

# TWIST AND TURN: THE STORY OF HEX

## THE CLASSIC 2-PLAYER CONNECTION GAME

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- 1 PRE-
- 2 EARLY
- 3 MID
- 4 RECENT

## THANK YOU

- joint: Arneson, Henderson, Toft [Rex]
- also van Rijswijck, Bjänsson, Johanson, Kan
- also UofA GAMES, Schaeffer, Müller, Stewart
- Natural Sciences and Engineering Research Council of Canada

## PREHISTORY

- 15 000 years ago last ice age ends
- 10 000 years ago agriculture  
hunt/gather → reap/sow  
stories around fire → games around hearth

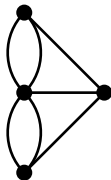
## PREHISTORY

- 5 000 years ago pre-??? Iraq
- 4 000 years ago pre-go China
- 3 400 years ago pre-checkers Egypt
- 1 500 years ago pre-chess India



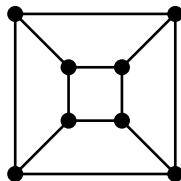
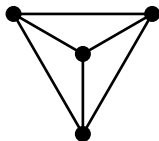
# GRAPH THEORY

1736 Euler Königsberg bridge



# GRAPH THEORY

1750 Euler polyhedron  $V - E + F = 2$



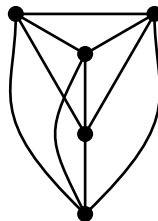
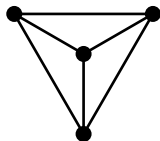
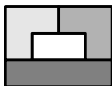




# GRAPH THEORY: 4-COLOR [TO 1940]

1840 Möbius 5 princes

$K_5$  not planar



# GRAPH THEORY: 4-COLOR [TO 1940]

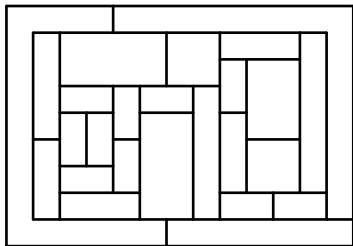
1852 Guthrie → brother → de Morgan

1879 Kempe 'proof'

1890 Heawood: Kempe counterexample, 5 color theorem

...

1940 known for maps with  $\leq 35$  regions







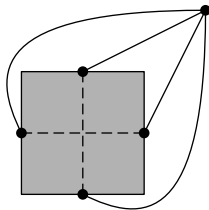
# HEIN'S INSPIRATION

## GAME CRITERIA

- fair *balanced/equal*
- progressive *no state repetition*
- final *finite*
- easy to comprehend *clear state evaluation*
- strategic
- decisive *no draw*
- no explicit math'l sol'n *not easy to solve*
  
- !!! 2 opposing-side chords of quadrilateral must cross
- no draw  $\Rightarrow$  hexagonal grid

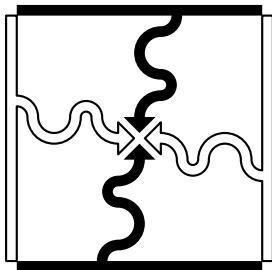


# HEIN'S INSPIRATION





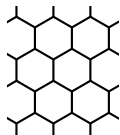
## HEIN'S INSPIRATION



OUTLINE  
PRE-  
EARLY  
MID  
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1942 HEIN GAME  
1948 NASH GAME  
1952 SHANNON COMPUTER  
1957,58 GARDNER COLUMNS

# HEIN'S INSPIRATION



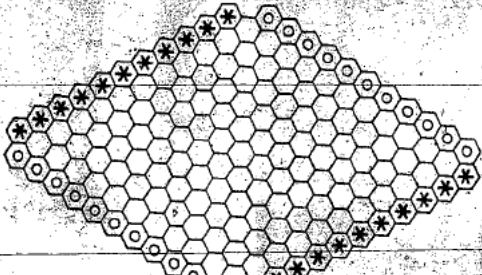
# HEIN'S COLUMNS

**TIKEN** **26. December 1942**

## Vil De lære Polygon?

Piet Hein har konstrueret et Spil, der med lige stor Glæde kan dyrkes af Skakesperten og den, der blot kan holde en Blyant

„Politiken“ udskriver i Dag en Præmieopgave, der vil volde Hovedbrud for Begyndere



ten kan afbryde Forbindelsen ved at besætte det mellemste gode Felt; dens Anvendelse beror paa Bøkkerens Placering i den videre Omegn. I det hele taget viser det sig snart nødvendigt at tage en stærk Del af Spillebrættet med Hensigt.

