Investigating NeST Graphs[*]

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Abstract. We investigate NeST graphs, namely, intersection tolerance graphs of neighborhood subtrees of a tree, introduced recently by Bibelnieks and Dearing.

We establish a "tolerance-free" characterization of NeST graphs. This in turn yields tolerance-free characterizations of two known subclasses of NeST graphs, namely, proper NeST graphs (in which no neighborhood subtree is properly contained within another) and fixed tolerance NeST graphs (also called constant NeST graphs). Also, we introduce two new subclasses of NeST graphs, namely, fixed diameter NeST graphs and fixed distance NeST graphs.

We show that the class of fixed diameter NeST graphs (equivalent to unit NeST graphs) is exactly the class of proper NeST graphs, analogous to the case for interval graphs and in contrast to the case for interval tolerance graphs.

We show that the class of fixed distance NeST graphs is exactly the class of threshold tolerance graphs, introduced by Monma, Reed and Trotter. This responds to a question they raised concerning the characterization of threshold tolerance graphs as intersection graphs of subtrees of a tree, and implies polynomial recognition for fixed distance NeST graphs.

Finally, various inclusion questions are resolved. In particular, we answer a question of Bibelnieks and Dearing by showing the existence of a graph that is weakly triangulated but not NeST.

Key Words. NeST graph, interval tolerance graph, interval graph, threshold tolerance graph, weakly triangulated graph, fixed diameter NeST graph, proper NeST graph, fixed distance NeST graph, fixed tolerance NeST graph,

1. Introduction.

1.1. History and Motivation.

In 1957 Häjos asked which graphs can be represented as the intersection graphs of intervals on a line [Hj]. Since that time these *interval graphs* have been widely studied, in part due to their appearance in diverse areas of science, including archaeology (sequence dating), molecular biology (genetic mapping), and discrete mathematics (interval orders), to name only a few. An introduction to interval graphs, discussing both applications and theory, is given by Golumbic [Go].

In 1982 Golumbic and Monma introduced a generalization of interval graphs called *interval tolerance graphs*, or simply *tolerance graphs* [GM]. Informally, interval tolerance graphs are described by the following edge relation:

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Two vertices are adjacent if and only if the size of the overlap of their intervals (on a line) exceeds at least one of their tolerances.

Why generalize to interval tolerance graphs? First, the notion of tolerance often exists in physical measurement; thus interval tolerance graphs model many problems more accurately than interval graphs. Second, the class of interval tolerance graphs contains not only interval graphs, but also permutation graphs [GMT] as well as threshold graphs and the complement of threshold tolerance graphs [MRT]. Thus results on interval tolerance graphs apply to several classes of graphs.

In 1991 Bibelnieks and Dearing generalized interval tolerance graphs to a class of graphs they called *neighborhood subtree tolerance* (NeST) *graphs* [BD]. A formal definition of NeST graphs is given in the next section. Informally, NeST graphs generalize interval tolerance graphs by generalizing intervals on a line to neighborhood subtrees of a plane-embedded tree, where a neighborhood subtree is specified by its center and radius. Thus, NeST graphs are described by the following edge relation:

Two vertices are adjacent if and only if the size of the overlap of their neighborhood subtrees (in an embedded tree) exceeds at least one of their tolerances.

Why generalize to NeST graphs? Again, both applications and theory motivate the generalization. First, there exist problems that are best described by a model in which the underlying shared commodity is represented by a tree rather than a line. Some examples are presented in §1.2.

Second, the generalization of interval graphs obtained by replacing the line commodity model with a tree (yielding intersection graphs of subtrees of a tree, or *subtree graphs*) has been previously studied, so extending this model to include tolerances is a natural theoretical step. That this generalization is indeed a natural one is supported by Gavril's proof that the class of subtree graphs is exactly the well-known class of triangulated graphs [Ga].

Bibelnieks and Dearing point out that when subtree graphs are generalized by introducing tolerances, all graphs are realized. If an interesting class of graphs is to be obtained in the generalization of interval graphs to the intersection tolerance graphs of subtrees of an embedded tree then some restriction must be put on the kinds of subtrees that are permitted. The restriction they chose is that of the neighborhood subtree, a subtree studied earlier by Tamir in relation to certain planar location problems [T].

The above discussion is summarized in Figure 1.1. The only undefined class in this figure is the class of *neighborhood subtree graphs* (to be be discussed in section 8), namely, the intersection graphs of neighborhood subtrees in an embedded tree. Note that in Figure 1.1 the top three classes of graphs are classes of intersection graphs,

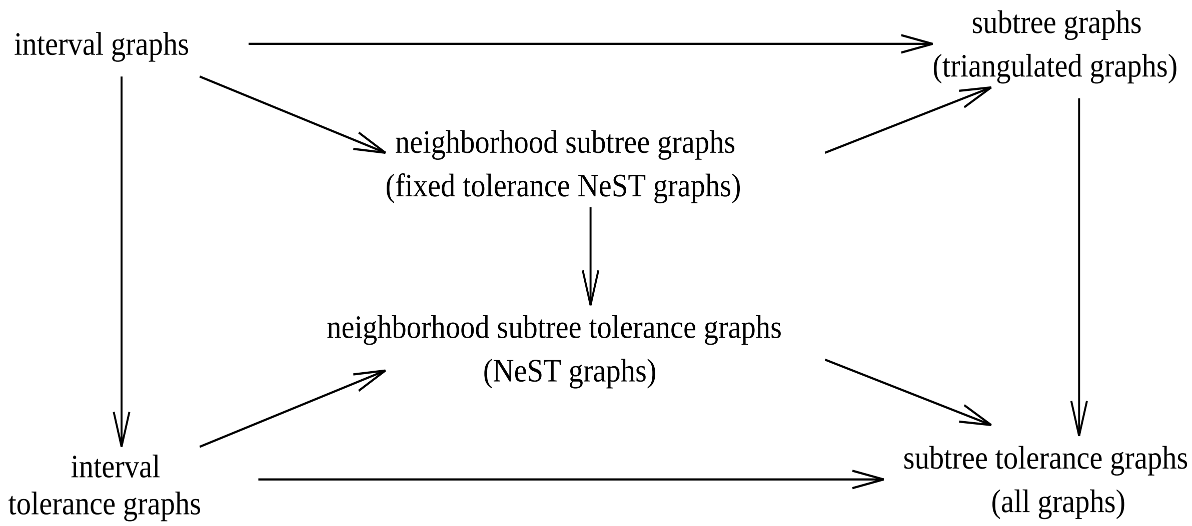

FIG. 1. *Generalizing interval graphs and interval tolerance graphs (arrows indicate set inclusion).*

and that the bottom three are obtained via the introduction of tolerances.

1.2. Applications. In this section we give examples of applications that can be modelled by NeST graphs. Definitions used here are presented formally in §2.

