} - d(c_y, c_{\dot{y}})\}.$$

Since all neighborhood subtree diameters are equal and all neighborhood subtrees are maximal, $|T_x| = 2r$, for all $x \in V(G)$. This gives us

$$xy \in E(G) \Leftrightarrow 2r - d(c_x, c_y) > min\{2r - d(c_x, c_{\dot{x}}), 2r - d(c_y, c_{\dot{y}})\}$$

which simplifies to

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) < min\{d(c_x, c_{\dot{x}}), d(c_y, c_{\dot{y}})\}. \tag{3.1}$$

This is equivalent to

$$xy \in E(G) \Leftrightarrow \begin{cases} d(c_x, c_y) < d(c_x, c_p), \text{ for all } p \in M_x \\ \text{or} \\ d(c_x, c_y) < d(c_y, c_q), \text{ for all } q \in M_y \end{cases}.$$

Hence, let $(T, V(G), c')$ be the $V(G)$ where $c'$ is the locator defined by $c'_x = c_x$.

($\Leftarrow$) Let $(T', V(G), c')$ be a $V(G)$-tree such that (3.1) holds. We construct a tolerance-free NeST representation, $(T, S)$, by defining

- $T = (\bigcup_{x \in V(G)} L_x) \cup T'$ where, for each $x \in V(G)$, $L_x$ is a line segment of sufficient length attached to $T'$ at $c'_x$,

- $c_x = c'_x$, for all $x \in V(G)$ and

- $r_x = r$, for all $x \in V(G)$, where $r$ is sufficiently large so that $T_{ab} \neq \emptyset$, for all $a, b \in V(G)$.

We show that $(T, S)$ is a proper, tolerance-free NeST representation of $G$.

$(T, S)$ is proper since, for all $x \in V(G)$, there exists $m \in L_x$ such that $m \in T_x$ but $m \notin T_y$, for all $y \neq x \in V(G)$. $(T, S)$ being proper allows us to use Lemma 3.3.3 to expand

$$xy \in E(G) \Leftrightarrow 2r - d(c_x, c_y) > min\{2r - d(c_x, c_{\dot{x}}), 2r - d(c_y, c_{\dot{y}})\}$$

to

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}.$$

Hence, $(T, S)$ is a proper, tolerance-free NeST representation of $G$. □

The above characterization of proper NeST graphs is both tolerance-free and radius-free in the sense that it depends only upon the embedded tree and the location of vertices in the embedded tree. This motivates the following definition:

**Definition 3.3.8** *If $G$ is a graph and $(T, V(G), c)$ is a $V(G)$*-tree *which satisfies*

$$xy \in E(G) \Leftrightarrow \begin{cases} d(c_x, c_y) < d(c_x, c_p), \text{ for all } p \in M_x \\ \qquad\qquad or \\ d(c_x, c_y) < d(c_y, c_q), \text{ for all } q \in M_y \end{cases}$$

*then $(T, V(G), c)$ is called a* proper $V(G)$-tree representation *of $G$.*

## 3.4 Fixed Distance NeST Graphs

In this section we introduce our second new subclass of NeST graphs obtained by the restriction that all neighborhood subtree centers be equidistant in the NeST representation. Our main result is that this new subclass is the same as the class of threshold tolerance graphs introduced by Monma, Reed and Trotter.

**Definition 3.4.1** *A graph $G$ is a* fixed distance NeST graph *if it has a standard NeST representation, called a* fixed distance NeST representation*, in which the distance between every pair of distinct neighborhood subtree centers is the same.*

The definition of fixed distance NeST graphs differs from the other three subclasses studied here since it requires that fixed distance NeST representations be standard. For fixed diameter, proper and fixed tolerance NeST representations this is not required since the redefinition of the neighborhood subtrees in these NeST representations as standard neighborhood subtrees does not change the fact that the representation is fixed diameter, proper or fixed tolerance. However, redefining a neighborhood subtree as a standard neighborhood subtree usually requires that the neighborhood subtree center be repositioned, hence, the property of equidistant neighborhood subtree centers may be lost.

A *star* is a an acyclic graph composed of a vertex, called the *center*, which is adjacent to all other vertices in the graph, hence, stars are trees. An *embedded star*, $T$, is an embedding of a star $\mathcal{T}$. We call the point in $T$ which corresponds to the center of $\mathcal{T}$ the *center* of $T$.

**Definition 3.4.2** *A* star NeST representation *is a NeST representation* $(T, S, \mathcal{T})$ *where*

- *$T$ is an embedded star,*
- *all neighborhood subtree centers are equidistant from each other.*

*A graph is a* star NeST graph *if it has a star NeST representation.*

**Lemma 3.4.3** *A graph $G$ is a fixed distance NeST graph if and only if it is a star NeST graph.*

*Proof:* If $G$ has a star NeST representation, then all neighborhood subtree centers are equidistant, hence, $G$ is a fixed distance NeST graph.

Conversely, suppose $G$ has a fixed distance NeST representation $(T, S, \mathcal{T})$. Define the embedded tree

$$T' = \bigcup_{x,y \in V(G)} P(c_x, c_y)$$

and add line segments of sufficient length to each endpoint of $T'$. Let $(T', S', \mathcal{T})$ be the NeST representation where each $c'_x$ is located a distance $d(c_x, c)$ from the center of $T'$, where $c$ is the center of $T$. Let $r'_x = r_x$, for all $x \in V(G)$.