En anden Erfaring, som kommer senere, men som man kan lette Spillet's Begyndelse ved at røbe, er, at det betaler sig at begynde i hvert Fald nogenlunde paa Midten. En rimelig, men paa ingen Maade nødvendig Aabning af Spillet er denne:

Paa Spillebrættet i Midten er Hvid begyndt i Midterfeltet. Ses her Sort sat i Kontaktfeltet til det ned imod Midten af Hvids Front og derved gjort to nyttige Felter, som staar i Vinkelstilling til Midterfeltet, usikre. Hvid har ogsaa vejet et Felt i Kontakt med sin første. Og nu svarer Sort med at besætte et Vinkelfelt, som vilde være meget nyttigt for Hvid. Hvor skal nu Hvid sætte? Der er forskellige gode Muligheder.

Saadan er dette Spil nu begyndt. Nu kan enhver fortsætte. Der er altsaa Hvids Tur. Man skal ikke være udspekuleret fra Begyndelsen. Der er ingen bedre Vej til at lære Spillet end at spille les.



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# HEIN'S COLUMNS











# HEIN'S GRUKS

## PROBLEMS

Problems worthy  
of attack  
prove their worth  
by hitting back.

## HEIN'S GRUKS

### CONSOLATION GROOK

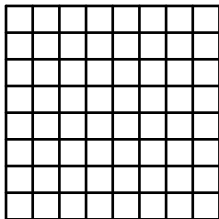
Losing one glove  
is certainly painful,  
but nothing  
    compared to the pain,  
of losing one,  
throwing away the other,  
and finding  
    the first one again.

# NASH'S GAME

## PRINCETON 1948-9

- Nash new game → Gale
- Gale board → Fine Hall

# NASH'S GAME















# NASH'S THEOREM

## N-BY-N HEX IS 1ST-PLAYER WIN

- lemma: extra X-cell ok for player X
- lemma: no draws in Hex
- suppose P2 has win strategy S2
- then P1 can move anywhere, forget move, and follow S2
- thus P1 has win strategy, contradiction  $\square$

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# NASH AT RAND

## PROJECT RAND

DOCUMENT

SOME GAMES AND MACHINES FOR PLAYING THEM

John Nash

D-1164

2 February 1952

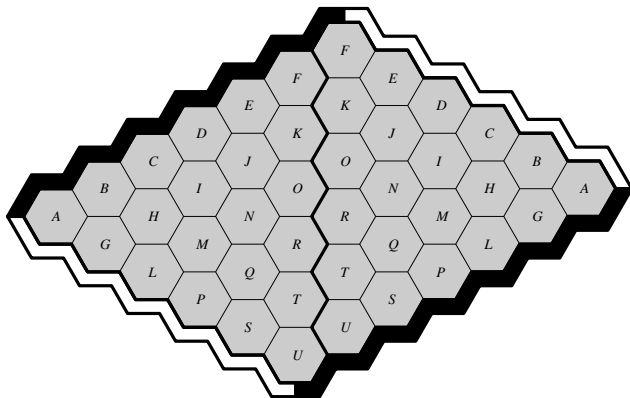
Assigned to

*Math R. Tucker*





# N-BY-(N+1) HEX IS LARGER-SIDE WIN



## SHANNON'S ANALOG COMPUTERS

### 1951 BIRD CAGE (A.K.A. GALE, BRIDG-IT) MACHINE

- bird cage: players occupy *edges*
- board  $\leftrightarrow$  electrical network
- apply black side-to-side voltage
- black edge  $\leftarrow$  short/contract edge
- white edge  $\leftarrow$  cut edge
- computer move: take edge with max voltage drop
- almost always won with first move