Application 1. Let R be an acyclic, connected road system on which centers for various emergency response vehicles, say ambulances, are located. Let A be the set of ambulance centers. An administrator wishes to decide the *service radius*, r_p, for each ambulance center p in A, where distances are measured along the road system. The service radius together with the location of an ambulance center defines an *area of responsibility*, R_p, for each ambulance center, which is a neighborhood subtree of R. The conditions that must be met when assigning service radii are:

- Each point in R is within the responsibility of at least one ambulance center.
- The service radius of any ambulance center does not exceed a certain upper bound r. The upper bound reflects the policy that no person should wait longer than a prespecified amount of time for ambulance response.
- The size of the overlap of the area of responsibility of any two ambulance centers does not exceed some upper bound t. This tolerance for overlap is introduced to enforce economy in the assignment of service radii to the ambulance centers. That is, it would be inefficient to allow two ambulance centers to have areas of responsibility with overlap more than t.

Let G be a graph where $V(G) = A$ and $E(G) = \emptyset$. The administrator must find a NeST representation $(R, S, \mathcal{T})$ of G, where S is the set of ambulance center areas of responsibility and $\tau_p = t$ for each p in A such that

- S covers R and
- for all $p \in A$, $r_p \leq r$.

Observe that the third condition above is satisfied if $(R, S, \mathcal{T})$ is a NeST representation of G. An approximate solution to the problem is one in which $|E(G)|$ is minimized.

Application 2. Let R be a road system as in Application 1; assume here that the emergency response service is the fire department. Let $f_1, \ldots, f_m$ be the location of m fire stations on R and let $s_1, \ldots, s_n$ be the location of n suburbs on R. We assume that each fire station f_i has a service radius r_{f_i} and each suburb s_i can be represented as a neighborhood subtree of R with radius r_{s_i}.

A fire insurance company wishes to sell a new policy to the home owners of the n suburbs. The policy offers better rates to those home owners who live in a suburb which falls completely within the area of responsibility of more than one fire station. Furthermore, the the greater the number of fire stations which are responsible for a suburb the better the policy rates for the home owners in that suburb. The following model will aid the insurance company in determining which policy a particular home owner is eligible for. Assign an infinite (that is, very large) tolerance to each fire station. Let the tolerance of any suburb be exactly twice the radius of that suburb. Define a graph G where the vertices of G are the fire stations along with the suburbs by

$$xy \in E(G) \quad \Leftrightarrow \quad |R_x \cap R_y| \geq min\{t_x, t_y\},$$

where R_x, t_x, R_y and t_y are the neighborhood subtrees and tolerances associated with x and y. We observe that

- $f_i f_j \notin E(G)$ for all f_i and f_j,
- $s_i s_j \notin E(G)$, since $|R_{s_i} \cap R_{s_j}| \geq min\{t_{s_i}, t_{s_j}\}$ implies that $R_{s_i} \subseteq R_{s_j}$ or $R_{s_j} \subseteq R_{s_i}$, which implies that $s_i \equiv s_j$ (unless we allow suburbs within suburbs) and
- $f_i s_j \in E(G) \quad \Leftrightarrow \quad |R_{f_i} \cap R_{s_j}| \geq min\{t_{f_i}, t_{s_j}\} = 2r_j \quad \Leftrightarrow \quad R_{s_j} \subseteq R_{f_i} \quad \Leftrightarrow \quad s_j$ is completely within the area of responsibility of station f_i.

We can see that the model is a NeST representation of the graph G, in which the degree of vertex s_i is the number of fire stations which share responsibility for s_i and the neighbors of vertex f_j are the suburbs f_j is responsible for.

The insurance company would like to determine the best policy (from the home owner's point of view) for which a customer is eligible given the above representation, regardless of whether the customer lives in a suburb or in a rural district along R. To model this we alter the above representation. Remove all suburbs from the representation and allow all fire station tolerances to be arbitrarily small (but still nonnegative). Let H be the graph where $V(H)$ is the set of fire stations and $E(H)$ is defined by

$$xy \in E(H) \Leftrightarrow |R_x \cap R_y| \geq min\{t_x, t_y\}.$$

That is, the vertices f_i and f_j form an edge in H if and only if their areas of responsibility overlap. Observe that if some home owner is within the area of responsibility of k fire stations, then those k fire stations form a complete subgraph of H. Hence, determining the best possible policy is equivalent to solving the maximum clique problem for H. Since NeST graphs are perfect[1] [BD], this can be solved in polynomial time [GLS].

1.3. Overview. This paper is motivated by a desire to learn more of the structure of NeST graphs. We begin by trying to simplify the definition; our first result is a characterization of NeST graphs in which tolerances do not appear, but are instead implicitly determined by other parameters. This characterization is the foundation for most other results in the paper. It is presented in §4 after a formal definition of NeST graphs is presented in §2, and a canonical form for NeST representations is described in §3.

From the perspective of our "tolerance-free" characterization, we consider four subclasses of NeST graphs, namely,

- *fixed diameter NeST graphs*, in which all neighborhood subtrees have the same diameter,
- *proper NeST graphs*, in which no neighborhood subtree is properly contained in another neighborhood subtree,
- *fixed distance NeST graphs*, in which neighborhood subtree centers are equidistant, and
- *fixed tolerance NeST graphs*, in which all neighborhood subtrees have the same tolerance.

These are presented in §5 through §8 respectively.

In §9 we examine the relationship between NeST graphs and weakly triangulated graphs, and in §10 we present our conclusions.

2. Defining NeST Graphs. In this section we give a precise definition of NeST graphs. To do so we must define the notions of embedding tree, neighborhood subtree, and neighborhood subtree size.

[1] A graph G is *perfect* if, for all induced subgraphs G' of G, the chromatic number of G' equals the maximum clique size of G'. See [BC] or [Go] for more on the topic of perfect graphs.

We use standard graph theory terminology [BM]. An undirected simple graph G is specified by its vertex set $V(G)$ and its edge set $E(G)$, which is an irreflexive and symmetric relation on $V(G)$ (in other words, our graphs are simple, undirected, and loopless). We abbreviate "$\{x, y\} \in E(G)$" as "$xy \in E(G)$", and call x and y "neighbors". If $xy \notin E(G)$ then we call x and y non-neighbors. For any $x \in V(G)$ we define the *neighborhood* of x to be $N_x = \{y \in V(G) : xy \in E(G)\}$ and the *non-neighborhood* of x to be $M_x = \{y \in V(G) - \{x\} : xy \notin E(G)\}$. If $W \subseteq V(G)$ then G_W denotes the subgraph of G induced by W, that is, $V(G_W) = W$ and $E(G_W) = \{xy \in E(G) : x, y \in W\}$.

