On Bull-free, Weakly Triangulated, and Perfectly Orderable Graphs*

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Abstract. In support of a recent conjecture by de Figueiredo, Maffray, and Porto, we show that certain bull-free weakly triangulated graphs are perfectly orderable.

Key Words. bull-free graph, weakly triangulated graph, perfectly orderable graph, box partition

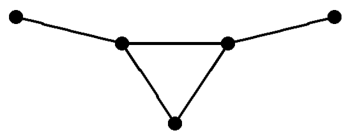

FIG. 1. *The bull.*

1. Introduction. The *bull* is the graph shown in Figure 1; a *hole* is an induced cycle with at least five vertices. Chvátal posed the following in 1989 [C2].

CONJECTURE 1.1. *Every bull-free graph with no odd hole and no anti-hole is perfectly orderable.*

Recently de Figueiredo, Maffray, and Porto made substantial progress on this conjecture, reducing it to the following conjecture [FMP] [FMP2]. (The broom is the graph shown in Figure 2 and the anti-broom its complement; P_k is the induced path with k vertices; box partition is defined later.)

CONJECTURE 1.2. *Every bull-free, P_6-free, broom-free, anti-broom-free weakly triangulated graph whose complement has a box partition is perfectly orderable.*

In this paper we show that Conjecture 1.2 holds in the case where G does not contain a certain graph with eight vertices.

2. Box partitions. A key idea in the work of de Figueiredo, Maffray, and Porto is the notion of a *box partition*, namely a partition of the vertex set V of a graph G into non-empty sets $\{B^0_1, \ldots, B^0_r, B^1_1, \ldots B^1_s\}$ called *boxes*, such that

- each box B^t_j induces a connected subgraph of G,
- for $t = 0, 1$, for $j \neq k$, B^t_j is B^t_k-null,

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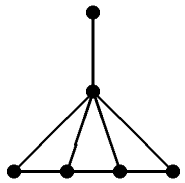

FIG. 2. *The broom.*

- for each j and t such that the set M_j^t of vertices partial on B_j^t is non-empty, there are vertices w and z of $V - B_j^t - M_j^t$ such that w is B_j^t-universal and M_j^t-null and z is B_j^t-null and M_j^t-universal.

3. A doubly lexical ordering. Let $\mathcal{W}$ be the set of graphs which are weakly triangulated, bull-free, P_6-free, broom-free, anti-broom-free and admit a box partition. Let $\overline{\mathcal{W}}$ be the set of complements of graphs in $\mathcal{W}$.

LEMMA 3.1. *Let*

- *G be a graph in $\overline{\mathcal{W}}$,*
- *$\{B_1^0, \ldots, B_s^0, B_1^1, \ldots, B_t^1\}$ be a box partition of G,*
- *G^* be the bipartite graph obtained from G by removing all edges with both ends in B^0 or B^1,*
- *$(v_1^0, \ldots, v_\alpha^0)$ and $(v_1^1, \ldots, v_\beta^1)$ be labellings of vertices of B^0 and B^1 respectively, such that the associated $B^0 \times B^1$ adjacency matrix of G^* is doubly lexically ordered.*

Then the following hold, for $t = 0$ and 1.

1. *$G[B^t]$ is P_4-free.*
2. *If there are vertices v_j^t, v_k^t, v_p^{1-t} such that $j < k$ and v_p^{1-t} sees v_j^t and but not v_k^t, then v_j^t and v_k^t are in different components of $G[B^t]$, and there is some vertex v_q^{1-t} such that $p < q$, v_p^{1-t} and v_q^{1-t} are in different components of $G[B^{1-t}]$, and v_q^{1-t} misses v_j^t and sees v_k^t.*
3. *There are no vertices $v_j^t, v_k^t, v_p^{1-t}, v_q^{1-t}$ such that $j < k$, $p < q$, v_j^t sees v_p^{1-t} and v_q^{1-t}, and v_k^t sees v_p^{1-t} and misses v_q^{1-t}.*
4. *There are no vertices $v_j^t, v_k^t, v_p^{1-t}, v_q^{1-t}$ such that $j < k$, $p < q$, v_j^t sees v_k^t, v_p^{1-t} sees v_q^{1-t}, v_j^t misses v_p^{1-t} and sees v_q^{1-t}, and v_k^t sees both v_p^{1-t} but not v_q^{1-t},*
5. *If $(v_a^t, v_b^t, v_{x'}^{1-t}, v_c^t)$ and $(v_{a'}^{1-t}, v_{b'}^{1-t}, v_x^t, v_{c'}^{1-t})$ each induce a P_4 in G, and $a < b < c$ and $a' < b' < c'$, then*
   - *(a) if $x = a$, then $v_{x'}^{1-t} \neq v_{a'}^{1-t}$, and $v_{a'}^{1-t} < v_{x'}^{1-t}$;*
   - *(b) if $x < a$ and $x' < a'$, then the graph $G[\{v_x^t, v_a^t, v_b^t, v_c^t, v_{x'}^{1-t}, v_{a'}^{1-t}, v_{b'}^{1-t}, v_{c'}^{1-t}\}]$ is as shown in Figure 4.*