## SHANNON'S ANALOG COMPUTERS

### 1952 HEX MACHINE

- board  $\leftrightarrow$  2-dimensional potential field
- black cell  $\leftarrow$  positive charge
- white cell  $\leftarrow$  negative charge
- computer move: certain saddle point
- computer positionally strong, tactically weak

# SHANNON'S ANALOG COMPUTERS

## GAG MACHINE

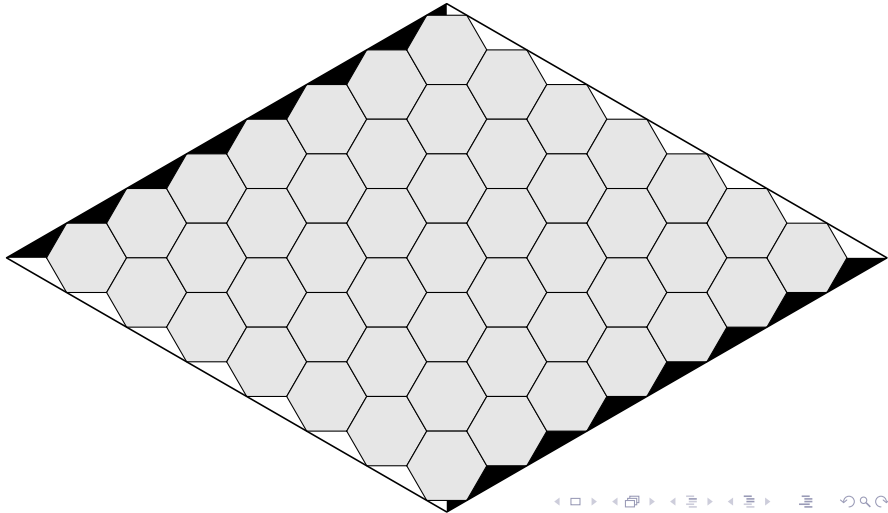
- 7x8 board disguised to look regular
- played pairing strategy, always won



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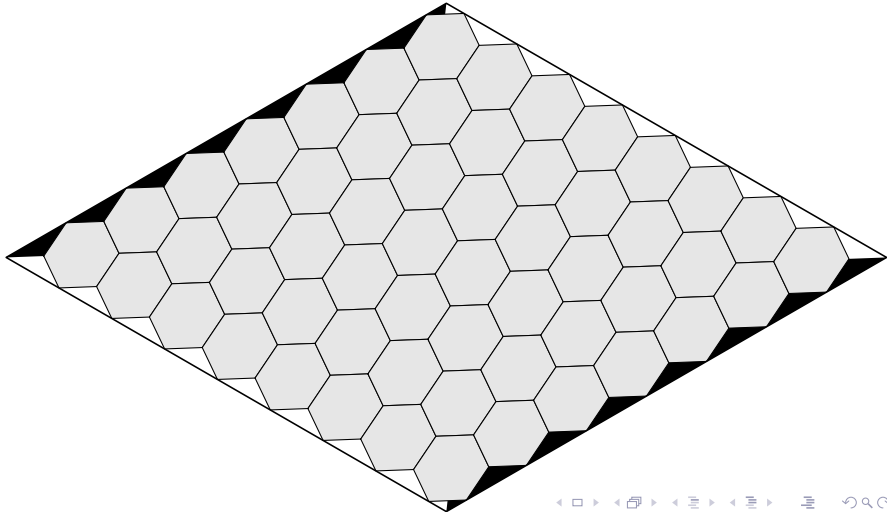
# SHANNON'S ANALOG COMPUTERS