Obviously $T'$ is an embedded star and all neighborhood subtree centers are equidistant. It follows from $r'_x = r_x$ and $d(c'_x, c'_y) = d(c_x, c_y)$, for all $x, y \in V(G)$ and the maximality of each $T'_x$ (since each $T_x$ is maximal) that $|T'_{xy}| = |T_{xy}|$. Hence, $(T', S', \mathcal{T})$ is a star NeST representation of $G$. □

In light of the above lemma, we shall henceforth refer to both star NeST graphs and fixed distance NeST graphs as fixed distance NeST graphs.

**Definition 3.4.4** *A* radius-only representation *of a graph $G$ is a set of nonnegative numbers $R = \{r_v : v \in V(G)\}$ such that*

$$xy \in E(G) \quad \Leftrightarrow \begin{cases} r_x > r_{y^*}, \text{ for all } y^* \in M_y, \\ \qquad\quad or \\ r_y > r_{x^*}, \text{ for all } x^* \in M_x. \end{cases}$$

The following theorem is a key result of this section.

**Theorem 3.4.5** *A graph is a fixed distance NeST graph if and only if it has a radius-only representation.*

*Proof:* ($\Rightarrow$) Let $G$ have a tolerance-free, star NeST representation $(T, S)$. We will assume, without loss of generality, that all neighborhood subtree intersections are nonempty. Let $R = \{r_x : x \in V(G)\}$.

We wish to show that

$$xy \in E(G) \Leftrightarrow \begin{cases} r_x > r_{y^*}, \text{ for all } y^* \in M_y, \\ \qquad\qquad or \\ r_y > r_{x^*}, \text{ for all } x^* \in M_x, \end{cases}$$

thereby proving that $R$ is a radius-only representation of $G$.

Suppose that $xy \in E(G)$. We have $|T_{xy}| > |T_{x\dot{x}}|$ or $|T_{xy}| > |T_{y\dot{y}}|$; assume the former without loss of generality. Observe that since neighborhood subtree centers are equidistant we have that

$$|T_{ab}| > |T_{ac}| \Rightarrow r_b > r_c.$$

As a consequence, we have $r_y > r_{\dot{x}}$.

If $M_x = \emptyset$ then $r_y > r_{x^*}$, for all $x^* \in M_x$, is trivially true. Let $z \in M_x$. If $|T_{x\dot{x}}| > |T_{xz}|$ then $r_{\dot{x}} > r_z$ and, by transitivity, $r_y > r_z$. Otherwise, $|T_{x\dot{x}}| = |T_{xz}|$ and so $r_y > r_{\dot{x}}$ implies $r_y > r_z$.

Suppose that $xy \notin E(G)$. We need to show $r_x \leq r_{y^*}$, for some $y^* \in M_y$ or $r_y \leq r_{x^*}$, for some $x^* \in M_x$. But since $x \in M_y$ and $y \in M_x$ this is trivially satisfied.

($\Leftarrow$) Let $G$ have a radius-only representation $R'$. We may assume that $r'_x \neq r'_y$ for all $x \neq y$. For simplicity, suppose $V(G) = \{1, \ldots, |V(G)|\}$ and $R'$ is indexed such that

$r'_x > r'_y \Leftrightarrow x > y$. Pick some $m > 0$ and assign values to the radii $\{r_x : x \in V(G)\}$ by the formula $r_k = m + k(\frac{m}{|V(G)|})$, $k = 1, \ldots, |V(G)|$. It follows that, $r'_x > r'_y \Leftrightarrow r_x > r_y$.

Let $(\mathcal{T}, w)$ be a weighted star with $|V(G)|$ edges each of weight $3m$. Let $T$ be a weight preserving embedding of $\mathcal{T}$. Place all neighborhood subtree centers a distance $m$ from the center of $T$ such that no two neighborhood subtrees are located at the same point of $T$. Hence, $(T, S)$ is a tolerance-free star NeST representation. By our assignment of values to radii, $m < r_x \leq 2m$, for all $x \in V(G)$. It follows that, for all $x, y \in V(G)$, $T_x \nsubseteq T_y$, $T_y \nsubseteq T_x$ and $T_{xy} \neq \emptyset$. By Lemma 3.3.3, $|T_{xy}| = r_x + r_y - d(c_x, c_y) = r_x + r_y - 2m$, for all $x, y \in V(G)$. It follows that, for all $x, y, z \in V(G)$, $|T_{xz}| > |T_{yz}| \Leftrightarrow r_x > r_y$.

We show that $(T, S)$ is a tolerance-free, star NeST representation of $G$.

Suppose that $xy \in E(G)$. We have $r_x > r_{y^*}$, for all $y^* \in M_y$ or $r_y > r_{x^*}$, for all $x^* \in M_x$; assume the former without loss of generality. It follows that $|T_{xy}| > |T_{yy^*}|$, for all $y^* \in M_y$. In particular, we have $|T_{xy}| > |T_{y\dot{y}}|$.