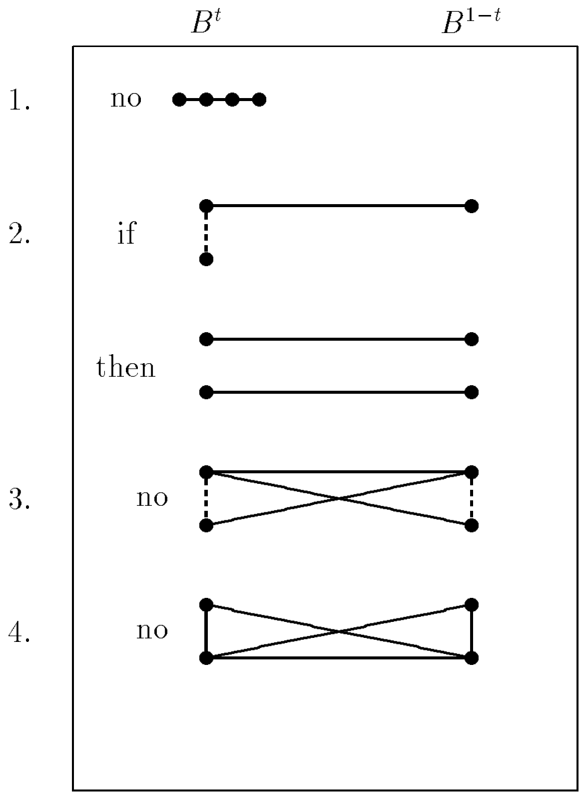

FIG. 3. *Lemma illustration.*

Proof. Let $\mathcal{F}$ be the class of graphs which are weakly triangulated, bull-free, broom-free, anti-broom-free and admit a box partition. In [FMP] it is shown that every box in a box partition of a graph in $\mathcal{F}$ is P_4-free. Now 1 follows from the fact that $\mathcal{W}$ is a subset of $\mathcal{F}$.

Consider case 2. Let v_j^t, v_k^t, v_p^{1-t} be as described. In [FMP] it is shown that for any two vertices in the same box of a box partition of a graph in $\mathcal{F}$, the neighborhoods in the rest of the graph (that is, excluding the box containing them) of the two vertices are comparable (that is, one is a subset of the other). Since the labelling of vertices described in the lemma is doubly lexical, the adjacency vector of v_j^t (in G^*) is lexically less than the adjacency vector of v_k^t, which with the preceding implies that the neighborhood of v_j^t in B^{1-t} must be a subset of the neighborhood of v_k^t in B^{1-t}. It follows that v_j^t and v_k^t are in different boxes of $[B^t]$, that is, in different components of $G[B^t]$. Now the existence of a vertex v_q^{1-t} with $p < q$ that misses v_j^t and sees v_k^t follows from the fact that v_j^t is lexically less than v_k^t. Arguing as with v_j^t and v_k^t, it follows that v_p^{1-t} and v_q^{1-t} are in different components of $G[B^{1-t}]$. Thus 2 holds.