Let $\mathcal{T}$ be a tree (a connected and acyclic graph) and let T be an embedding of $\mathcal{T}$ in the plane. We call T an *embedded tree*. The notion of tree embedding we use is formally defined in [T]. We denote by $P(x, y)$ the unique path in T between the points x and y. Let $d(x, y)$ be a measure of the length of $P(x, y)$. We call $x \in T$ an *endpoint* of T if x corresponds to an endvertex of $\mathcal{T}$.

Define the *neighborhood subtree* of T with center $c \in T$ and radius $r \geq 0$, denoted $T(c, r)$, to be the set of points $\{x \in T : d(x, c) \leq r\}$.

Now we define our notion of neighborhood subtree size. Let S be any connected subset of points of the embedded tree T. Define the function $|\cdot|$ so that

$$|S| = \begin{cases} max\{d(p_1, p_2) : p_1, p_2 \in S\}, & \text{if } S \neq \phi \\ 0, & \text{if } S = \phi. \end{cases}$$

For any neighborhood subtree $T(c, r)$, we call $|T(c, r)|$ the *diameter* of the neighborhood subtree. Observe that $|T(c, r)|$ is the length of a longest path in $|T(c, r)|$.

DEFINITION 2.1. *A graph G is a* neighborhood subtree tolerance (NeST) graph *if there exists an embedded tree T, a set $S = \{T(c_v, r_v) : v \in V(G)\}$ of neighborhood subtrees of T and a set $\mathcal{T} = \{\tau_v : v \in V(G)\}$ of positive numbers called* tolerances *such that the following edge condition holds:*

$$xy \in E(G) \Leftrightarrow |T(c_x, r_x) \cap T(c_y, r_y)| \geq min\{\tau_x, \tau_y\}.$$

The triple $(T, S, \mathcal{T})$ is called a neighborhood subtree tolerance (NeST) representation *of G. G is called the graph* associated *with the NeST representation $(T, S, \mathcal{T})$.*

Throughout the paper we shall use T_x as shortened notation for the neighborhood subtree $T(c_x, r_x)$. We shall also denote the intersection of T_x and T_y by T_{xy}. In referring to a NeST representation $(T, S, \mathcal{T})$, it is assumed that T_x, c_x, r_x and τ_x denote the representation's neighborhood subtree, center, radius and tolerance respectively, for each vertex x of the associated graph.

Our definition of diameter differs from that found in [BD], where $|T(c, r)|$ is defined to be $2r$. Our reasons for this slight change are as follows.

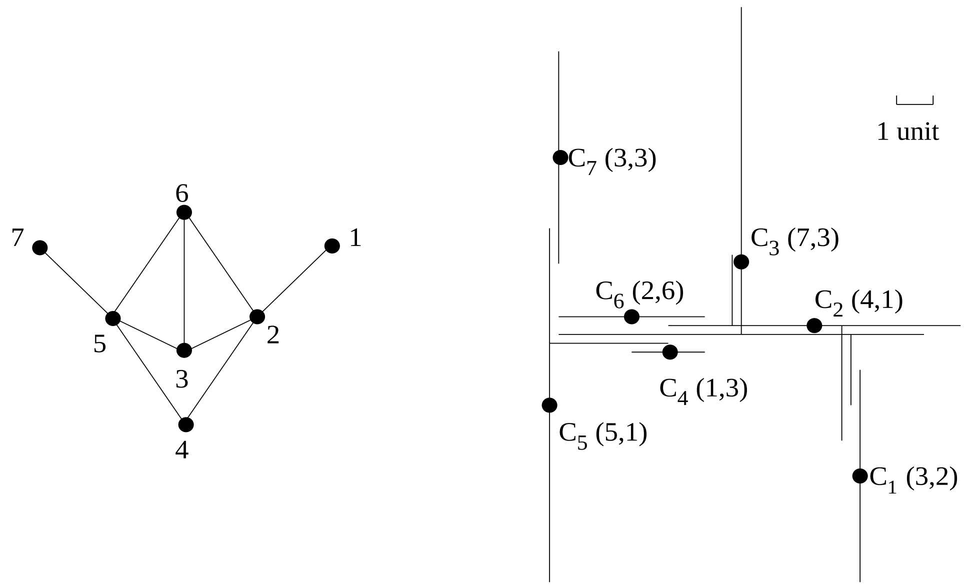

FIG. 2. *A graph and its NeST representation. Pairs denote radius and tolerance values.*

First, consider the embedded tree T consisting of a line segment two units long, with endpoint p and midpoint m. Let $T(p,1)$ and $T(m,1)$ be two neighborhood subtrees defined on T. The definition of diameter in [BD] gives us $|T(p,1)| = 2 = |T(m,1)|$. However, $T(m,1)$ is obviously twice as large as $T(p,1)$. Defining the diameter of a neighborhood subtree to be the length of its longest path gives the more satisfactory $|T(p,1)| = 1$ and $2 = |T(m,1)|$.

In general, when a neighborhood subtree center is located near an endpoint of the embedded tree it may happen that the length of a longest path in the neighborhood subtree is less than twice the neighborhood subtree's radius. In this case the diameter of the neighborhood subtree is not accurately measured by twice its radius.

Second, defining diameter only on the space of neighborhood subtrees, as in [BD], causes an awkward special case: the diameter of the empty set is not defined (since a neighborhood subtree can never be empty). On the other hand, since diameter is used to measure neighborhood subtree intersections it is necessary that it be defined for empty intersections. Our definition of diameter extends to cover the special case where the set being measured is empty.

In section 3 we will see that our definition of diameter is essentially the same as that used in [BD] when restricted to measuring the size of what we call standard neighborhood subtrees.

3. Standard Neighborhood Subtrees. In this section we motivate, define and explore what we call standard neighborhood subtrees. It will be shown that

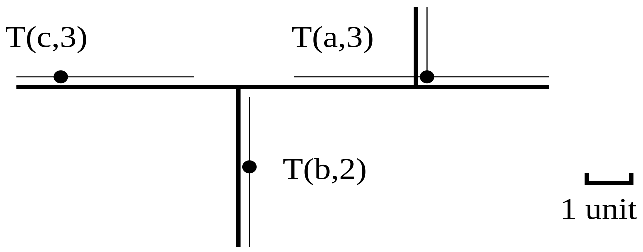

FIG. 3. *Maximal and Truncated Neighborhood Subtrees.*

standard neighborhood subtrees have practical and theoretical advantages over arbitrary neighborhood subtrees.