Suppose that $xy \notin E(G)$. We have that $r_x \leq r_{y^*}$, for some $y^* \in M_y$, and $r_y \leq r_{x^*}$, for some $x^* \in M_x$. Since our radii are distinct, $r_x < r_{y^*}$, for some $y^* \in M_y$, and $r_y < r_{x^*}$, for some $x^* \in M_x$. It follows that $|T_{xy}| < |T_{yy^*}|$, for some $y^* \in M_y$, and $|T_{xy}| < |T_{xx^*}|$, for some $x^* \in M_x$. However, $|T_{x\dot{x}}| \geq |T_{xx^*}|$, for all $x^* \in M_x$, and $|T_{y\dot{y}}| \geq |T_{yy^*}|$, for all $y^* \in M_y$. We conclude that $|T_{xy}| \leq |T_{y\dot{y}}|$ and $|T_{xy}| \leq |T_{x\dot{x}}|$. $\square$

**Theorem 3.4.6** *If $G$ is a fixed distance NeST graph then $G$ is a proper NeST graph.*

*Proof:* In the proof of Theorem 3.4.5, we constructed a tolerance-free, star NeST representation of a fixed distance NeST graph where all neighborhood subtree centers were distance $2m$ apart. The neighborhood subtree radii had lower and upper bounds of $m$ and $2m$, respectively. Hence, no neighborhood subtree properly contained another neighborhood subtree. Thus the constructed tolerance-free, star NeST representation is proper. $\square$

Monma, Reed and Trotter ask whether *threshold tolerance graphs* can be characterized as intersection graphs of subtrees in a tree [16]. In response, we shall show that threshold tolerance graphs are exactly fixed distance NeST graphs.

**Definition 3.4.7** *$G$ is a* threshold tolerance graph *if there is a pair of sets of numbers, $(A, B)$, where $A = \{a_x : x \in V(G)\}$ and $B = \{b_x : x \in V(G)\}$, such that the following edge condition holds:*

$$xy \in E(G) \quad \Leftrightarrow \quad \begin{cases} a_x > b_y \\ \quad or \\ a_y > b_x. \end{cases}$$

We call $(A, B)$ a *threshold tolerance representation* of $G$.

We use a theorem due to Mike Saks (see Theorem 2.5 in Monma, Reed and Trotter [16]), which can be restated as follows:

**Theorem [Saks]** *A graph $G$ is a threshold tolerance graph if and only if there exists a total order of $V(G)$, $>$, such that*

$$xy \in E(G) \quad \Leftrightarrow \begin{cases} x > M_y \\ \quad or \\ y > M_x, \end{cases}$$

*where $x > M_y$ means $x > z$, for all $z \in M_y$, similarly for $y > M_x$.*

**Theorem 3.4.8** *$G$ is a fixed distance NeST graph if and only if $G$ is a threshold tolerance graph.*

*Proof:* By Saks' theorem, it follows that $G$ is a threshold tolerance graph if and only if $G$ admits a radius-only representation. The result follows by Theorem 3.4.5. $\square$

Monma, Reed and Trotter [16] describe a polynomial time recognition algorithm for threshold tolerance graphs. This, together with Theorem 3.4.8, implies polynomial recognition of fixed distance NeST graphs. Furthermore, the proof of Theorem 3.4.5 describes a (polynomial) algorithm by which a fixed distance NeST representation can be constructed from the radius-only representation outputted by the recognition algorithm.

## 3.5 Fixed Tolerance NeST Graphs

The final class of graphs we examine is the class of fixed tolerance NeST graphs, introduced by Bibelnieks and Dearing[2] [4]. The relationship between fixed tolerance NeST graphs and NeST graphs is analogous to the relationship between interval graphs and tolerance graphs (see Figure 1.1). The consideration of this relationship leads to a characterization of fixed tolerance NeST graphs.

**Definition 3.5.1** *A graph $G$ is a* fixed tolerance NeST graph *if there exists a NeST representation of $G$ where all tolerances are equal. Such a representation is called a* fixed tolerance NeST representation *of $G$.*

Before reaching our characterization we present a definition and two theorems.

**Definition 3.5.2** [19] *$G$ is a* neighborhood subtree graph *if there exists an embedded tree $T$ and a set, $S$, of neighborhood subtrees of $T$ indexed by $V(G)$ such that the*

[2]Bibelnieks and Dearing actually call these graphs *constant NeST graphs*. However, to maintain consistency within our subclass nomenclature, we refer to them as fixed tolerance NeST graphs.

*following edge condition holds:*

$$xy \in E(G) \;\Leftrightarrow\; T_{xy} \neq \emptyset.$$

$(T, S)$ *is called a* neighborhood subtree representation *for* $G$ *and* $G$ *the graph* associated *with* $(T, S)$.

Bibelnieks and Dearing prove the following theorem which characterizes fixed tolerance NeST graphs as those graphs with neighborhood subtree representations. Hence, this model is generalized to NeST representations by the incorporation of tolerances, just as interval representations are generalized to tolerance representations by the incorporation of tolerances.