To see 3, first observe that the graph G^* has no hole. (Any such hole has the

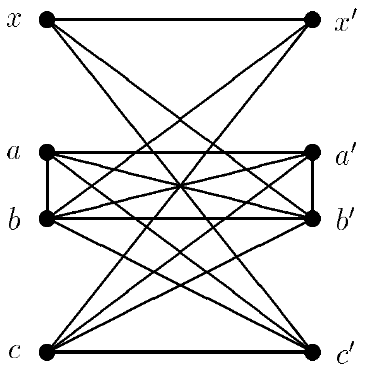

FIG. 4.

form $(x_1, y_1, \ldots, x_z, y_z)$, where $x_1, \ldots, x_z$ is a subset of B^0 and $y_1, \ldots, y_z$ is a subset of B^1. Since $z \geq 3$, no two vertices of X have comparable neighborhoods in B^1, so by 2 no two vertices of X are in the same box of B^0. Similarly, no two vertices of Y are in the same box of B^1. Thus each pair of vertices with both ends in X or Y are adjacent in G, and the hole is an induced cycle of G, a contradiction since G is weakly triangulated.) Thus G^* is a hole-free bipartite graph (also called chordal bipartite graph), and so the doubly lexical ordering of G^* is such that the adjaceny matrix does not include $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ as an induced submatrix (see for example [AF] or [L]). Thus 3 holds.

Argue 4 by contradiction; assume $v_j^t, v_k^t, v_p^{1-t}, v_q^{1-t}$ are as described. Vertices v_j^t and v_k^t are adjacent and so in the same box, say B_f^t. Observe that v_p^{1-t} is partial on B_f^t. From the definition of box partition (and the fact that (B^0, B^1) is a box partition of G), there are vertices w and z such that w sees z, w sees v_j^t and v_k^t and misses v_p^{1-t}, and z misses v_j^t and v_k^t and sees v_p^{1-t}. Now these vertices induce either a graph with a hole (if w misses v_q^{1-t}) or $\overline{P}_6$ (if w sees v_q^{1-t}), contradiction. Thus 4 holds.

Consider 5(a). Let $v_x, v_a, v_b, v_c, v_{x'}, v_{a'}, v_{b'}, v_{c'}$ correspond to the associated superscripted vertices, and suppose that $v_x = v_a$. Then v_b misses $v_{a'}$ (otherwise, if v_b misses $v_{b'}$ then 2 is contradicted by $v_{a'}, v_{b'}, v_b$; if v_b sees $v_{b'}$ then 3 is contradicted by $v_{a'}, v_{b'}, v_a, v_b$), so $v_{x'} \neq v_{a'}$. Also, $v_{b'}$ sees v_b (use 2 with $v_{b'}, v_a, v_b$), so $v_{x'}$ sees at least one of $v_{a'}$ or $v_{b'}$ (otherwise $\{v_{a'}, v_{b'}, v_{x'}, v_a, v_b\}$ induces a bull), so $v_{x'}$ is in the box of B^{1-t} containing $v_{a'}$ and $v_{b'}$, and $a' < x'$ (use 2 with $v_{x'}, v_{a'}, v_b$). Thus 5(a) holds.

Consider 5(b). We begin by proving that

$$v_x^t \text{ is in a different component of } G[B^t] \text{ from } v_a^t \text{ and } v_b^t. \qquad (*)$$

Assume the contrary. Using 2, it follows that v_x misses $v_{x'}$, and that v_a and v_b see $v_{b'}$ and $v_{c'}$. Then v_a misses $v_{a'}$. (Suppose the contrary, namely that v_a see $v_{a'}$. Using 2 with $v_a, v_b, v_{a'}$ it follows that $v_{a'}$ sees v_b. Using 4 with $v_x, v_a, v_{a'}, v_{b'}$ it follows that v_x misses v_a; replacing v_a with v_b in this argument, it follows that v_x misses v_b. Now $v_{x'}$ misses $v_{a'}$, for otherwise $\{v_{x'}, v_{a'}, v_x, v_a, v_b\}$ induce a bull. Now $v_{x'}$ misses $v_{b'}$, for otherwise $\{v_{x'}, v_{a'}, v_{b'}, v_x, v_b\}$ induce a bull. Now $v_{x'}$ misses $v_{c'}$, for otherwise $\{v_{x'}, v_{a'}, v_{c'}, v_x, v_b\}$ induce a bull. Now $\{v_{x'}, v_{a'}, v_{b'}, v_{c'}, v_a\}$ induces a bull, contradiction.)

