Weakly Triangulated Graphs

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A graph is triangulated if it has no chordless cycle with four or more vertices. It follows that the complement of a triangulated graph cannot contain a chordless cycle with five or more vertices. We introduce a class of graphs (namely, weakly triangulated graphs) which includes both triangulated graphs and complements of triangulated graphs (we define a graph as weakly triangulated if neither it nor its complement contains a chordless cycle with five or more vertices). Our main result is a structural theorem which leads to a proof that weakly triangulated graphs are perfect. © 1985 Academic Press, Inc.

Claude Berge defined a graph G to be *perfect* if, for each induced subgraph F of G, the chromatic number of F equals the largest number of pairwise adjacent vertices in F. A part of Berge's inspiration came from previous results on *triangulated* graphs, defined as graphs containing no chordless cycles with at least four vertices: Hajnal and Surányi [8] proved that complements of triangulated graphs are perfect, and Berge [1] proved that triangulated graphs are perfect. We shall call a graph *weakly triangulated* if it has no induced subgraph isomorphic to a chordless cycle with five or more vertices, or to the complement of such a cycle. It is easy to see that the chordless cycle with five vertices is isomorphic to its complement, and that the complement of every chordless cycle with at least six vertices contains a chordless cycle with four vertices; hence triangulated graphs are weakly triangulated, and complements of triangulated graphs are weakly triangulated. Our main result states that weakly triangulated graphs are perfect.

Our key tool is a lemma involving the notion of a *star-cutset*; this is a cutset C such that some vertex in C is adjacent to all the remaining vertices in C. The term *minimal imperfect graph*, used in the lemma, refers to an imperfect graph G such that every proper induced subgraph of G is perfect.

THE STAR-CUTSET LEMMA (CHVÁTAL [6]). *If G is a minimal imperfect graph then neither G nor its complement $\bar{G}$ has a star-cutset.*

Chvátal conjectured that G or $\bar{G}$ has a star-cutset whenever G is a weakly triangulated graph with at least three vertices. This conjecture will be proved as our Theorem 2. A preliminary result of independent interest is presented first.

THEOREM 1. *Let N be a minimal cutset of a weakly triangulated graph G, and let N induce a connected subgraph of $\bar{G}$. Then each connected component of $G - N$ includes at least one vertex adjacent to all the vertices of N.*

Proof of Theorem 1. We first show that

every two non-adjacent vertices in N
have a common neighbour in each component of $G - N$. (1)

For this purpose, consider arbitrary non-adjacent vertices x and y in N, and an arbitrary component A of $G - N$. Since the cutset N is minimal, each vertex in N has at least one neighbour in A; now connectedness of A implies the existence of a path from x to y with all interior vertices in A; the shortest such path P is chordless. The same argument, applied to another component B of $G - N$, shows the existence of a chordless path Q from x to y with all interior vertices in B. The two paths P and Q combine into a chordless cycle in G; since G contains no chordless cycle with five or more vertices, each of the two paths must have only one interior vertex. In particular, the interior vertex of P is a common neighbour of x and y in A, and (1) is proved.

Next, let us show that

the theorem holds whenever no two vertices in N are adjacent. (2)

To prove (2), we use induction on $|N|$. When $|N| = 1$, the conclusion follows from the fact that the cutset N is minimal. When $|N| = 2$, the conclusion is guaranteed by (1). When $|N| \geq 3$, choose distinct vertices x, y, z in N and consider an arbitrary component A of $G - N$. Note that $N - x$ is a minimal cutset of $G - x$, and that $(G - x) - (N - x) = G - N$. Hence the induction hypothesis guarantees the existence of a vertex u in A that is adjacent to all vertices in $N - x$. By the same argument, some vertex v in A is adjacent to all vertices in $N - y$, and some vertex w in A is adjacent to all the vertices in $N - z$. We will show that at least one of the vertices u, v, w is adjacent to all the vertices in N. Assuming the contrary, note that u, v, w must be distinct. Now u cannot be adjacent to v (else y, u, v, x and any common neighbour of x and y in $G - N - A$, whose existence is guaranteed by (1), would induce a chordless cycle in G); by the same argument, u cannot be adjacent to w, nor v to w. But then x, w, y, u, z, v induce a chordless cycle in G. This contradiction completes the proof of (2).

To prove the theorem in its full generality, we again use induction on $|N|$. When $|N| \leq 2$, the conclusion follows from (2). When $|N| \geq 3$, we may assume that at least two vertices in N are adjacent (else the conclusion is guaranteed by (2) again). Now we claim that N includes distinct vertices x and y such that

(i) x and y are adjacent in G, and
(ii) both $N - x$ and $N - y$ induce connected subgraphs of $\bar{G}$.