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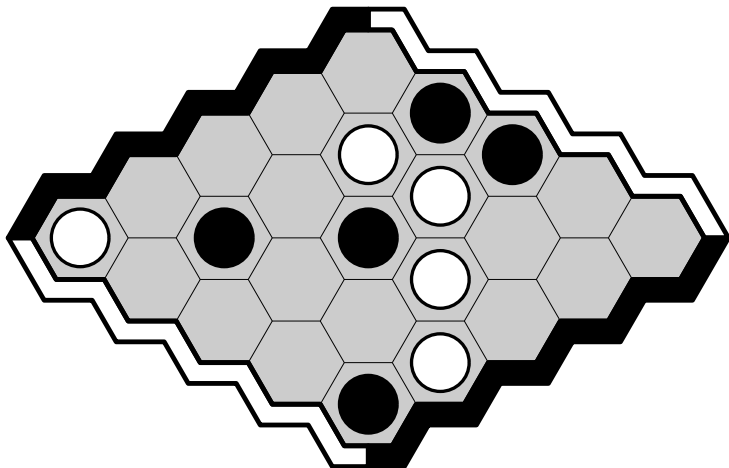
# SHANNON'S ANALOG COMPUTERS



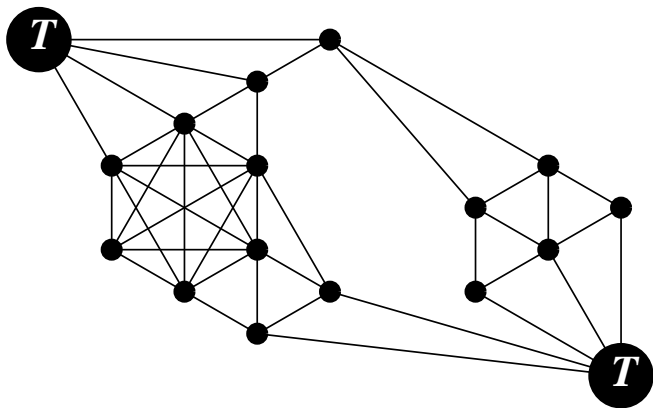
## BEYOND HEX: SHANNON SWITCHING GAME

- play on any graph
- two marked vertices
- black move: 'short' any vertex (make nbrs clique)
- white move: 'cut' any vertex (delete)
- black wins iff two marked vertices are shorted (connected)
  
- generalizes Hex

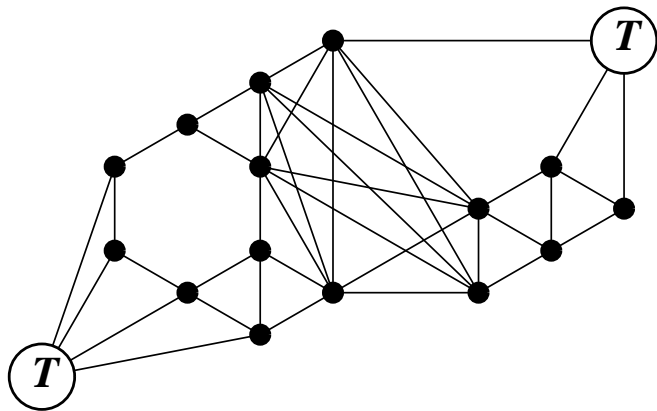
## BEYOND HEX: SHANNON SWITCHING GAME



# BEYOND HEX: SHANNON SWITCHING GAME



# BEYOND HEX: SHANNON SWITCHING GAME









## GARDNER'S COLUMNS

### 1957 JULY SCIENTIFIC AMERICAN MATHEMATICAL GAMES

- concerning the game of Hex, which may be played on the tiles of the bathroom floor

