Probing the 4-3-2 Edge Template in Hex

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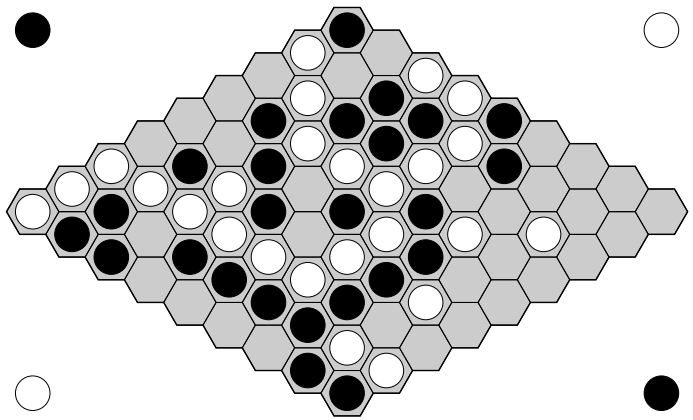
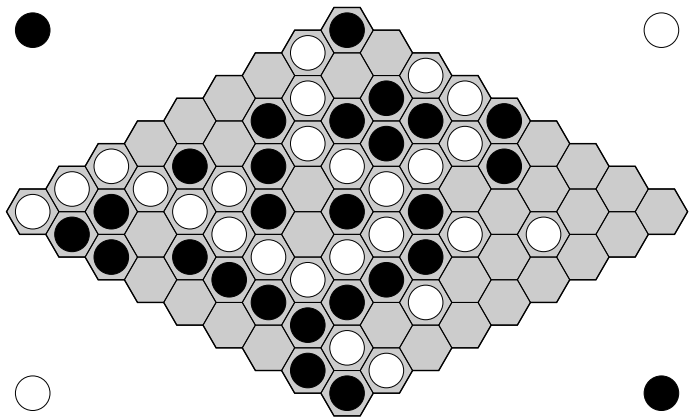
Hex Basics



Rules

- Two players alternate turns playing on any empty cell
- Stones are permanent (no moving or capturing)
- Goal is to connect your two sides of the board

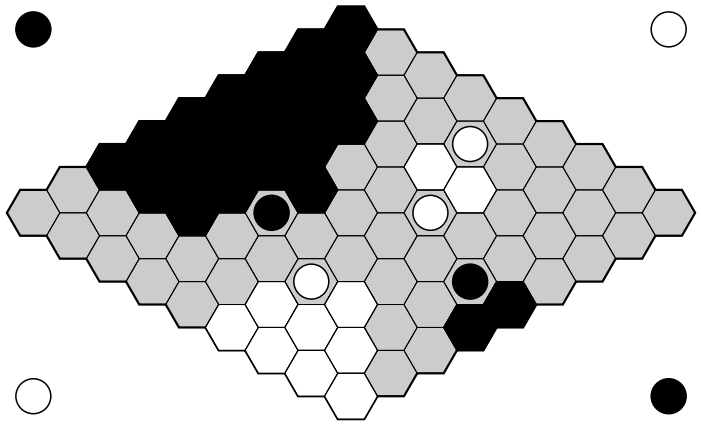
Hex Basics



Theoretical Properties

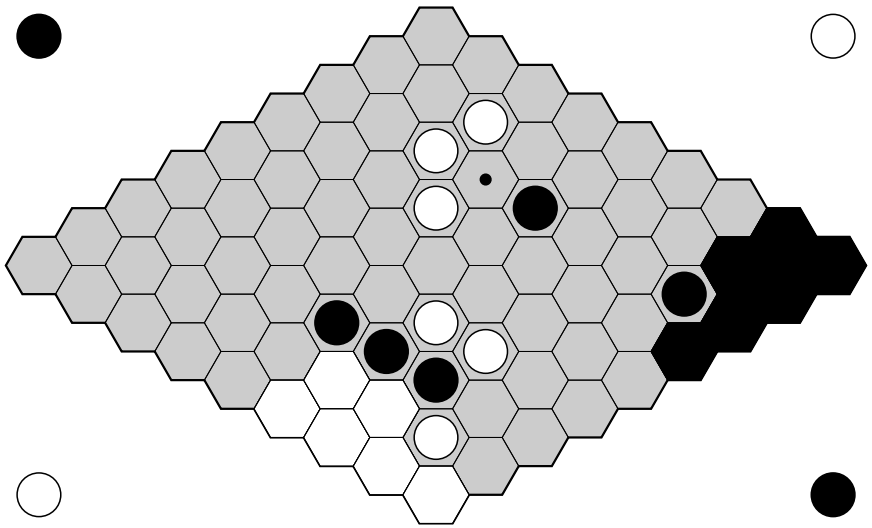
- An extra stone of your color is never a disadvantage
- First-player win: strategy-stealing argument
- PSPACE-hard to determine winner in arbitrary position