(To justify this claim, we only need choose x and y so that, in the subgraph of $\bar{G}$ induced by N, the shortest path from x to y is as long as possible.) Consider an arbitrary component A of $G - N$. By the induction hypothesis, A includes vertices u and v such that u is adjacent to all the vertices in $N - x$ and v is adjacent to all the vertices in $N - y$. We will show that at least one of the vertices u and v is adjacent to all the vertices in N. Assuming the contrary, note that u and v must be distinct. By (i), the shortest path P from x to y in the subgraph of $\bar{G}$ induced by N has at least one interior vertex. Now u and v must be adjacent: else u, v and P would induce a chordless cycle in $\bar{G}$. Next, the argument showing the existence of v in A shows also the existence of a vertex r in $G - N - A$ such that r is adjacent to all the vertices in $N - y$. If r is not adjacent to y then u, r and P induce a chordless cycle in G; else u, r, v and P induce a chordless cycle in $\bar{G}$. This contradiction completes the proof.

THEOREM 2. *If G is a weakly triangulated graph with at least three vertices then G or $\bar{G}$ has a star-cutset.*

Proof of Theorem 2. The star-cutset may be found as follows. Choose an arbitrary vertex w in G. For each vertex x other than w, put x in the set M if x is adjacent to w; else put x in the set N. If N is empty then stop: $\{u\}$ is a star-cutset in G for every vertex v in M. If M is empty then stop: $\{v\}$ is a star-cutset in $\bar{G}$ for every vertex v in N.

Now, both M and N are nonempty. If M induces a disconnected subgraph of G then stop: $\{w\} \cup N$ is a star-cutset in G. If N induces a disconnected subgraph of $\bar{G}$ then stop: $\{w\} \cup M$ is a star-cutset in $\bar{G}$.

Now, M induces a nonempty connected subgraph of G and N induces a nonempty connected subgraph of $\bar{G}$. If some vertex v in N is adjacent to no vertex in M then stop: $\{w\} \cup (N - \{v\})$ is a star-cutset in G. In the other case, each vertex in N is adjacent to at least one vertex in M; note that N is a minimal cutset in G. Now, Theorem 1 guarantees that some vertex u in M is adjacent to all the vertices in N. Stop: $\{w\} \cup (M - \{u\})$ is a star-cutset in $\bar{G}$.

COROLLARY. *All weakly triangulated graphs are perfect.*

Let P_k stand for the chordless path with k vertices. Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to have the same P_4-structure if, for some bijection $f: V_1 \to V_2$, a subset S of V_1 induces a P_4 in G_1 if and only if $f(S)$ induces a P_4 in G_2. Note that P_4 is isomorphic to its own complement, and so every graph G has the P_4-structure of $\bar{G}$. The Semi-Strong Perfect Graph Theorem, conjectured by Chvátal [4] and proved recently by Reed [10], is as follows: if a graph G has the P_4-structure of a perfect graph then G is perfect. A special case of this theorem is implied by the following result, which follows easily from Theorem 1 and a result by Chvátal [4] concerning the P_4-structures of chordless cycles: if a graph G has the P_4-structure of a triangulated graph then G is weakly triangulated.

In the rest of the paper we show how weakly triangulated graphs relate to certain other classes of perfect graphs. The class of *perfectly orderable* graphs, introduced by Chvátal [3], consists of those graphs characterized by the existence of a linear order $<$ on the set of vertices such that no chordless path with vertices a, b, c, d and edges ab, bc, cd has $a < b$ and $d < c$. A *clique* of a graph is a set of pairwise adjacent vertices. A *stable set* of a graph is a set of pairwise nonadjacent vertices. The class of perfectly orderable graphs is contained in the class of *strongly perfect* graphs, introduced by Berge and Duchet [2]. These are graphs G such that every induced subgraph H of G has a stable set meeting all maximal cliques in H. Triangulated graphs and complements of triangulated graphs are perfectly orderable, and hence strongly perfect; however, this is not true of weakly triangulated graphs. The graph in Fig. 1 is weakly triangulated but not strongly perfect.