### 1958 OCT SCIENTIFIC AMERICAN MATHEMATICAL GAMES

- 4 mathematical diversions involving concepts of topology

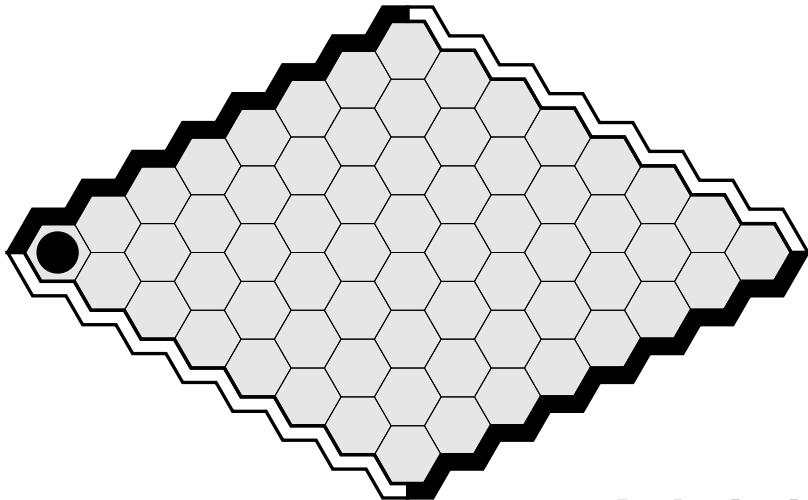
# LEHMAN'S BRIDG-IT SOLUTION

- poly-time algorithm to find winning move
- maintain 2 edge-disjoint spanning trees
- Bridg-It less fun

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**1969 BECK OPENING**  
1975 SCHENSTED & TITUS BOOK  
1975 EVEN & TARJAN PSPACE-COMPLETE  
1977 BERGE PROBLEMS  
1979 GALE NO-DRAW PROOF

## BECK OPENING: N-BY-N ACUTE CORNER LOSES



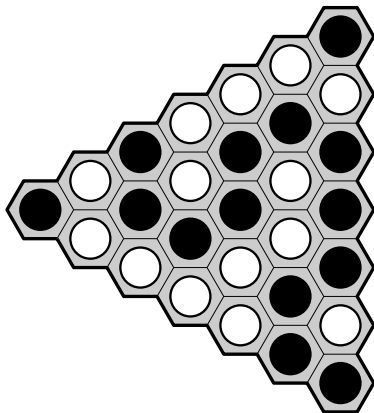
## SCHENSTED & TITUS

## MUDCRACK Y & POLY-Y

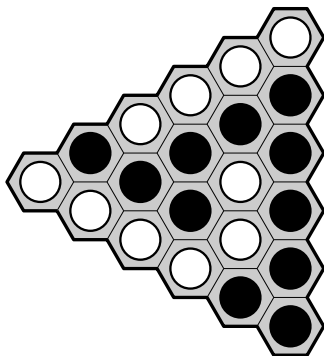
### PROVERBS

- play the best ... 1st player wins ... handicapping
- two-way stretch ... best offense is a good defense
- be relevant ... double trouble
- you can't see the whole sky through a bamboo tube
- waste not, want not ... shun the worthless triangle
- beware the square ... don't trust the pentagon
- luck is a many-sided region ... breaking mirrors is bad luck
- never try to cut a bamboo joint
- equivalent patterns ... the aim of the game

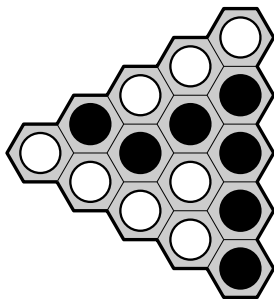
## SCHENSTED'S Y REDUCTION: NO DRAWS



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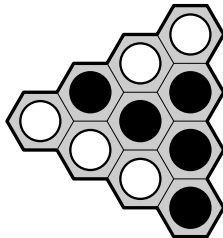
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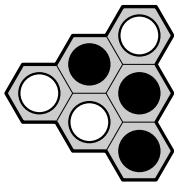
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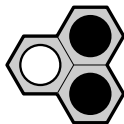
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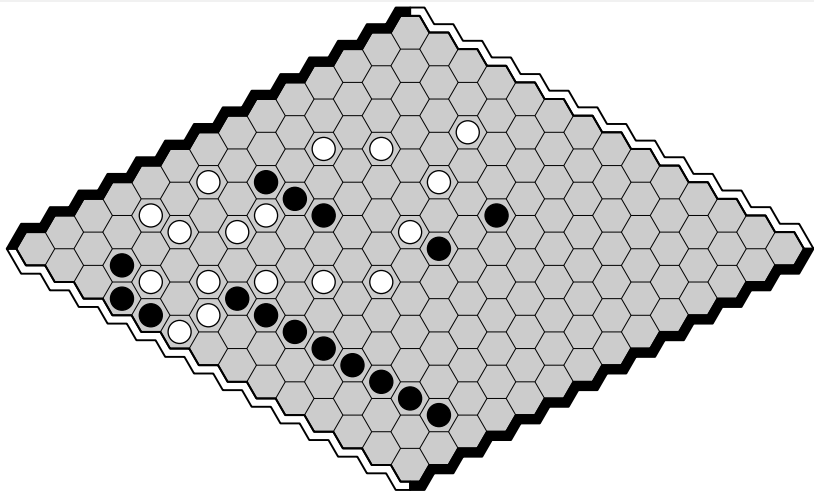
## HEX PSPACE-COMplete