Then v_b misses $v_{a'}$ (by 4 with $v_a, v_b, v_{a'}, v_{b'}$). By repeating the argument we have used to this point with v_x, v_a, v_b, v_c interchanged with $v_{x'}, v_{a'}, v_{b'}, v_{c'}$, it follows that $v_{x'}$ is in a separate box from $v_{a'}$ and $v_{b'}$ (assuming the contrary we would conclude that v_a misses $v_{b'}$, which we already know not to be the case). Thus $v_{x'}$ misses $v_{a'}$ and $v_{b'}$. But then $\{v_{x'}, v_{a'}, v_{b'}, v_a, v_b\}$ induces a bull, contradiction. Thus $(*)$ holds, and in particular

$$v_x \ \text{ misses } \ v_a, v_b \ \text{ and } \ v_{x'} \ \text{ misses } \ v_{a'}, v_{b'} \ .$$

Now we claim that

$$v_b \ \text{ sees } \ v_{b'} \ .$$

Argue by contradiction: suppose v_b misses $v_{b'}$. Then

- v_b misses $v_{a'}$ (by 2 with $v_{b'}$),
- $v_{b'}$ misses v_a (by 2 with v_b),
- v_a misses $v_{a'}$ (by 2 with $v_{b'}$,
- v_x misses $v_{x'}$ (by 3 with $v_b, v_{b'}$),
- v_c misses v_x (assuming the contrary, then v_c sees $v_{b'}$ (by 2 with v_x), v_c misses $v_{a'}$ (by 4 with $v_x, v_{b'}$, and $\{v_x, v_c, v_{x'}, v_{a'}, v_{b'}\}$ induces a bull, contradiction),
- $v_{c'}$ misses $v_{x'}$ (symmetric to above),
- v_c misses $v_{a'}$ (assuming the contrary, then v_c sees $v_{b'}$ (by 2 with $v_{a'}$), and $\{v_x, v_c, v_{x'}, v_{a'}, v_{b'}\}$ induces a bull, contradiction),
- $v_{c'}$ misses v_a (symmetric to above),
- some v_d with $c < d$ sees $v_{a'}$ and misses $v_c, v_{x'}$ (by 2),
- v_d sees $v_{b'}$ (by 2 with $v_{a'}$),
- v_d misses v_x (by 4 with $v_{a'}, v_{b'}$),
- v_d sees $v_{c'}$ (by 3 with $v_x, v_{b'}$),
- v_d misses v_a and v_b (by 2 with $v_{x'}$),
- some $v_{d'}$ with $c' < d'$ sees v_a and misses $v_{c'}, v_x$ (by 2),
- $v_{d'}$ sees v_b, v_c and misses $v_{x'}, v_{a'}, v_{b'}$ (symmetric to above),
- v_d misses $v_{d'}$ (else $\{v_a, v_b, v_{d'}, v_d, v_{x'}\}$ induces a bull),
- v_c misses $v_{c'}$ (by 3 with $v_d, v_{d'}$),
- v_b misses $v_{c'}$ (by 3 with $v_d, v_{d'}$), and
- v_c misses $v_{b'}$ (by 3 with $v_d, v_{d'}$),

- some v_e with $d < e$ sees $v_{d'}$ and misses $v_d, v_{c'}$ (by 2),
- v_e misses v_x (by 2 with $v_{c'}$),
- v_e misses $v_{a'}, v_{b'}$ (by 3 with $v_d.v_{c'}$),
- v_e misses $v_{x'}$ (in selecting v_d, we may assume that v_d is the lexically largest vertex in B^t which sees $v_{a'}$ and misses $v_{x'}$; by 2 this implies that for every f with $d < f$, v_f sees either both or neither of $v_{a'}, v_{x'}$),
- v_e misses v_a, v_b, v_c (by 2 with $v_{x'}$),

but then $\{v_{x'}, v_a, v_b, v_{d'}, v_e\}$ induces a bull,contradiction.

Next, we claim that

$$v_b \quad \text{sees} \quad v_{a'} .$$

Again, argue by contradiction: suppose v_b misses $v_{a'}$. Then

- v_a misses $v_{a'}$ (by 2 with v_b),
- v_a misses $v_{b'}$ (else bull with $v_b, v_{a'}, v_{x'}$),
- some v_d with $b < d$ sees $v_{a'}$ and misses $v_b, v_{x'}$ (by 2),
- v_d sees $v_{b'}$ (by 2 with $v_{a'}$),
- v_d misses v_a (by 2 with $v_b, v_{x'}$,
- v_d misses v_x (by 4 with $v_{a'}, v_{b'}$),
- some $v_{d'}$ with $b' < d'$ sees v_a and misses $v_{b'}, v_x$ (by 2),
- $v_{d'}$ sees v_b and misses $v_{a'}, v_{x'}$ (symmetric to above),
- v_d sees $v_{d'}$ (3 with $v_b, v_{b'}$)