Dirac [7] showed that every minimal cutset in a triangulated graph is a clique. Weakly triangulated graphs do not have this property. In fact, it is easy to construct weakly triangulated graphs with no clique cutset. Let G be any graph with some clique cutset C, and let S be any graph with at least two nonadjacent vertices. Let G' be the graph obtained from G by substituting the graph S for some vertex c in C. Then the cutset C' of G' corresponding to the cutset C in G is not a clique cutset. In fact, any weakly triangulated graph (with at least one clique cutset) can be transformed into a weakly triangulated graph with no clique cutset by repeatedly performing the above procedure (G' will be weakly triangulated if and only if G and S are weakly triangulated). A *homogeneous set* H in a graph G is a
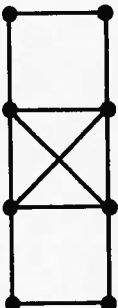

FIGURE 1

proper subset of the vertices of G, such that H has at least two vertices, and every vertex of G not in H is adjacent to either all or none of the vertices of H. Note that the above procedure for eliminating a clique cutset creates a homogeneous set (the vertices of S form a homogeneous set in G'). However, there are weakly triangulated graphs with no clique cutset, no clique cutset in the complement, and no homogeneous set. The smallest such graph appears in Fig. 2.

A vertex x is said to be *dominated* by a vertex y if every vertex z (different from x and y) that is adjacent to x is also adjacent to y. Call a graph with no dominated vertex *domination-free*. It is easy to see that if G (with at least three vertices) has a dominated vertex, then either G or $\bar{G}$ has a star cutset. We close this paper with the description of a domination-free weakly triangulated graph W. Our search for such a graph was inspired by Mahadev [9].

The set of vertices of W is the union of the sets $X = \{x_0, x_1, x_2, ..., x_{11}\}$ and $Y = \{y_0, y_1, y_2, ..., y_{11}\}$. The only edges of W with both endpoints in X are (x_{3k}, x_{3k+1}) and (x_{3k+1}, x_{3k+2}), for $k = 0, 1, 2, 3$. The only edges of $\bar{W}$ with both endpoints in Y are (y_{3k}, y_{3k+1}) and (y_{3k+1}, y_{3k+2}), for $k = 0, 1, 2, 3$. Finally, for $k = 0, 1, 2, 3$, (all indices are modulo 12)

the only edge of W between
$\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k}, x_{3k+1}, x_{3k+2}\}$ is (y_{3k}, x_{3k}),

the only edge of W between
$\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+3}, x_{3k+4}, x_{3k+5}\}$ is (y_{3k}, x_{3k+3}),

the only edge of $\bar{W}$ between
$\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+6}, x_{3k+7}, x_{3k+8}\}$ is (y_{3k}, x_{3k+7}),

the only edge of W between
$\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+9}, x_{3k+10}, x_{3k+11}\}$ is (y_{3k+1}, x_{3k+9}).
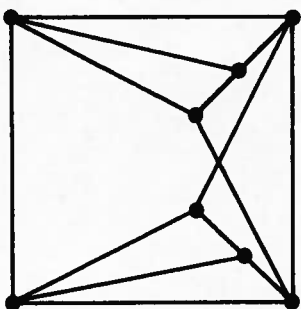

FIGURE 2

TABLE I

$a_{ij} = 1$ if and only if x_i is adjacent to y_j in W

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1 0 0 0 0 0 1 0 0 1 1 1
0 0 0 0 0 1 1 1 1 1 1 1
0 0 0 0 1 1 1 1 1 1 1 1
1 1 1 0 1 1 1 1 0 1 1 1
0 1 1 1 0 1 1 1 0 0 1 1
1 1 1 1 0 0 1 1 0 0 0 1
1 1 1 1 0 0 0 1 1 0 0 0
1 1 1 1 0 0 0 0 1 1 0 0
1 1 1 1 1 0 0 0 1 1 1 0
0 1 1 1 0 1 1 1 0 0 0 0
0 0 1 1 0 1 1 1 1 0 0 0
0 0 0 1 1 1 1 1 1 1 0 0
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Table I lists that part of the adjacency matrix of W representing edges of the form (x_i, y_j). Fig. 3 is a drawing of the subgraph of W induced by $X \cup \{y_{3k}, y_{3k+1}, y_{3k+2}\}$.

Note that W is self-complementary: the permutation P defined by $P(x_i) = y_i$ and $P(y_i) = x_{i+3}$ for $i = 0, 1, ..., 11$ sends edges of W onto edges of $\overline{W}$ and vice versa.

Since W is self-complementary, in order to prove that W is weakly triangulated it is sufficient to show that W has no chordless cycle C with at least 5 vertices. Argue by contradiction: suppose that W contains such a C. Recall that

(i) the subgraph of W induced by X consists of four disjoint P_3's,

(ii) the subgraph of $\overline{W}$ induced by Y consists of four disjoint P_3's.