Virtual Connections



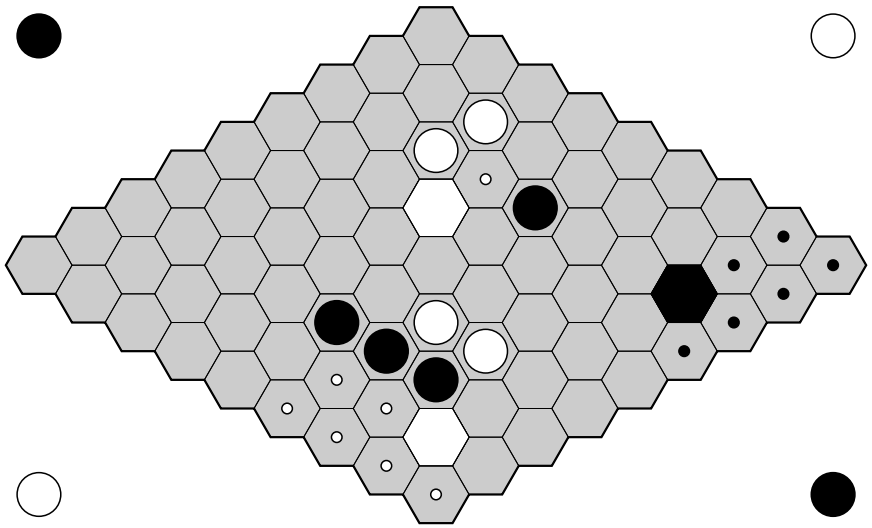
- Virtual connection (VC): a second-player strategy to connect two cells
- Carrier set: empty cells needed for a VC
- Edge template: VC where one endpoint is an edge
- Probe: a move into the carrier set of an opponent's VC

Fill-in



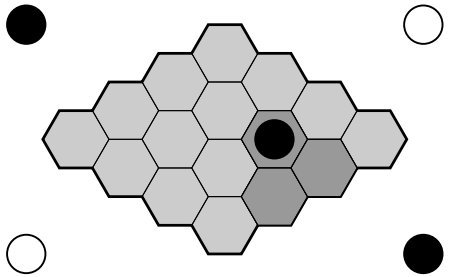
- Fill-in: stones that can be added to the board without changing its win/loss value
- Dead: fill-in that can be assigned to either color
- Captured: fill-in assigned to a particular player

Domination



- A move is capture-dominated by another if the latter gets a superset of the fill-in
- No other forms of domination in Hex were known

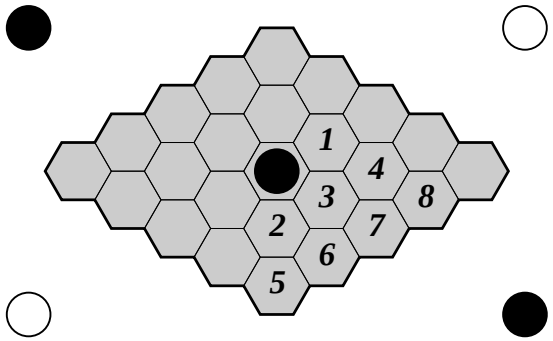
Bridge Edge Template



Bridge edge template:

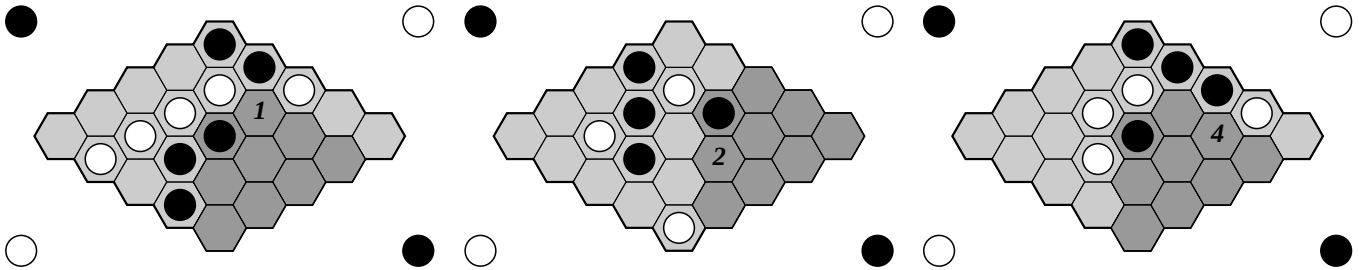
- Captures both carrier cells
- Improves the VCs that can be found
- Helps solve Hex faster