**Theorem 3.5.3** [4] *The class of fixed tolerance NeST graphs is exactly the class of neighborhood subtree graphs.*

**Theorem 3.5.4** *If* $G$ *is a fixed tolerance NeST graph then there is a neighborhood subtree representation of* $G$ *in which all neighborhood subtree radii are equal and no neighborhood subtree is properly contained within another.*

*Proof:* Let $(T, S)$ be a neighborhood subtree representation of $G$.

We may assume that all neighborhood subtrees in $S$ are standard neighborhood subtrees.

We define a new neighborhood subtree representation $(T', S')$, where $r = max\{r_x : x \in V(G)\}$, as follows:

- $T' = (\bigcup_{x \in V(G)} L_x) \cup T$ where, for each $x \in V(G)$, $L_x$ is a line segment of sufficient length attached to $T$ at $c_x$,

- for all $x \in V(G)$, $c'_x$ is located on $L_x$ such that $d(c_x, c'_x) = r - r_x$ and
- $r_x = r$, for all $x \in V(G)$.

Observe that $T'_x \cap T = T_x$, for all $x \in V(G)$. It follows that $(T', S')$ is also a neighborhood subtree representation of $G$. Furthermore, all neighborhood subtree radii are equal. Finally, for each $x \in V(G)$, there is a point $p \in L_x$ such that $d(c'_x, p) = r$, and so, $p \in T_x - T_y$, for all $y \neq x \in V(G)$. It follows that $(T', S')$ is proper. $\square$

**Theorem 3.5.5** *$G$ is a fixed tolerance NeST graph if and only if there is a $V(G)$-tree $(T, V(G), c)$, and a positive constant $k$, such that*

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq k.$$

*Proof:* $(\Rightarrow)$ Let $G$ have a neighborhood subtree representation $(T, S)$ where all neighborhood subtree radii are equal and no neighborhood subtree is properly contained within another. Lemma 3.3.3 informs us that, for any neighborhood subtree intersection $T_{xy}$:

$$|T_{xy}| = \begin{cases} r_x + r_y - d(c_x, c_y) & \text{if } T_{xy} \neq \emptyset, \\ 0 & \text{if } T_{xy} = \emptyset. \end{cases}$$

Hence, the edge condition $xy \in E(G) \Leftrightarrow T_{xy} \neq \emptyset$ can be refined to

$$xy \in E(G) \Leftrightarrow r_x + r_y - d(c_x, c_y) \geq 0$$

and since $r_x = r$, for all $x \in V(G)$,

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq 2r.$$

Thus, we define the $V(G)$-tree $(T, V(G), c')$, where $c'_x = c_x$, for all $x \in V(G)$,

$$xy \in E(G) \Leftrightarrow d(c'_x, c'_y) \leq k$$

where $k = 2r$.

($\Leftarrow$) Let $(T, V(G), c)$ be a $V(G)$-tree, such that

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq k,$$

for some $k > 0$.

We form a neighborhood subtree representation $(T, S)$ by defining, for all $x \in V(G)$, $T_x = T(c_x, r)$ where $r = k/2$. It follows that

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq 2r$$

and so

$$xy \in E(G) \Leftrightarrow T_{xy} \neq \emptyset.$$

Hence, $(T, S)$ is a neighborhood subtree representation of $G$, and so, $G$ is a fixed tolerance NeST graph. $\square$

Theorem 3.5.5 may be restated in the following way: $G$ is a fixed tolerance NeST graph if and only if there exists an embedded tree and a placement of the vertices of $G$ in the embedded tree such that $x$ and $y$ are neighbors in $G$ if and only if they are "close" in the embedded tree.

# Chapter 4

# Class Inclusions

## 4.1 The Sunflower Graph

Bibelnieks and Dearing [4] proved that NeST graphs are weakly triangulated, and so perfect, via a result of Hayward [13]. They further asked whether this inclusion is proper. In this section we show that this is indeed the case, by proving that the graph in Figure 4.1, the *sunflower graph*, is weakly triangulated but not a NeST graph.

To prove this result we will employ the characterization of proper NeST graphs which we obtained in section 3.3. One of our motivations for studying subclasses of NeST graphs is that the study of subclasses can often participate in the resolution of important open questions concerning the superclass. In this particular case, our investigation of proper NeST graphs will help resolve the open question of whether NeST graphs are properly contained in weakly triangulated graphs or not. In this sense, we are reaping the benefits of our previous work.

**Definition 4.1.1** *A graph isomorphic to* $(\{1,2,3,4\},\{\{1,2\},\{3,4\}\})$ *is called a* $2K_2$.

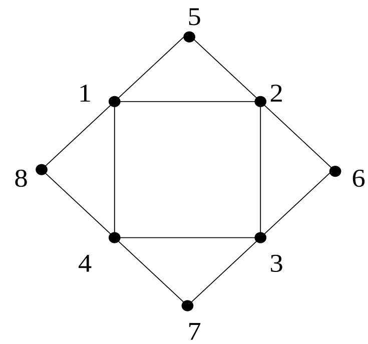



Figure 4.1: The sunflower graph.