A neighborhood subtree $T(c,r)$ is called *maximal* if $|T(c,r)| = 2r$. The following lemma illustrates why the adjective "maximal" is appropriate:

LEMMA 3.1. *For any neighborhood subtree $T(c,r)$, $|T(c,r)| \leq 2r$.*

Proof. Since d is a metric on the embedded tree T, the triangle inequality holds, that is, $d(i,j) \leq d(i,k) + d(j,k)$ for all points $i,j,k \in T$. Suppose that $|T(c,r)| > 2r$, then there is a path $P(a,b)$ in $T(c,r)$ such that $d(a,b) > 2r$. However, $r \geq d(a,c)$ and $r \geq d(b,c)$ imply $2r \geq d(a,c) + d(b,c) \geq d(a,b) > 2r$, a contradiction. □

A neighborhood subtree which is not maximal will be called *truncated*. The existence of truncated neighborhood subtrees in a NeST representation is undesirable for the following reasons:

- The diameter of a maximal neighborhood subtree $T(c,r)$ depends only upon the value of r and can be computed in constant time. The diameter of a truncated neighborhood subtree depends upon r, c and the embedded tree T and cannot be computed in constant time. Hence, truncated neighborhood subtrees require more time and information to measure than maximal neighborhood subtrees.
- When reasoning about NeST representations which may have truncated neighborhood subtrees, we cannot make assumptions about the diameters of neighborhood subtrees. This makes analysis much more complicated. However, when all neighborhood subtrees are maximal, it can be assumed that neighborhood subtree diameters are twice the radii. This greatly simplifies analysis. In particular, this assumption will simplify much of the reasoning presented in this paper.

The complications incurred by the presence of truncated neighborhood subtrees can be avoided by the use of standard neighborhood subtrees. We shall see that every NeST graph has a NeST representation in which each neighborhood subtree is a standard neighborhood subtree. Furthermore, standard neighborhood subtrees are always maximal.

DEFINITION 3.2. *We call* $T(m(c,r), l(c,r))$ *the* standard neighborhood subtree *of* $T(c,r)$ *where*

1. $m(c,r)$ *is the midpoint of a longest path in* $T(c,r)$ *and*
2. $l(c,r)$ *is half the length of a longest path in* $T(c,r)$.

LEMMA 3.3. *The standard neighborhood subtree of a neighborhood subtree is uniquely defined.*

Proof. Let $T(c,r)$ be a neighborhood subtree. To show that $T(m(c,r), l(c,r))$ is uniquely defined it is sufficient to show that $m(c,r)$ and $l(c,r)$ are uniquely defined. Obviously, $l(c,r)$ is uniquely defined since it is the length of a longest path in $T(c,r)$.

To see that $m(c,r)$ is also uniquely defined, suppose that $P(a,b)$ and $P(x,y)$ are two longest paths in $T(c,r)$ with midpoints c_1 and c_2 where $d(c_1,c_2) > 0$. We have that $|P(a,b)| = 2l(c,r) = |P(x,y)|$. We have either $P(a,c_1) \cap P(c_1,c_2) = \{c_1\}$ or $P(b,c_1) \cap P(c_1,c_2) = \{c_1\}$. Similarly we have either $P(x,c_2) \cap P(c_1,c_2) = \{c_2\}$ or $P(y,c_2) \cap P(c_1,c_2) = \{c_2\}$. Without loss of generality assume that $P(a,c_1) \cap P(c_1,c_2) = \{c_1\}$ and $P(x,c_2) \cap P(c_1,c_2) = \{c_2\}$. It follows that $P(a,x) = P(a,c_1) \cup P(c_1,c_2) \cup P(x,c_2)$ which implies that $|P(a,x)| = 2l(c,r) + d(c_1,c_2) > 2l(c,r)$. This contradicts the maximality of $|P(a,b)|$ and $|P(x,y)|$. By contradiction, $d(c_1,c_2) = 0$. Hence, the midpoints of all longest paths in $T(c,r)$ are identical which implies that $m(c,r)$ is uniquely defined. □

The following lemma demonstrates that a neighborhood subtree and its standard neighborhood subtree are identical point sets.

LEMMA 3.4. $T(c,r) = T(m(c,r), l(c,r))$, *for any neighborhood subtree* $T(c,r)$.

Proof. Let $P(a,b)$ be a longest path in $T(c,r)$. First we show that $T(m(c,r), l(c,r)) \subseteq T(c,r)$. We have that $m(c,r)$ is on $P(a,c)$ or $P(b,c)$, hence, $r \geq d(a,c) = d(c,m(c,r)) + d(a,m(c,r))$ or $r \geq d(b,c) = d(c,m(c,r)) + d(b,m(c,r))$. In either case we have $r \geq d(c,m(c,r)) + l(c,r)$. If $x \in T(m(c,r), l(c,r))$ then $l(c,r) \geq d(m(c,r),x)$ and so $r \geq d(c,m(c,r)) + d(m(c,r),x)$. This proves that $x \in T(c,r)$.

To see that $T(c,r) \subseteq T(m(c,r), l(c,r))$ first assume that $x \in T(c,r) - T(m(c,r), l(c,r))$. This assumptions gives us $d(m(c,r),x) > l(c,r)$. Now, $d(a,x) = d(a,m(c,r)) + d(m(c,r),x)$ or $d(b,x) = d(b,m(c,r)) + d(m(c,r),x)$. Thus, either $d(a,x) = l(c,r) + d(m(c,r),x) > 2l(c,r)$ or $d(b,x) = l(c,r) + d(m(c,r),x) > 2l(c,r)$. In either case the maximality of $|P(a,b)|$ in $T(c,r)$ is contradicted. This proves that $T(c,r) \subseteq T(m(c,r), l(c,r))$. □

LEMMA 3.5. *A neighborhood subtree is maximal if and only if it is a standard neighborhood subtree.*

Proof. ($\Rightarrow$) Let $T(c,r)$ be a maximal neighborhood subtree in some NeST representation. Let $P(a,b)$ be a maximal path in $T(c,r)$. By Lemma 3.4, $P(a,b)$ is maximal in $T(m(c,r), l(c,r))$ as well. Hence, $l(c,r) = r$. Since $d(a,c) = d(b,c) = r = |P(a,b)|/2$, c is the midpoint of $P(a,b)$. Similarly, $m(c,r)$ is the midpoint of $P(a,b)$.

($\Leftarrow$) Let $T(c,r)$ be a standard neighborhood subtree. By definition, $|T(c,r)| = 2r$, hence $T(c,r)$ is maximal. □

LEMMA 3.6. *If G is a NeST graph then G has a NeST representation in which $c = m(c,r)$ and $r = l(c,r)$ for each neighborhood subtree $T(c,r)$.*

Proof. Suppose G has a NeST representation $(T, S, \mathcal{T})$. Replace all neighborhood subtrees in S with their standard neighborhood subtrees. By Lemma 3.3 this can be done uniquely. Furthermore, $(T, S, \mathcal{T})$ remains a NeST representation of G since, by Lemma 3.4, $T(c,r) = T(m(c,r), l(c,r))$ for all neighborhood subtrees $T(c,r)$. □

COROLLARY 3.7. *If G is a NeST graph then G has a NeST representation where all neighborhood subtrees are maximal.*

Proof. By Lemma 3.6, G has a NeST representation in which all neighborhood subtrees are standard neighborhood subtrees. Hence, by Lemma 3.5 G has a NeST representation where all neighborhood subtrees are maximal. □

A consequence of corollary 3.7 is that we may restrict ourselves, without loss of generality, to those NeST representations whose neighborhood subtrees are maximal.