but then $\{v_d, v_{d'}, v_a, v_b, v_{x'}\}$ induces a bull, contradiction. Arguing symmetrically,

$$v_a \quad \text{sees} \quad v_{b'} ,$$

and so (by 4 with $v_b, v_{b'}$)

$$v_a \quad \text{sees} \quad v_{a'} .$$

To finish the proof of 5(b), observe that

- v_c sees $v_{a'}, v_{b'}$ (by 3 with $v_b, v_{x'}$),
- $v_{c'}$ sees v_a, v_b (by 3 with $v_{b'}, v_x$),
- v_c sees $v_{c'}$ (by 3 with $v_b, v_{b'}$),
- v_c misses v_x (by 4 with $v_{a'}, v_{b'}$),
- $v_{c'}$ misses $v_{x'}$ (by 4 with v_a, v_b),
- v_x sees $v_{x'}$ (else bull with $v_a, v_b, v_{b'}$).

$\square$

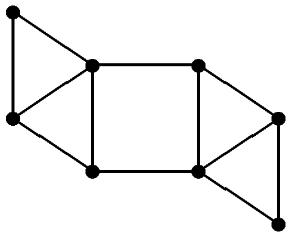

FIG. 5. *The graph Q_8.*

4. The main result. Let Q_8 be the graph shown in Figure 5.

THEOREM 4.1. *Let G be a graph in $\mathcal{W}$ which does not contain Q_8. Then G is perfectly orderable.*

The theorem follows from the correctness of the following algorithm.

ALGORITHM 4.2.

Input. Any graph $G = (V, E)$ in $\mathcal{W}$ which is Q_8-free.

Output. A perfect ordering of G.

1. *Find a box partition $\{B_0^1, \ldots, B_0^s, B_1^1, \ldots, B_t^1\}$ of $\overline{\mathrm{G}}$.*
2. *Find labellings of the vertices of B^0 and B^1 such that the adjacency matrix $B^0 \times B^1$ of the bipartite graph $\overline{\mathrm{G}}^*$ obtained by removing all edges from $\overline{\mathrm{G}}[B^0]$ and $\overline{\mathrm{G}}[B^1]$ is doubly lexically ordered.*
3. *For $t = 0$ and 1, for all edges $(v_{x'}^{1-t}, v_a^t)$ such that there is some P_4 $(v_{x'}^{1-t}, v_a^t, v_c^t, v_b^t)$ of G with $v_a^t < v_b^t < v_c^t$, orient the edge so that $v_a^t < v_{x'}^{1-t}$.*
4. *Extend the acyclic orientation of G defined by the previous two steps into a total order of the vertices, and return this order.*

Proof of correctness of Algorithm 4.2. Step 1 is well defined, since by definition $\overline{G}$ admits a box partition. Step 2 is well defined, as explained in the proof of Lemma 3.1. Step 3 is well defined, since 5(a) of Lemma 3.1 implies that an edge will be oriented in at most one direction by this step.

We claim that the orientation of G after step 3 is acyclic. Consider any (not necessarily induced) cycle $C = (c_1, c_2, \ldots, c_k)$ of G. If the edges of the cycle do not contain any cross edges (namely, edges with one vertex in B^0 and the other in B^1), then the cycle is acyclically oriented since vertices of B^0 and B^1 are each linearly ordered by the doubly lexical ordering. Suppose then that the cycle includes at least two cross edges.

If the cycle includes two consecutive cross edges (c_{j-1}, c_j) and (c_j, c_{j+1}), then as a consequence of 5(a) of Lemma 3.1 the two edges are oriented so that $c_{j-1} < c_j$ and $c_{j+1} < c_j$, or $c_j < c_{j-1}$ and $c_j < c_{j+1}$, and the claim holds in this case.