4-3-2 Edge Template



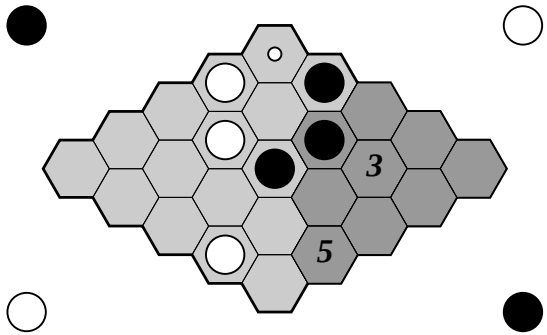
- 4-3-2 edge template is common
- Carrier not captured, but maybe some probes are inferior
- Pruning probes would improve Hex solvers and players

Unique Winning Moves



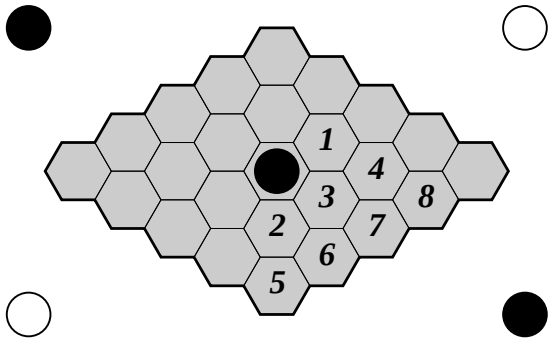
- Probes 1, 2, 4 can be unique winning move
  $\Rightarrow$ Probes 1, 2, 4 are not always inferior

Other Probes



- Probes 3 and 5 can be only winning 4-3-2 probes
- Seems contrived: merely delays win by threatening a second winning path
- Probes 6, 7, 8 never identified as unique winning moves

Conjecture



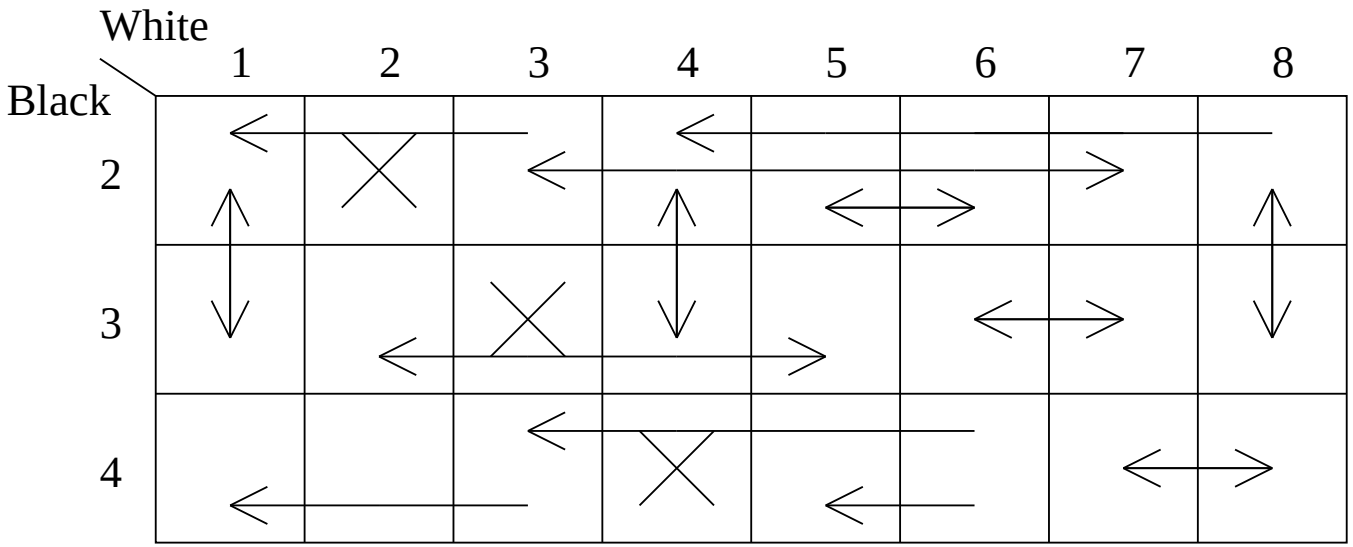
- **Conjecture:** Probes 3, 5, 6, 7, 8 are always inferior moves
- Conjecture is still open
- Conjecture holds under certain constraints