As discussed in section 2, the definition of diameter found in [BD] differs from the definition of diameter found here. However, when measuring the size of maximal neighborhood subtrees there is no practical difference between the definition found in [BD] and ours.

In closing we make the following definition:

DEFINITION 3.8. *A* standard NeST representation *is a NeST representation in which all neighborhood subtrees are standard neighborhood subtrees.*

4. The Role of Tolerances. Our aim in this section is to simplify NeST representations by eliminating tolerances. Note that the removal of tolerances is done in an effort to better understand the class of NeST graphs. We are not suggesting a redefinition of NeST representations without tolerances as tolerances are an important modelling component of NeST representations. The "tolerance-free" characterization of NeST graphs we achieve suggests a simpler yet sufficient description of NeST graphs and will be used in the study of various subclasses of NeST graphs discussed later.

Given a graph G with NeST representation $(T, S, \mathcal{T})$ we define the set $B(x)$, for every vertex $x \in V(G)$, by

$$B(x) = \{z \in M(x) : |T_{xz}| \geq |T_{xy}|, \text{ for all } y \in M(x)\}.$$

$B(x)$ is the set of non-neighbors of x that maximize the size of their neighborhood subtree intersections with the neighborhood subtree of x. Observe that $B(x)$ may be empty if $M(x) = \emptyset$.

DEFINITION 4.1. *The pair (T, S), where T is an embedded tree and S is a set of neighborhood subtrees in T, is a* tolerance-free NeST representation *of the graph G if the following edge condition holds:*

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\},$$

where for a vertex z, $\dot{z}$ is any vertex in $B(z)$, and $|T_{z\dot{z}}| = 0$ whenever $B(z)$ is empty.

Before stating and proving the main result of this section, we define the *perturbation number* of a tolerance-free NeST representation of a graph G:

DEFINITION 4.2. *The perturbation number, $\rho(R, G)$, of a tolerance-free NeST representation $R = (T, S)$ of a graph G is defined by*

$$\rho(R, G) = \begin{cases} min(D), & \text{if } D \neq \emptyset \\ 0, & \text{if } D = \emptyset \end{cases}$$

where

$$D = \{|T_{xy}| - |T_{x\dot{x}}| : |T_{xy}| > |T_{x\dot{x}}|, \ \ xy \in E(G)\}.$$

Stating that G is a NeST graph is equivalent to stating that there is a NeST representation $(T, S, \mathcal{T})$ such that G and $(T, S, \mathcal{T})$ satisfy the following relation:

$$xy \in E(G) \quad \Leftrightarrow \quad |T_{xy}| \geq min\{\tau_x, \tau_y\}.$$

There is redundancy in this characterization. In particular, the set of tolerances $\mathcal{T}$ is redundant. The following theorem proves that this redundancy exists and can be removed.

THEOREM 4.3. *$(T, S, \mathcal{T})$ is a NeST representation of G if and only if (T, S) is a tolerance-free NeST representation of G.*

Proof. ($\Leftarrow$) Let G have a tolerance-free NeST representation (T, S). Define a NeST representation $(T, S, \mathcal{T})$ where each tolerance $\tau_x \in \mathcal{T}$ is defined by

$$\tau_x = |T_{x\dot{x}}| + \epsilon$$

and ϵ is given by

$$\epsilon = \begin{cases} \rho((T, S), G)/2, & \text{if } \rho((T, S), G) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

To prove that $(T, S, \mathcal{T})$ is a NeST representation of G we must verify that

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\},$$

for all $x, y \in V(G)$.

Suppose that $xy \in E(G)$. From the definition of a tolerance-free NeST representation it follows that $|T_{xy}| > |T_{x\dot{x}}|$ or $|T_{xy}| > |T_{y\dot{y}}|$; assume the former without loss of generality. Since $|T_{xy}| > |T_{x\dot{x}}|$ it follows that $\rho((T, S), G) > 0$ and $|T_{xy}| - |T_{x\dot{x}}| \geq \rho((T, S), G) > \epsilon$. Hence, $|T_{xy}| > |T_{x\dot{x}}| + \epsilon = \tau_x$.

Suppose that $xy \notin E(G)$. From the definition of a tolerance-free NeST representation it follows that $|T_{xy}| \leq |T_{x\dot{x}}|$ and $|T_{xy}| \leq |T_{y\dot{y}}|$. Since $\epsilon > 0$ it follows that $|T_{xy}| < |T_{x\dot{x}}| + \epsilon = \tau_x$ and $|T_{xy}| < |T_{y\dot{y}}| + \epsilon = \tau_y$.

($\Rightarrow$) Let G have a NeST representation $(T, S, \mathcal{T})$. We claim that (T, S) is a tolerance-free NeST representation of G. To prove this we must verify that

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\},$$

for all $x, y \in V(G)$.

Suppose that $xy \in E(G)$. It follows that either $|T_{xy}| \geq \tau_x$ or $|T_{xy}| \geq \tau_y$; assume the former without loss of generality. Observe that for all $z \in M_x$, $\tau_x > |T_{xz}|$. In particular, $\tau_x > |T_{x\dot{x}}|$. It follows that $|T_{xy}| > |T_{x\dot{x}}|$.

Suppose that $xy \notin E(G)$. By definition of $\dot{x}$ and $\dot{y}$, $xy \notin E(G)$ implies that $|T_{xy}| \leq |T_{x\dot{x}}|$ and $|T_{xy}| \leq |T_{y\dot{y}}|$. □

In closing, we make three observations.

1. Theorem 4.3 and its proof characterize the role of tolerances in a NeST representation as that of placeholders. It is not necessary that these placeholders be specified explicitly as in a NeST representation: tolerances can be implicitly defined by tolerance-free NeST representation. This simplification of NeST representations, though not surprising, will prove to be useful.
2. The above result proves that a tolerance-free NeST representation (T, S) of a graph G is obtained from a NeST representation $(T, S, \mathcal{T})$ of G simply by dropping $\mathcal{T}$ from the triple $(T, S, \mathcal{T})$. Conversely, the proof of Theorem 4.3 describes a method by which we can reconstruct a set of tolerances $\mathcal{T}$ from a tolerance-free NeST representation (T, S) of G to obtain a NeST representation $(T, S, \mathcal{T})$ of G. It is important to realize that the embedded tree T and the set of neighborhood subtrees S in the tolerance-free NeST representation and the NeST representation are identical.
3. For every NeST representation there is a unique associated graph. That is, the triple $(T, S, \mathcal{T})$ and the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\}$$

completely specify a unique graph G. This is not true for tolerance-free NeST representations. That is, the pair (T, S) and the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

may be satisfied by several different graphs G.