- 1975 Even & Tarjan Shannon v-switching PSPACE-complete
- 1981 Stefan Reisch *Hex ist PSPACE-vollständig*
- 2000 Clay Math Inst \$1 000 000 P vs NP

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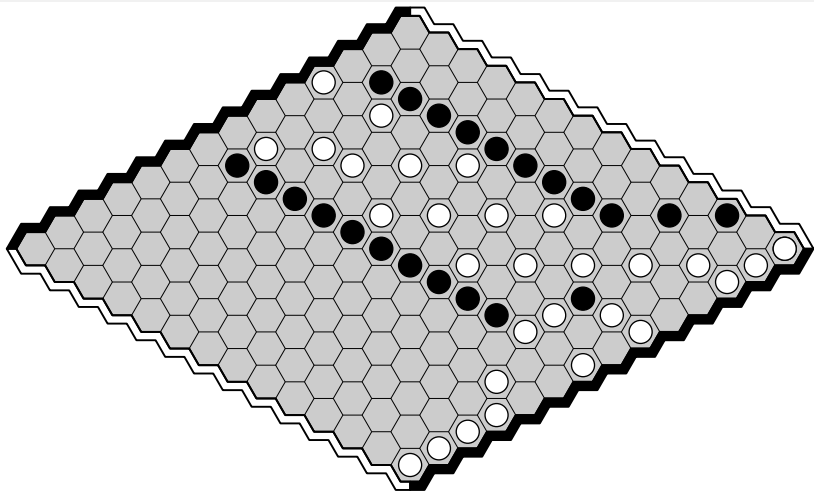
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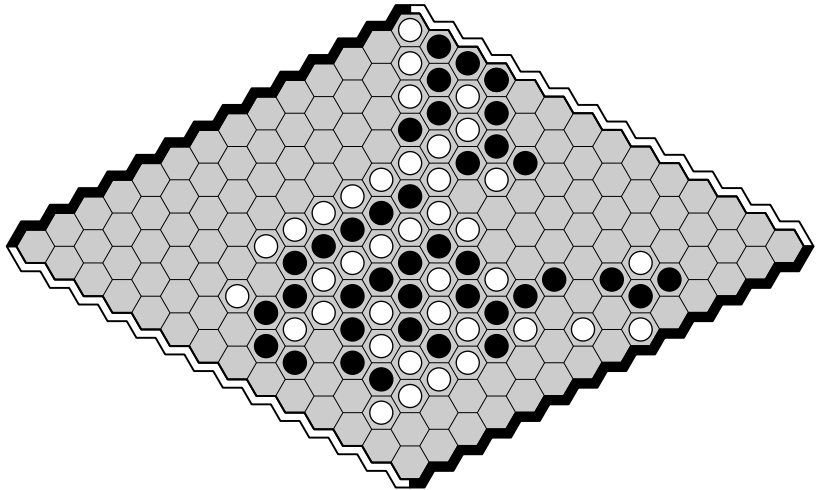
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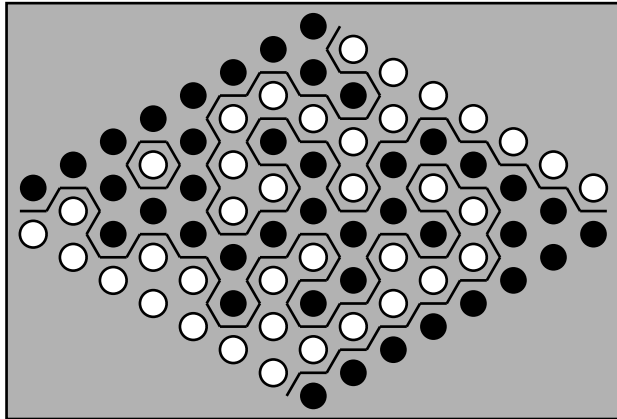
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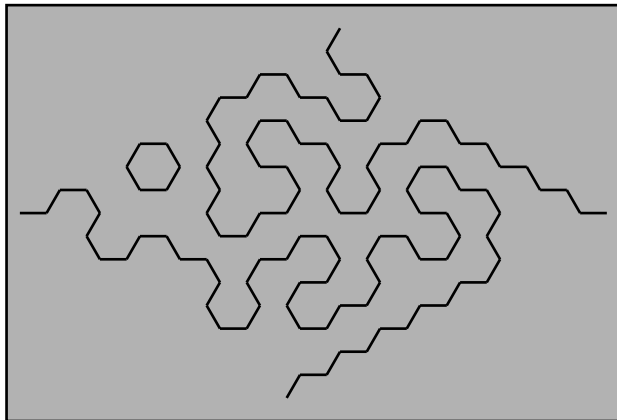