The $2K_2$ structure of a proper NeST graph, along with our characterization of proper NeST graph's, results in the following structural property:

**Lemma 4.1.2** *Suppose $(T, V(G), c)$ is a proper $V(G)$-tree representation of $G$. If $W = \{1,2,3,4\} \subseteq V(G)$ and $G_W$ is a $2K_2$, where $12, 34 \in E(G)$, then $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$ in $T$.*

*Proof:* The following edge condition is satisfied, for all $x, y \in V(G)$:

$$xy \in E(G) \Leftrightarrow \begin{cases} d(c_x, c_y) < d(c_x, c_p), \text{ for all } p \in M_x \\ \text{or} \\ d(c_x, c_y) < d(c_y, c_q), \text{ for all } q \in M_y \end{cases}.$$

Since 12 and 34 are edges we have

$$\begin{gathered}(d(c_1, c_2) < d(c_1, c_3) \ \ \textit{and} \ \ d(c_1, c_2) < d(c_1, c_4)) \ \ \textit{or} \\ (d(c_1, c_2) < d(c_2, c_3) \ \ \textit{and} \ \ d(c_1, c_2) < d(c_2, c_4)) \end{gathered} \tag{4.1}$$

and

$$\begin{gathered}(d(c_3, c_4) < d(c_1, c_3) \ \ \textit{and} \ \ d(c_3, c_4) < d(c_2, c_3)) \ \ \textit{or} \\ (d(c_3, c_4) < d(c_1, c_4) \ \ \textit{and} \ \ d(c_3, c_4) < d(c_2, c_4)). \end{gathered} \tag{4.2}$$

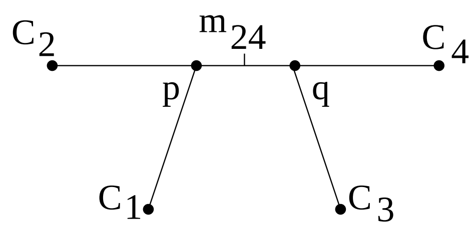



Figure 4.2: $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$

Without loss of generality, assume that

$$d(c_1, c_2) < d(c_1, c_3) \;\; and \;\; d(c_1, c_2) < d(c_1, c_4). \tag{4.3}$$

The two clauses of (4.2) are symmetric about vertices 3 and 4. Hence, without loss of generality, assume $d(c_3, c_4) < d(c_1, c_3)$ and $d(c_3, c_4) < d(c_2, c_3)$.

Let $m_{24}$ denote the midpoint of $P(c_2, c_4)$ in $T$ and $p$ be the unique intersection $P(c_1, c_2)$, $P(c_1, c_4)$ and $P(c_2, c_4)$. Similarly, let $q$ be the unique intersection of $P(c_2, c_3)$, $P(c_3, c_4)$ and $P(c_2, c_4)$. Since $d(c_1, c_2) < d(c_1, c_4)$, $p \in P(c_2, m_{24})$. Since $d(c_3, c_4) < d(c_2, c_3)$, $q \in P(c_4, m_{24})$. Furthermore, $d(p, q) > 0$. This situation is depicted in Figure 4.2. It follows that $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$ in $T$. $\square$

For brevity, we will denote $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$, in an embedded tree, by $12 \bullet 34$.

**Lemma 4.1.3** *The sunflower graph is not a proper NeST graph.*

*Proof:* Let $G$ be the sunflower graph and assume that $G$ has a proper $V(G)$ representation $(T, V(G), c)$ such that the discrete tree associated with $T$ has only degree 1 or 3 vertices (Lemma 2.3.6). Before performing a case-by-case analysis, we make two observations. Firstly, for any four vertices $1, 2, 3$ and $4$ exactly one of $12 \bullet 34$, $13 \bullet 24$ and $14 \bullet 23$ is in $T$. This is a consequence of $T$'s discrete tree having only degree 1

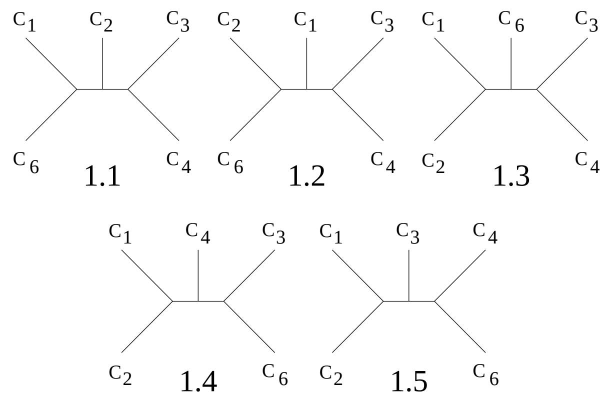



Figure 4.3: Structures of $H_1$

or 3 vertices. With respect to $1, 2, 3, 4 \in G$, the first and last of these three cases are symmetric, hence, we address the first two cases only in our analysis.

Secondly, there are eight $2K_2$'s in $G$. These are the subgraphs induced by the vertex sets $\{1,3,5,7\}$, $\{1,3,5,6\}$, $\{1,3,7,8\}$, $\{1,3,6,8\}$, $\{2,4,5,8\}$, $\{2,4,5,7\}$, $\{2,4,6,8\}$ and $\{2,4,6,7\}$. By Lemma 4.1.2, we can conclude that the following are true of $T$: $15 \bullet 37$, $15 \bullet 36$, $18 \bullet 37$, $18 \bullet 36$, $25 \bullet 48$, $25 \bullet 47$, $26 \bullet 48$ and $26 \bullet 47$.