5. Fixed Diameter NeST Graphs. We begin our study of subclasses of NeST graphs with the class of fixed diameter NeST graphs. Fixed diameter NeST graphs are those graphs which have a NeST representation in which all neighborhood subtrees have the same diameter. We shall show that fixed diameter NeST graphs are exactly unit NeST graphs (fixed diameter NeST graphs where all neighborhood subtree diameters have unit size); thus fixed diameter NeST graphs are to NeST graphs as unit interval graphs are to interval graphs, and as unit interval tolerance graphs are to interval tolerance graphs. More importantly, we shall also show that fixed diameter NeST graphs are proper NeST graphs (defined below).

Fixed diameter NeST graphs are of practical interest as well. Many applications modelled by NeST graphs require that all neighborhood subtrees be equal in size.

DEFINITION 5.1. *A graph G is a* fixed diameter NeST graph *if there exists a NeST representation of G in which all neighborhood subtree diameters are equal. Such a NeST representation is called a* fixed diameter NeST representation. *G is a* unit NeST graph *if it has a NeST representation where all neighborhood subtree diameters have unit size. Such a NeST representation is called a* unit NeST representation.

Before we continue with our discussion of fixed diameter NeST graphs, we introduce the concept of *scaling* a NeST representation. If $(T, S, \mathcal{T})$ is a NeST representation then we scale $(T, S, \mathcal{T})$ by a positive constant k, denoted $k(T, S, \mathcal{T})$, if

- the embedded tree T is expanded (or contracted) uniformly by a factor k, such that the distance between any two points a and b in the expanded tree is equivalent to $kd(a, b)$ where $d(a, b)$ is the distance from a to b in T,
- all neighborhood subtree radii are multiplied by a factor k and
- all neighborhood subtree tolerances are multiplied by a factor k.

The following lemma confirms the intuition that the scaled NeST representation $k(T, S, \mathcal{T})$ is a NeST representation of G if and only if $(T, S, \mathcal{T})$ is a NeST representation of G.

LEMMA 5.2. *$(T, S, \mathcal{T})$ and $k(T, S, \mathcal{T})$ are NeST representations of the same graph.*

Proof. Let $(T, S, \mathcal{T})$ be a NeST representation of G and let $(T', S', \mathcal{T}') = k(T, S, \mathcal{T})$. We first demonstrate that $|T'_{xy}| = k|T_{xy}|$ for all $x, y \in V(G)$.

Suppose $T_x \subset T_y$ (symmetric with $T_y \subset T_x$). It follows that $T'_x \subset T'_y$ as well. Hence, $|T_{xy}| = |T_x|$ and $|T'_{xy}| = |T'_x|$. Let $P(a, b)$ be a maximal path in T_x. Since T

is expanded uniformly, $P'(a,b)$ is maximal in T'_x. It follows that $|T'_x| = |P'(a,b)| = d'(a,b) = kd(a,b) = k|P(a,b)| = k|T_x|$.

Suppose that $T_x \not\subset T_y$ and $T_y \not\subset T_x$. If $T_{xy} = \emptyset$ then $T'_{xy} = \emptyset$ and we are done. Otherwise, $T_{xy} = T(c,r)$ where c is the point on $P(c_x, c_y)$ such that $d(c_x, c) = 1/2(r_x - r_y + d(c_x, c_y)$ and $r = 1/2(r_x + r_y - d(c_x, c_y))$ [BD]. It follows that $|T_{xy}| = r_x + r_y - d(c_x, c_y)$. It is also the case that $T'_x \not\subset T'_y$ and $T'_y \not\subset T'_x$. Hence, $|T'_{xy}| = r'_x + r'_y - d'(c_x, c_y) = kr_x + kr_y - kd(c_x, c_y) = k|T_{xy}|$.

Now that we have proven that $|T'_{xy}| = k|T_{xy}|$ for all $x, y \in V(G)$, we have

$$|T_{xy}| \geq min\{\tau_x, \tau_y\}$$

if and only if

$$k|T_{xy}| \geq min\{k\tau_x, k\tau_y\}$$

if and only if

$$|T'_{xy}| \geq min\{\tau'_x, \tau'_y\}.$$

This proves that $(T, S, \mathcal{T})$ and $k(T, S, \mathcal{T})$ are both NeST representations of G. □

COROLLARY 5.3. *G is a fixed diameter NeST graph if and only if it is a unit NeST graph.*

Proof. Let $(T, S, \mathcal{T})$ be a fixed diameter NeST representation of G where all neighborhood subtree diameters have value D. Since scaling preserves the ratio of neighborhood subtree diameters, $(1/D)(T, S, \mathcal{T})$ is a fixed diameter NeST representation of G where all diameters have unit size.

The converse is true by definition. □

Proper NeST graphs are those graphs which have a NeST representation where no neighborhood subtree is properly contained within another. This class of graphs is formally introduced in section 6. Proper interval and proper interval tolerance graphs have analogous definitions. It is known that unit interval and unit interval tolerance graphs are proper interval [R] and proper interval tolerance graphs [GMT], respectively. Below we show that the same is true for NeST graphs. That is, fixed diameter NeST graphs (i.e. unit NeST graphs) are proper NeST graphs.

THEOREM 5.4. *A fixed diameter NeST representation is a proper NeST representation.*

Proof. Let $(T, S, \mathcal{T})$ be a fixed diameter NeST representation. Suppose that $T_x \subset T_y$ for some $x, y \in V(G)$ and let $D = |T_x| = |T_y|$. Consider the standard neighborhood subtrees $T(m(c_x, r_x), l(c_x, r_x))$ and $T(m(c_y, r_y), l(c_y, r_y))$ of T_x and T_y. For brevity, let $c'_x = m(c_x, r_x)$, $r'_x = l(c_x, r_x)$, $c'_y = m(c_y, r_y)$ and $r'_y = l(c_y, r_y)$. Recall from section 3 that $T_x = T(c'_x, r'_x)$ and $T_y = T(c'_y, r'_y)$. Hence, $T(c'_x, r'_x) \subset T(c'_y, r'_y)$.

Now, c'_x is the midpoint of the path $P(a, b)$ where a and b are maximally distant points in $T(c'_x, r'_x)$. However, since $|T(c'_x, r'_x)| = D = |T(c'_y, r'_y)|$ it follows that a and b are maximally distant in $T(c'_y, r'_y)$ as well. Since $T(c'_y, r'_y)$ is a standard neighborhood subtree, it follows from Lemma 3.3 that $c'_y = c'_x$ and $r'_y = r'_x$. However, this implies that $T(c'_x, r'_x) = T(c'_y, r'_y)$ which contradicts $T(c'_x, r'_x) \subset T(c'_y, r'_y)$. We conclude that $(T, S, \mathcal{T})$ is a proper NeST representation. □

In the next section on proper NeST graphs, we will again encounter fixed diameter NeST graphs as they play a central role in the understanding of proper NeST graphs.