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# GALE'S NO-DRAW PROOF



# COMPUTER PLAYERS

## ICGA CGO HEX TOURNAMENTS

- 2000 London Hexy Queenbee Killerbee
- 2003 Graz Six Mongoose
- 2004 Ramat-Gan Six Mongoose
- 2006 Turin Six Wolve HexKriger
- 2008 Beijing Wolve MoHex Six Yopt
- 2009 Pamplona MoHex Wolve Six Yopt
- 2010 Kanazawa MoHex Wolve MIMHex Yopt

# COMPUTER PLAYERS

## SHANNON TO WOLVE

- Shannon bird-cage eval'n
- Hexy: + limited search, and-or virtual connection algebra
- Six: + vc lazy queue processing (+ vcs through edge)
- Wolve: + inferior cell engine, capture in vc engine, solver

## MOHEX

- same ic engine, vc engine, solver as Wolve
- Monte Carlo tree search

# SOLVING HEX OPENINGS

## HUMAN PROOFS

- |                |       |     |
|----------------|-------|-----|
| ● 2001 Yang    | 17/49 | 7x7 |
| ● 2002 Yang    |       | 8x8 |
| ● 2003 Yang    |       | 9x9 |
| ● 2004 Noshita |       | 7x7 |
| ● 2005 Noshita |       | 8x8 |
| ● 2006 Mishima |       | 8x8 |

# SOLVING HEX OPENINGS

## COMPUTER PROOFS

- |   |           |
|---|-----------|
| ● 1995 Enderton                                 | 6x6       |
| ● 2000 van Rijswijk                             | 6x6       |
| ● 2003 H Björnsson Johanson Kan Po van Rijswijk | all 7x7   |
| ● 2007 Rasmussen Maire Hayward                  | all 7x7   |
| ● 2009 Arneson H Henderson                      | all 8x8   |
| ● 2010 A H H                                    | 53/31 9x9 |

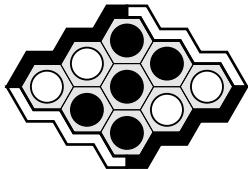
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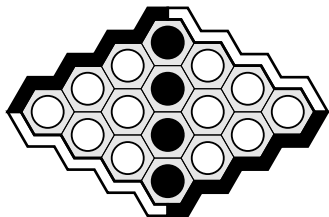