We now begin a case-by-case analysis:

**Case 1:** Suppose that $12 \bullet 34$. Let $H_1$ be the subtree of $T$ formed by the union of all pairwise paths between the neighborhood centers associated with vertices 1, 2, 3, 4 and 6:

$$H_1 = \bigcup_{x,y \in \{1,2,3,4,6\}} P(c_x, c_y).$$

$H_1$ has five nonisomorphic structures which appear in Figure 4.3.

Structures 1.1 and 1.5, as well as 1.2 and 1.4, are symmetric with respect to $G$. Hence, we address structures 1.1, 1.2 and 1.3 only. Suppose that $H_1$ is structured as 1.1. $18 \bullet 36$ implies that $24 \bullet 68$. However, this contradicts $26 \bullet 48$. Suppose that $H_1$ is structured as 1.2. $15 \bullet 36$ and $18 \bullet 36$ imply $24 \bullet 58$. This contradicts $25 \bullet 48$.

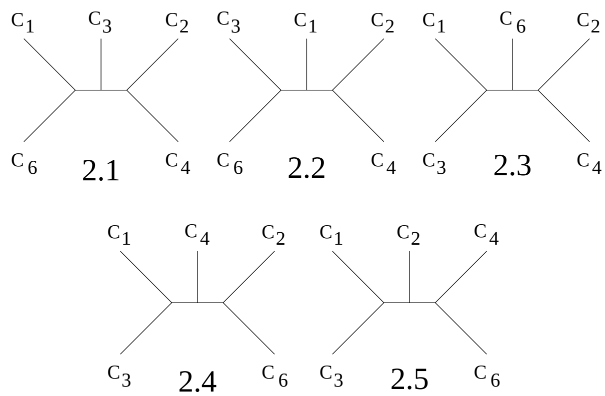



Figure 4.4: Structures of $H_2$

Finally, suppose that $H_1$ is structured as 1.3. $18 \bullet 36$ implies $28 \bullet 46$. This contradicts $26 \bullet 48$. By elimination, $12 \bullet 34$ is not possible.

**Case 2:** Suppose that $13 \bullet 24$. Let $H_2$ be the subtree of $T$ formed by the union of all pairwise paths between the neighborhood centers associated with vertices $1, 2, 3, 4$ and 6. $H_2$ has five nonisomorphic structures which appear in Figure 4.4.

Structures 2.1 and 2.5, as well as 2.2 and 2.4, are symmetric with respect to $G$. Hence we will address structures 2.1, 2.2 and 2.3 only. Suppose that $H_2$ is structured as 2.1. $18 \bullet 36$ implies that $24 \bullet 68$. However, this contradicts $26 \bullet 48$. Suppose that $H_2$ is structured as 2.2. $26 \bullet 48$ and $26 \bullet 47$ imply $13 \bullet 78$. This contradicts $18 \bullet 37$. Finally, suppose that $H_2$ is structured as 2.3. $18 \bullet 36$ implies $24 \bullet 68$. This contradicts $26 \bullet 48$. Thus, $13 \bullet 24$ is not possible.

Since both cases result in contradictions, $G$ is not a proper NeST graph. $\square$

**Lemma 4.1.4** *If $G$ is a NeST graph, but not a proper NeST graph, then there exists $x, y \in V(G)$ such that $xy \notin E(G)$ and $N_x \subseteq N_y$.*

*Proof:* Let $G$ be a NeST graph but not a proper NeST graph. Let $G$ have a tolerance-free NeST representation $(T, S)$. Define, for each $x \in V(G)$, $P_x = \{z \in V(G) :$

$T_x \subset T_z\}$. Since $(T,S)$ is not a proper NeST representation it follows that, for some $x \in V(G)$, $P_x \neq \emptyset$.

Suppose that, for all $x$ where $P_x \neq \emptyset$, we have $P_x \subseteq N_x$. Let $P_w \neq \emptyset$ and derive a new tolerance-free NeST representation $(T', S')$ for $G$ as follows:

- $T' = T \cup L$ where $L$ is sufficiently long line segment attached to $T$ at $c_w$ and
- $S' = S$, except that $c'_w$ is located on $L$ such that $d(c_w, c'_w) = \delta$ where $\delta > max\{r_z - d(c_w, c_z) : z \in P_w\}$ and $r'_w = r_w + \delta$.

Observe that for $y \notin P_w$, $|T'_{wy}| = |T_{wy}|$. For $y \in P_w$, $|T'_{wy}| \geq |T_{wy}|$ but $wy \in E(G)$ by assumption. Hence, the edge condition

$$xy \in E(G) \Leftrightarrow |T'_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

is satisfied. Furthermore, in $(T', S')$ we have $P'_w = \emptyset$. We can repeat this process, for all $w$ such that $P_w \neq \emptyset$, to achieve a proper NeST representation for $G$. This contradicts the hypothesis that $G$ is not a proper NeST graph. We conclude that there exists $x, y \in V(G)$ such that $y \in P_x$ and $xy \notin E(G)$.