6. Proper NeST Graphs. In this section we explore proper NeST graphs which were informally introduced in section 5. Proper NeST graphs are analogous to proper interval and proper interval tolerance graphs which are well-studied subclasses of interval and interval tolerance graphs, respectively.

Proper NeST graphs are of pratical relevence since they are a natural model for many applications, however, our focus here will be on the theoretical aspects of proper NeST graphs.

DEFINITION 6.1. *A graph G is called a* proper NeST graph *if it has a NeST representation $(T, S, \mathcal{T})$ where no neighborhood subtree is properly contained within another neighborhood subtree. That is, for all $x, y \in V(G)$, $T_x \not\subset T_y$. We call a NeST representation with this property a* proper NeST representation.

Roberts [R] raised and resolved the question of whether unit interval graphs are the same as proper interval graphs (yes); Bogart, Isaak, Langley and Fishburn [BILF] resolved the analogous question, posed by Golumbic, Monma and Trotter, of whether unit interval tolerance graphs are the same as proper interval tolerance graphs (no). As the major result of this section, we raise and resolve the natural question of whether fixed diameter NeST graphs (that is, unit NeST graphs) are proper NeST graphs (yes).

We will close the section with a tolerance-free characterization of proper NeST graphs.

LEMMA 6.2. *If G is a proper NeST graph then there exists a proper NeST representation of G where all neighborhood subtree intersections are nonempty.*

Proof. We know that the result is true for NeST graphs in general [BD]. However, we must show that it holds for proper NeST graphs as well.

Suppose that G has a proper NeST representation $(T, S, \mathcal{T})$. The following procedure is used by Bibelnieks and Dearing [BD] to modify the NeST representation $(T, S, \mathcal{T})$ of a graph G so that all neighborhood subtree intersections are non-empty.

- Let $M = max\{d_{xy} - r_x - r_y : x, y \in V(G), x \neq y\}$.
- If $M \leq 0$ then stop,

- else add $M/2$ to all radii, add M to all tolerances and attach line segments of length $M/2$ to all endpoints of T.

The above procedure enlarges all of the neighborhood subtrees, and the embedded tree T, such that a proper NeST representation remains proper after being transformed by the procedure. $\square$

LEMMA 6.3. *If* $T_x \not\subset T_y$, $T_y \not\subset T_x$ *and* $T_{xy} \neq \emptyset$ *then* $|T_{xy}| = r_x + r_y - d(c_x, c_y)$.

Proof. The nonempty intersection of two neighborhood subtrees T_x and T_y is also a neighborhood subtree [BD]. Furthermore, if $T_x \not\subset T_y$ and $T_y \not\subset T_x$ then T_{xy} is a neighborhood subtree with radius $1/2(r_x + r_y - d(c_x, c_y))$. Thus $|T_{xy}| = r_x + r_y - d(c_x, c_y)$. $\square$

The main result of this section follows:

THEOREM 6.4. *If* G *has a proper NeST representation then* G *has a fixed diameter, standard NeST representation.*

Proof. Let $R = (T, S, \mathcal{T})$ be a proper NeST representation of G. We will assume, without loss of generality, that R is a standard NeST representation.

Let $r = max\{r_x : x \in V(G)\}$. We define a new NeST representation $R' = (T', S', \mathcal{T}')$ as follows:

1. $T' = (\bigcup_{x \in V(G)} L_x) \cup T$ where, for each $x \in V(G)$, L_x is a line segment of length $2r$ attached to T at c_x.
2. For all $x \in V(G)$, c'_x is located on L_x such that $d(c'_x, c_x) = r - r_x$.
3. For all $x \in V(G)$, $r'_x = r$ and $\tau'_x = \tau_x$.

Claim 1: All neighborhood subtree diameters are equivalent.

Since $r'_x = r'_y = r$ for all $x, y \in V(G)$, we need only show that all neighborhood subtrees in R' are maximal.

Consider $x \in V(G)$. Let $p \in L_x$ such that $d(c'_x, p) = r$. Let $P(a, b)$ be a maximal path in T_x. Since T_x is maximal, $d(a, c_x) = r_x$. Consider the path $P(a, p)$ in T'_x. $P(a, p) = P(a, c_x) \cup P(c_x, c'_x) \cup P(c'_x, p)$. Furthermore, $|P(a, p)| = r_x + (r - r_x) + r = 2r$. Hence, T'_x is maximal.

Claim 2: All neighborhood subtrees are standard neighborhood subtrees.

In Claim 1 we proved that all neighborhood subtrees were maximal. By Lemma 3.5, all neighborhood subtrees are standard neighborhood subtrees.

Claim 3: R' is a NeST representation of G.

It must be shown that, for all $x, y \in V(G)$, the following edge condition holds:

$$xy \in E(G) \Leftrightarrow |T'_{xy}| \geq min\{\tau'_x, \tau'_y\}.$$

By Lemma 6.2, we may assume that $|T'_{xy}| \neq \emptyset$. It then follows from Lemma 6.3 that $|T'_{xy}| = r'_x + r'_y - d(c'_x, c'_y) = 2r - d(c'_x, c'_y)$. Observe that $d(c'_x, c'_y) = d(c_x, c'_x) + d(c_x, c_y) + d(c_y, c'_y) = (r - r_x) + d(c_x, c_y) + (r - r_y)$. Thus, $|T'_{xy}| = 2r - (2r - r_x - r_y +$

$d(c_x, c_y)) = r_x + r_y - d(c_x, c_y)$. Again, by Lemmas 6.2 and 6.3, we may assume that $|T_{xy}| = r_x + r_y - d(c_x, c_y)$ and so $|T'_{xy}| = |T_{xy}|$.

Since $\tau'_x = \tau_x$ for all $x \in V(G)$, it follows from

$$xy \in E(G) \Leftrightarrow |T_{xy}| \geq min\{\tau_x, \tau_y\}$$

that

$$xy \in E(G) \Leftrightarrow |T'_{xy}| \geq min\{\tau'_x, \tau'_y\}$$

The result follows from claims 1,2 and 3. □

COROLLARY 6.5. *G is a proper NeST graph if and only if G is a fixed diameter NeST graph (i.e. unit NeST graph).*

Proof. Theorem 6.4 and Theorem 5.4. □

As mentioned earlier, interval graphs possess the property that unit interval graphs are equivalent to proper interval graphs. The generalization of interval graphs to interval tolerance graphs, by the introduction of tolerances, results in the loss of this property for interval tolerance graphs. That is, unit interval tolerance graphs are not equivalent to proper interval tolerance graphs. It is an interesting fact that this property is regained in the generalization of interval tolerance graphs to NeST graphs by generalizing the line to a tree.

The following definition will be utilized in our characterization of proper NeST graphs:

DEFINITION 6.6. *Let G be a graph. If T is an embedded tree and S is a set of points of T indexed by $V(G)$ then (T, S) is called a G-*tree.