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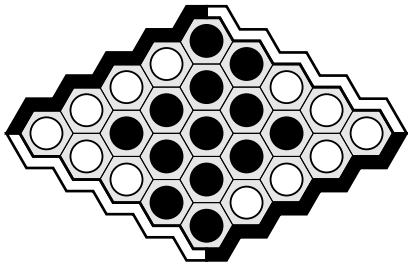




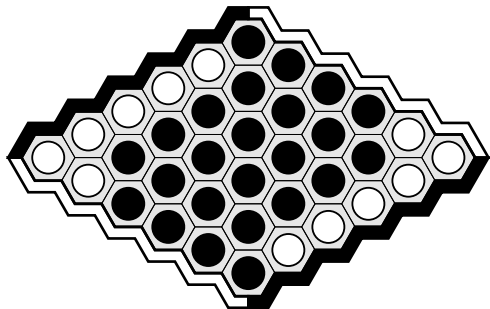
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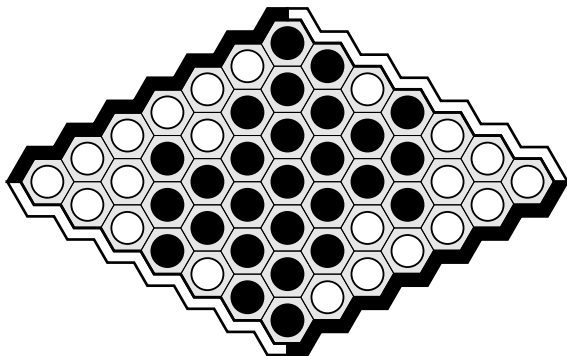
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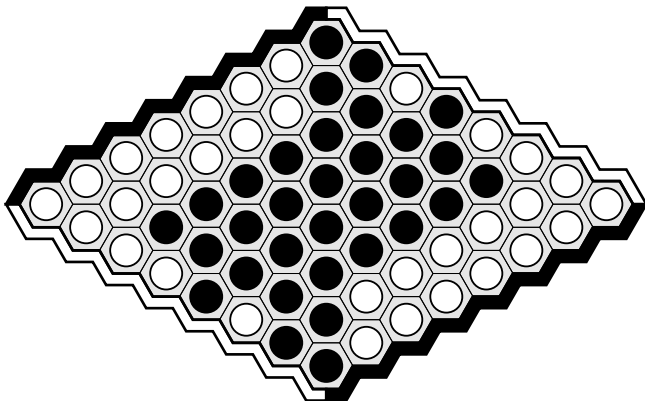
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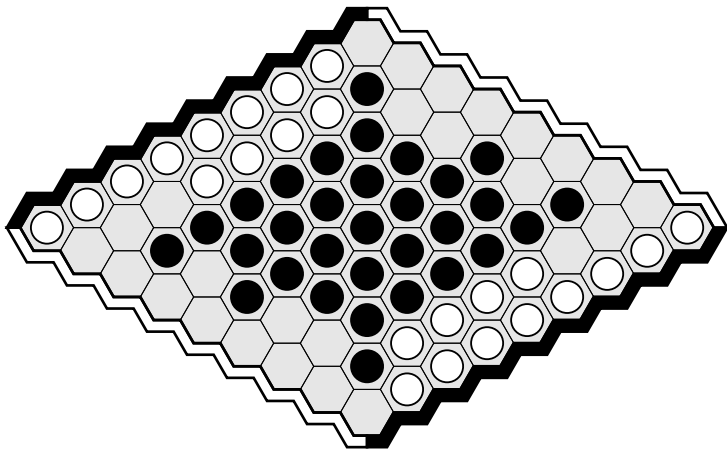
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# SOLVING HEX OPENINGS



## READING

- *Hex Strategy ...* Browne
- *Connection Games ...* Browne
- *Hexaflexagons, Probability, Paradoxes ...* Gardner
- *2nd SciAm Bk Math'l Puzzles & Diversions...* Gardner
- *Politiken* columns Hein
- *Everything You Always Wanted ... Hex ...* Maarup
- *A Beautiful Mind* Nasar
- *Mudcrack Y and Poly-Y* Schensted & Titus
- *Four Colors Suffice* Wilson