Let $x, y \in V(G)$ such that $y \in P_x$ and $xy \notin E(G)$. Let $xz \in E(G)$. We have that $|T_{x\dot{x}}| = |T_{xy}| = |T_x| \geq |T_{xz}|$ which implies $|T_{xz}| > |T_{z\dot{z}}|$. Since $T_x \subset T_y$ we have $|T_{yz}| \geq |T_{xz}| > |T_{z\dot{z}}|$. Hence, $yz \in E(G)$. We conclude that $N_x \subseteq N_y$. $\square$

**Lemma 4.1.5** *The sunflower graph is not a NeST graph.*

*Proof:* Let $G$ be the sunflower graph and suppose that $G$ is a NeST graph. Lemma 4.1.3 informs us that $G$ is not a proper NeST graph. By Lemma 4.1.4, there exists $x, y \in V(G)$ such that $xy \notin E(G)$ and $N_x \subseteq N_y$. A simple inspection of $G$ reveals that this is not true. By contradiction, $G$ is not a NeST graph. $\square$

**Lemma 4.1.6** *The sunflower graph is weakly triangulated.*

*Proof:* The sunflower graph does not contain an induced cycle on five or more vertices; neither does its complement. By definition, the sunflower graph is weakly triangulated. □

**Lemma 4.1.7** *The class of NeST graphs is a proper subclass of weakly triangulated graphs.*

*Proof:* Bibelnieks and Dearing prove that NeST graphs are weakly triangulated [4]. Lemmas 4.1.5 and 4.1.6 show this inclusion is proper. □

## 4.2 Subclass, Superclass

In this section we discuss class inclusions properties of NeST graphs, both known and unknown. Figure 4.5 summarizes our current knowledge and we will refer to it often.

As shown in section 4.1, NeST graphs are a proper subclass of weakly triangulated graphs. The sunflower graph is an example of a weakly triangulated graph which is not a NeST graph. It has been conjectured that the class of proper NeST graphs are identical to the class of NeST graphs [4]. However, this conjecture has not been resolved.

Let $G_1$ be the graph in Figure 4.5 which is a fixed tolerance NeST graph but not a fixed distance NeST graph. A fixed tolerance NeST representation is easily obtained for $G_1$. In fact, it is often the case that a NeST representation (or a restricted NeST representation such as a proper NeST representation) of a particular graph is simple to obtain, if one exists. The harder task is to prove that no representation

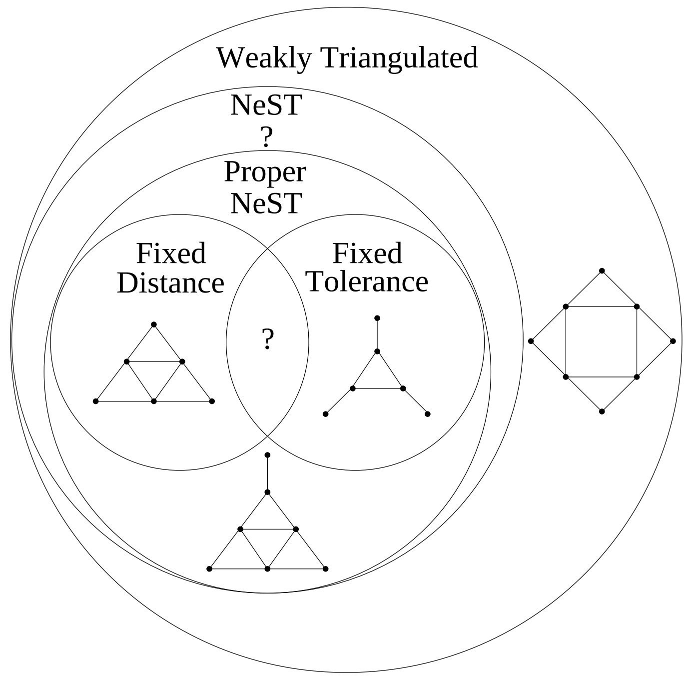



Figure 4.5: Classes of NeST graphs.

exists. Hence, here we will demonstrate nonmembership, as opposed to membership, of graphs in various NeST subclasses. To do this we will employ characterizations of NeST subclasses obtained in Chapter 3.

**Lemma 4.2.1** *$G_1$ is not a fixed distance NeST graph.*

*Proof:* Suppose $G_1$ is a fixed distance NeST graph. It follows from Theorem 3.4.5 that $G$ has a radius-only representation $R = \{r_v : v \in V(G)\}$. $R$ satisfies the following condition:

$$xy \in E(G) \;\; \Leftrightarrow \begin{cases} r_x > r_{y^*}, \text{ for all } y^* \in M_y, \\ \qquad\qquad \text{or} \\ r_y > r_{x^*}, \text{ for all } x^* \in M_x\,. \end{cases}$$

Let the vertices of $G_1$ which have degree 1 be labeled 1,2 and 3. Let the neighbors of these vertices be labeled 4,5 and 6, respectively. Since $14, 25, 36 \in E(G)$ we have the following:

$$r_1 > r_2, r_3 \quad or \quad r_4 > r_2, r_3, r_5, r_6\,,$$

$$r_2 > r_1, r_3 \quad or \quad r_5 > r_1, r_3, r_4, r_6 \text{ and}$$

$$r_3 > r_1, r_2 \quad or \quad r_6 > r_1, r_2, r_4, r_5\,.$$

If $r_1 > r_2, r_3$ then $r_5 > r_1, r_3, r_4, r_6$ and $r_6 > r_1, r_2, r_4, r_5$. But this gives a contradiction ($r_5 > r_6$ and $r_6 > r_5$). It must be that $r_4 > r_2, r_3, r_5, r_6$. It follows that $r_2 > r_1, r_3$ and $r_3 > r_1, r_2$. However, this too is a contradiction ($r_2 > r_3$ and $r_3 > r_2$).