x3k x3k+1 x3k+2 x3k+3 x3k+4 x3k+5 x3k+6 x3k+7 x3k+8 x3k+9 x3k+10 x3k+11
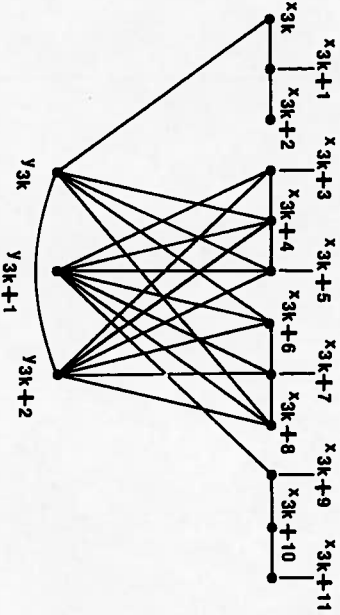
y3k   y3k+1   y3k+2

FIGURE 3

It is left to the reader to verify the following three claims:

(iii) W contains no chordless path (p_1, p_2, p_3, p_4) whose intersection with X is $\{p_2, p_3\}$;

(iv) W contains no chordless path with X is $\{p_2, p_3, p_4\}$;

(v) W contains no chordless cycle $(c_1, c_2, c_3, c_4, c_5)$ whose intersection X is $\{c_2, c_3, c_4\}$. From (v) and the fact that both W and C_5 are self-complementary, it follows that

(vi) W contains no chordless cycle $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with X is $\{c_1, c_3\}$.

Because of (i), C cannot be properly contained in X. Because of (ii), C cannot be properly contained in Y. Hence, let C_X be the subgraph of W induced by those vertices of C in X and C_Y be the subgraph of W induced by those vertices of C in Y. Both C_X and C_Y must consist of disjoint chordless paths. Because of (i), C_X contains no P_k with $k > 3$. Because of (iv) and (v), C_X contains no P_3. Because of (iii), C_X contains no P_2. Thus C_X consists of pairwise nonadjacent vertices. C_X cannot consist of a single vertex, because then C_Y would contain a P_k, with $k \geq 4$, contradicting (ii). Thus C_X consists of at least two non-adjacent vertices; hence C_Y consists of (at least two) disjoint chordless paths. But C_Y cannot contain three or more disjoint chordless paths, because then $\overline{C}_Y$ would contain a triangle, contradicting (ii). Thus C_Y consists of exactly two disjoint paths; now (ii) implies that one of these paths is an isolated vertex, and the other has two vertices (each subgraph of W induced by at least four vertices in Y is connected). But then the cycle would have to consist of exactly five vertices (c_1, c_2, c_3, c_4, c_5) whose intersection with Y is $\{c_2, c_4, c_5\}$, contradicting (vi). Thus, W is weakly triangulated.

To verify that W is domination-free, assume the contrary: some vertex u is dominated by a vertex v. First, consider the case when u is in X. By symmetry, we may assume that $u = x_i$ with $0 \leq i \leq 2$. To see that v cannot be in Y, consult Table II.

TABLE II

Neighbours of x_i Nonadjacent to y_j in W

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
x_0	x_1	x_1	x_1	x_1	x_1	y_8	y_8	y_7	y_7	y_{10}	y_{11}	y_{10}
x_1	x_2	x_2	x_2	x_2	x_1	y_8	y_7	y_8	y_7	y_{10}	y_{11}	y_{10}
x_2	x_1	x_1	x_1	x_1	x_1	y_7	y_8	y_7	y_{10}	y_{11}	y_{11}	y_{10}

Thus we must have $v = x_j$ for some j; consider the subgraph of W induced by X, we conclude easily that $0 \le j \le 2$. But now we only need observe that

y_0 is adjacent to x_0 and nonadjacent to x_1, x_2,

y_9 is adjacent to x_1, x_2 and nonadjacent to x_0,

y_6 is adjacent to x_2 and nonadjacent to x_1,

x_0 is adjacent to x_1 and nonadjacent to x_2.

Thus u cannot be in X.

Next, consider the case when u is in Y. By symmetry, we may assume that $u = y$, with $0 \le i \le 2$. To see that v cannot be in X, observe that u is adjacent to both x_4 and x_8, at least one of which is nonadjacent to v. The only remaining subcase, with u and v both in Y, is reduced to a previous subcase by considering the permutation P that sends W onto its complement: clearly, $P(v)$ is dominated by $P(u)$, and both $P(u)$ and $P(v)$ are in X. Thus W is domination-free.

Incidentally, W has neither a clique cutset nor a homogeneous set; neither is W strongly perfect. This is left to the reader to verify. (Using the algorithm due to Whitesides [11], it is easy to check that W has no clique cutset. Verifying that W has no homogeneous set seems to be a rather tedious task. To show that W is not strongly perfect, it suffices to show that the subgraph induced by $\{x_0, x_1, x_2, x_6, x_7, x_8, y_0, y_1, y_6, y_7\}$ is not strongly perfect.)

ACKNOWLEDGMENTS

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