Induced Path Domination

- One cell path-dominates another if the former appears on every minimal winning path that uses the latter
- **Lemma:** Path-domination implies move domination in Hex
  - Proof Sketch: If the path-dominated move wins, then use the same strategy for the path-dominating move replacing one cell with the other. The outcome must be identical since the path-dominated cell requires the other cell to contribute to a winning path.
- Mutual domination implies equivalence

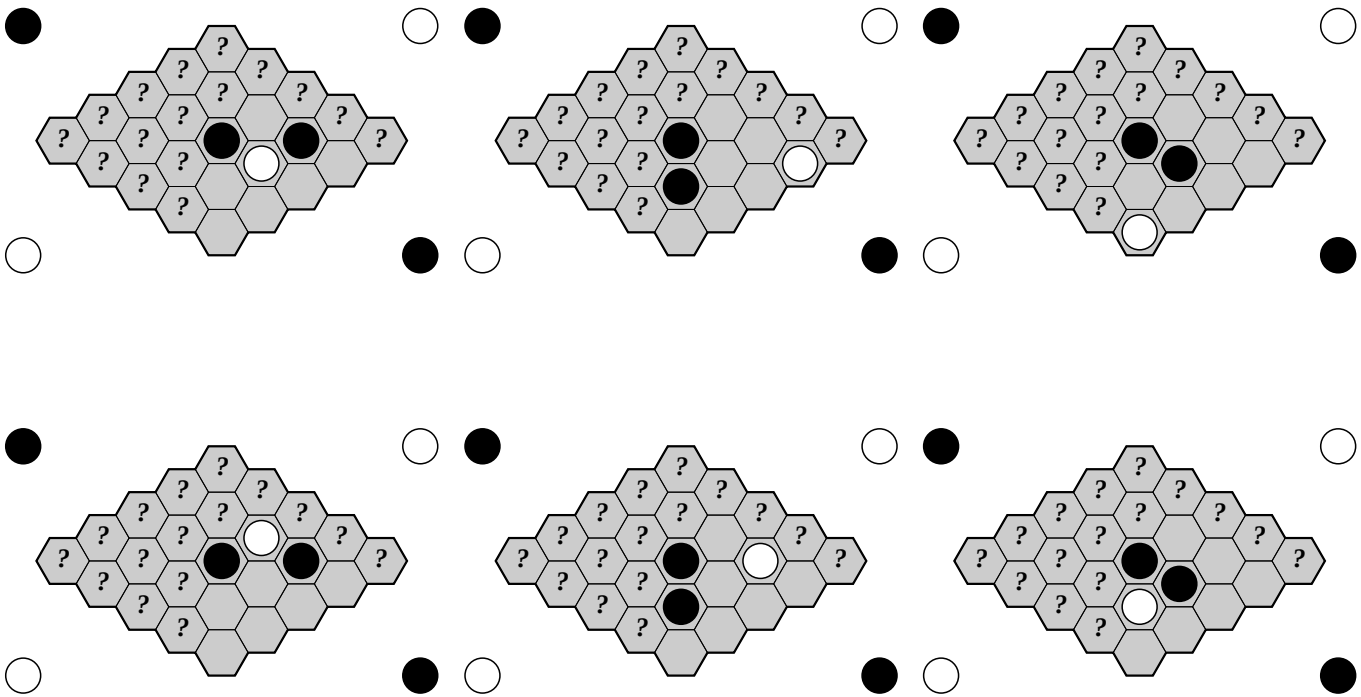
Induced Path State Domination

Induced path domination/equivalence gives many state domination/equivalence results:



Induced Path State Domination

Induced path state domination:



Conditional 4-3-2 Theorem

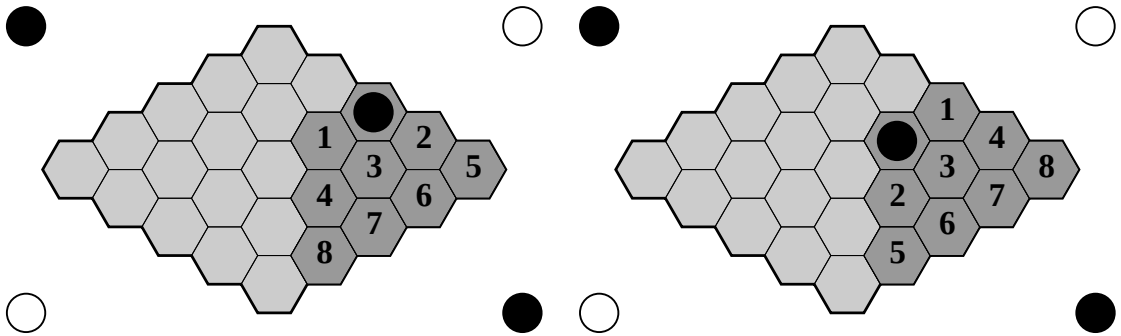
Using base cases:

- Capture-domination
- Induced path domination

we can inductively show:

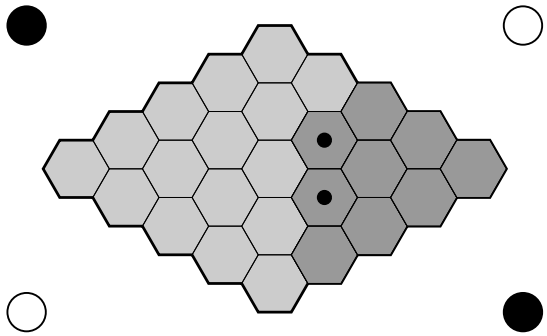
- **Theorem:** If probes 1 and 4 lose to maintenance of a 4-3-2 connection, then probes 3, 6, 7, 8 lose as well.

Acute Corner 4-3-2



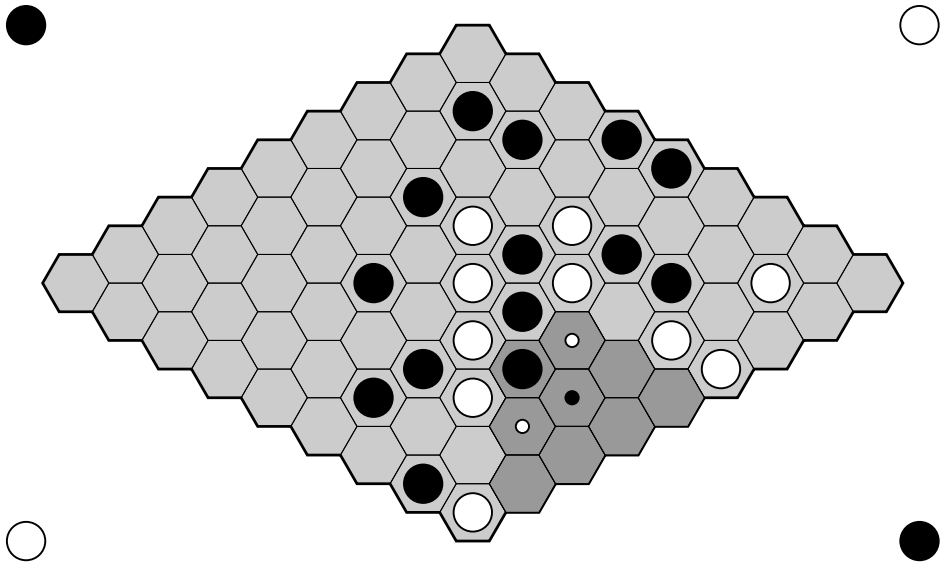
- **Lemma:** For the A3 4-3-2, probes 2,3,5 are inferior
- **Lemma:** For the B3 4-3-2, probes 1,3,4,5,6,7,8 are inferior

Acute Corner 4-3-2



- **Theorem:** In the acute corner 4-3-2 region, two cells dominate all others

Multi-Move Domination



- **Theorem:** Given a Black 4-3-2, if giving White probes 1 and 2 in exchange for Black 3 is a White loss, then probes 1, 2, 3, 5, 6, 7, 8 all lose for White

Applications

- Efficiently reduce moves to consider in both acute corners when solving or playing Hex
- Provide equivalent-value state transpositions within the solver database

Summary

- Edge templates are common, and provably pruning their probes is useful
- Induced path domination is a useful technique for proofs relating to Hex strategies
- Conditional move pruning can be done for all 4-3-2 edge templates
- Unconditional move pruning can be done for acute corner 4-3-2 edge templates

Future Work

- Weakening necessary conditions to prune 4-3-2 probes
- Resolving the (unconditional) conjecture
- Generalizing/automating this technique to other edge templates

Any Questions?

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