By contradiction, $G_1$ is not a fixed distance NeST graph. □

Let $G_2$ be the graph in Figure 4.5 which is a fixed distance NeST graph but not a fixed tolerance NeST graph. We will show that $G_2$ is not a fixed distance NeST graph. To do this we require a preliminary definitions.

**Definition 4.2.2** *A graph $G$ is a* trampoline *of size $n \geq 3$ if $V(G) = C \cup W$ where $C = \{c_1, c_2, \ldots, c_n\}$ is a clique, $W = \{w_1, w_2, \ldots, w_n\}$ is a stable set and $c_i w_i, c_{i+1} w_i \in E(G)$, for $1 \leq i \leq n$ and $c_{n+1} = c_1$.*

The following lemma is due to Bibelnieks and Dearing [4].

**Lemma 4.2.3** *If $G$ is a fixed tolerance NeST graph then $G$ does not have an induced subgraph isomorphic to a trampoline.*

It follows from Lemma 4.2.3 that $G_2$ is not a fixed tolerance NeST graph since it is the trampoline of size 3. We note that Lemma 4.2.3 can be obtained directly from our characterization of fixed tolerance NeST graphs.

The set of graphs obtained by intersecting fixed tolerance and fixed distance NeST graphs has not been characterized. Recall that threshold tolerance graphs are exactly the fixed distance NeST graphs (Theorem 3.4.8). It is easily seen that by fixing the tolerances results in the class of *threshold graphs* [16]. Hence, threshold graphs are a subclass of the intersection of fixed tolerance and fixed distance NeST graphs.

Let $G_3$ be the graph in Figure 4.5 which is a proper NeST graph but neither a fixed tolerance NeST graph nor a fixed distance NeST graph. To see that $G_3$ is neither a fixed tolerance NeST graph nor a fixed distance NeST graph, observe that both $G_1$ and $G_2$ are induced subgraphs of $G_3$. It follows that $G_3$ can be neither a fixed tolerance NeST graph nor a fixed distance NeST graph.

# Chapter 5

# Conclusions and Open Problems

We will summarize our main contributions and suggest open problems.

**Defining NeST graphs**

We give a rigorous definition of NeST graphs. In particular, we give a definition of neighborhood subtree diameter which avoids the problems that the definition of diameter in Bibelnieks and Dearing [4] is subject to.

**Refining NeST representations**

The generalization of the unbounded line by an embedded tree which has endpoints results in truncated neighborhood subtrees. We suggest two resolutions to this problem: standard NeST representations and extended NeST representations. We prove that the class of graphs associated with standard NeST representations is identical to the class associated with NeST representations. Hence, standard NeST representations are as general as NeST representations but possess nicer properties. Similarly, extended NeST representations and NeST representations are identical. However, extended NeST representations are a more natural generalization of interval representations and tolerance representations than are NeST representations.

We produce a tolerance-free NeST representation which forms the foundation for the characterizations of NeST graph subclasses presented in Chapter 3. A tolerance-free NeST representation can be associated with many graphs. An open problem is to investigate the equivalence relation on NeST graphs induced by tolerance-free NeST representations.

### Subclasses of NeST graphs

We have three main results. Our first result is that unit and proper NeST graphs are identical classes. This result is consistent with interval graphs. That is, unit and proper interval graphs are also identical classes. However, the same is not true of tolerance graphs: unit and proper tolerance graphs are not identical classes.

Our second result is in response to an open problem from the literature: fixed distance NeST graphs and threshold tolerance graphs are identical classes.

Our third result is the development of tolerance-free characterizations for proper (and hence, fixed diameter) and fixed tolerance NeST graphs. We believe that these characterizations will be useful in determining if polynomial recognition algorithms exist for proper and fixed tolerance NeST graphs or proving these problems to be intractable. It is known that the realization of an arbitrary partial order on interleaf distances by a tree is NP-complete [14]. Our characterizations of the NeST subclasses examined require the realization of restricted partial orders on interleaf distances.

### Class Inclusions

Our strategy of refining and then examining subclasses of NeST graphs proved beneficial in resolving the open question of whether or not NeST graphs were a proper subclass of weakly triangulated graphs. It remains to be shown if a family of graphs which is weakly triangulated but not neighborhood subtree tolerance can be constructed.

A major open question in this area is whether proper NeST graphs are a proper subclass of NeST graphs or not. If proper NeST graphs are the same as NeST graphs then our characterization of proper NeST graphs is a strong characterization of NeST graphs.

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