Suppose finally that the cycle includes at least two cross edges, no two of which are consecutive. Let e be the cross edge on the cycle whose vertex in B^1 is lexically smallest (in the order of step 2), over all cross edges in the cycle. Let this vertex be c_j, and assume (by relabelling $C - c_j$ if necessary) that $e = (c_{j-1}, c_j)$. Let c_k be the next vertex in the cycle (after c_j) in a cross edge. Thus every vertex in $P_j = (c_j, c_{j+1}, \ldots, c_k)$ is in B^1. By choice of e, $c_j < c_k$. Similarly, let c_f be the last vertex in the cycle (before c_{j-1}) in a cross edge. Thus every vertex in $P_f = (c_f, c_{f+1}, \ldots, c_{j-1})$ is in B^0. If $c_{j-1} < c_f$, then some edge in P_f is oriented in the opposite direction from an edge in P_j, and C is acyclically oriented, and we are done. Suppose then that $c_f < c_{j-1}$. Since (c_f, c_{f-1}) is a cross edge and c_{f-1} is in B^1, by our choice of e it follows that $c_j < c_{f-1}$. Now we are done unless $c_j < c_{j+1} < \ldots < c_k$ (otherwise C is acyclically oriented), and unless $c_f < c_{f+1} < \ldots < c_{j-1}$. Furthermore, since $c_f < c_{f+1}$, we are done unless the cross edge (c_f, c_{f-1}) is oriented in step 3 so that $c_{f-1} < c_f$ (otherwise the cycle is acyclically oriented), and since $c_j < c_{j+1}$, unless the cross edge (c_{j-1}, c_j) is oriented in step 3 so that $c_{j-1} < c_j$.

Now from the manner in which cross edges are oriented in step 3 it follows that in $\overline{G}$ there are P_4's $(v^0_a, v^0_b, v^1_{x'}, v^0_c)$ and $(v^1_{a'}, v^1_{b'}, v^0_x, v^1_{c'})$, with $v^1_{x'} = c_j$, $v^0_a = c_{j+1}$, $v^0_x = c_f$, $v^1_{a'} = c_{f-1}$, and $x < a < b < c$ and $x' < a' < b' < c'$. It follows by 5(b) of Lemma 3.1 that the graph $\overline{G}[\{v^t_x, v^t_a, v^t_b, v^t_c, v^{1-t}_{x'}, v^{1-t}_{a'}, v^{1-t}_{b'}, v^{1-t}_{c'}\}]$ is as shown in Figure 4. But this is a contradiction, since the graph in Figure 4 is the complement of Q_8. Thus the graph G is acyclically oriented at the end of Step 3.

The correctness of step 4 follows from the fact that any acyclic orientation can be extended into a total vertex ordering (for example, via topological sorting).

It remains only to show that the ordering of the graph is a perfect order. Let $P = (v_a, v_b, v_c, v_d)$ be a P_4 of G. It is sufficient to show that P is oriented so that $v_b < v_a$ or $v_c < v_d$ (or both).

Since $G[B^0]$ and $G[B^1]$ are each P_4-free, P contains at least one vertex from each of B^0 and B^1. Also, because in $\overline{G}$ vertices within a box have comparable neighborhoods in the rest of the graph, P cannot be oriented such that the two end vertices are on one side of the partition and the two middle vertices the other. There are four possible orientations of P to consider.

- v_a and v_c are in B^t, and v_b and v_d in B^{1-t}.
  Since in $\overline{G}$ vertices within a box have comparable neighborhoods in the rest of the graph, and because of the doubly lexical ordering, it must be that $v_c < v_a$ and $v_b < v_d$. Now the fact that the vertices are linearly ordered implies that $v_b < v_a$ or $v_c < v_d$.
- v_a and v_b are in B^t, and v_c and v_d in B^{1-t}.
  Since $\overline{G}^*$ is chordal bipartite, the doubly lexical ordering is Γ-free, so $v_b < v_a$ or $v_c < v_d$.

- v_a, v_b, v_c are in B^t, and v_d in B^{1-t}.
  Since in $\overline{G}$ vertices within a box have comparable neighborhoods in the rest of the graph, $v_c < v_a$. If in addition $v_a < v_b$, then the conditions for step 3 of the algorithm apply, and $v_c < v_d$.
- v_a, v_b, v_d are in B^t, and v_c in B^{1-t}.
  Since in $\overline{G}$ vertices within a box have comparable neighborhoods in the rest of the graph, $v_b < v_a$.

Thus P is always oriented as desired, and the order returned by the algorithm is a perfect order. □

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