Informally, a G-tree is an embedded tree in which the vertices of G have been located. This structure should not seem new to the reader; essentially it is a tolerance-free NeST representation where all neighborhood subtree radii information is absent.

We now state and prove our characterization of proper NeST graphs.

THEOREM 6.7 (PROPER NEST GRAPH CHARACTERIZATION). *G is a proper NeST graph if and only if there exists a G-tree (T, S) such that*

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) < min\{d(c_x, c_{\dot{x}}), d(c_y, c_{\dot{y}})\},$$

where $S = \{c_z : z \in V(G)\}$ and for $z \in V(G)$, $\dot{z}$ is any non-neighbor of z such that $d(c_z, c_{\dot{z}}) \leq d(c_z, c_w)$, for all $w \in M_z$. In the case that $M_z = \emptyset$, $d(c_z, c_{\dot{z}})$ is defined to be infinite.

Proof. $(\Rightarrow)$ Suppose G is a proper NeST graph. By Theorem 6.4, let $(T, S, \mathcal{T})$ be a fixed diameter, standard NeST representation of G. We may assume that all neighborhood subtree intersections are nonempty by Lemma 6.2. Let (T, S) be the corresponding tolerance-free NeST representation of G.

(T, S) satisfies the edge condition

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

which, by application of Lemma 6.3, can be expanded to

$$xy \in E(G) \Leftrightarrow r_x + r_y - d(c_x, c_y) > min\{r_x + r_{\dot{x}} - d(c_x, c_{\dot{x}}), r_y + r_{\dot{y}} - d(c_y, c_{\dot{y}})\}.$$

Since all neighborhood subtree diameters are equivalent and all neighborhood subtrees are maximal, we have that $|T_x| = 2r$ for all $x \in V(G)$. This gives us

$$xy \in E(G) \Leftrightarrow 2r - d(c_x, c_y) > min\{2r - d(c_x, c_{\dot{x}}), 2r - d(c_y, c_{\dot{y}})\}$$

which simplifies to

(1) $$xy \in E(G) \Leftrightarrow d(c_x, c_y) < min\{d(c_x, c_{\dot{x}}), d(c_y, c_{\dot{y}})\}.$$

Define T' to be the following subtree of T:

$$\bigcup_{x,y \in V(G)} P(c_x, c_y)$$

and let S' denote the set of neighborhood subtree centers in S. By definition, (T', S') is a G-tree, furthermore, since (T, S) satisfies (1), (T', S') also satisfies (1).

Note that in section 4, $\dot{z}$ is defined to be any non-neighbor of z which satisfies $|T_{z\dot{z}}| \geq |T_{zw}|$, for all $w \in M_z$. Here, this is equivalent to $d(c_z, c_{\dot{z}}) \leq d(c_z, c_w)$, for all $w \in M_z$, since all neighborhood subtree radii are equivalent. Furthermore, when $M_z = \emptyset$, $|T_{z\dot{z}}|$ is define to be 0 which corresponds to the definition of $d(c_z, c_{\dot{z}})$ being infinite when $M_z = \emptyset$.

($\Leftarrow$) Let (T', S') be a G-tree, where $S' = \{c'_x : x \in V(G)\}$, such that (1) holds. We construct a tolerance-free NeST representation, (T, S), by defining

- $T = (\bigcup_{x \in V(G)} L_x) \cup T'$ where, for each $x \in V(G)$, L_x is a line segment of arbitrarily length attached to T' at c'_x,
- $c_x = c'_x$, for all $x \in V(G)$ and
- $r_x = r$, for all $x \in V(G)$, where r is large enough so that $T_{ab} \neq \emptyset$, for all $a, b \in V(G)$.

We will show that (T, S) is a proper, tolerance-free NeST representation of G.

By (1) we have that

$$xy \in E(G) \Leftrightarrow 2r - d(c_x, c_y) > min\{2r - d(c_x, c_{\dot{x}}), 2r - d(c_y, c_{\dot{y}})\}$$

Since $r_x = r_y = r$, for all $x, y \in V(G)$, all neighborhood subtrees in (T, S) are maximal and have the same diameter. Hence, by Theorem 5.4, (T, S) is a tolerance-free NeST representation. By Lemma 6.3, $|T_{xy}| = 2r - d(c_x, c_y)$, for all $x, y \in V(G)$. It follows that

$$xy \in E(G) \Leftrightarrow |T_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}.$$

Hence, (T, S) is a proper, tolerance-free NeST representation of G. □

The above characterization of proper NeST graphs is both tolerance-free and radius-free in the sense that it depends only upon the embedded tree and the placement of neighborhood subtree centers in the embedded tree. This motivates the following definition:

DEFINITION 6.8. *A G-*tree representation *of a proper NeST graph G is a G-tree (T, S) such that*

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) < min\{d(c_x, c_{\dot{x}}), d(c_y, c_{\dot{y}})\}$$

where $S = \{c_z : z \in V(G)\}$ and $\dot{z}$ is defined as in Theorem 6.7.

Hence, G is a proper NeST graph if and only if there is a G-tree representation of G.

7. Fixed Distance NeST Graphs. In this section we introduce our second new subclass of NeST graphs, defined by imposing the restriction that all neighborhood subtree centers be equidistant. Our main result in this section shows that this new subclass is equivalent to the class of threshold tolerance graphs introduced by Monma, Reed and Trotter.

DEFINITION 7.1. *A graph G is a* fixed distance NeST graph *if it has a NeST representation in which the distance between every pair of distinct neighborhood subtree centers is the same.*

A *star graph* is a tree with a vertex, called the *center*, which is adjacent to all other vertices in the graph. An *embedded star T* is a tree embedding of a star graph $\mathcal{T}$. We call the point in T which corresponds to the center of $\mathcal{T}$ the *origin* of T.

DEFINITION 7.2. *A* star NeST representation *is a NeST representation $(T, S, \mathcal{T})$ where*

- *T is an embedded star,*
- *no two neighborhood subtree centers are located at the same point in T and*
- *all neighborhood subtree centers are equidistant from the origin of T.*

A graph is a star NeST graph *if it has a star NeST representation.*

LEMMA 7.3. *A graph G is a fixed distance NeST graph if and only if it is a star NeST graph.*

Proof. If G is a star NeST graph, then the distance between any pair of distinct neighborhood subtree centers is twice the distance from any neighborhood subtree center to the origin of the embedded star, so G is a fixed distance NeST graph. On the other hand, a straightforward inductive argument shows that any fixed distance NeST graph has a star NeST representation. $\square$

In light of the above lemma, we shall henceforth refer to both star NeST graphs and fixed distance NeST graphs as fixed distance NeST graphs.

DEFINITION 7.4. *A* radius-only representation *of a graph G is a set of non-negative numbers $R = \{r_v : v \in V(G)\}$ such that*

$$xy \in E(G) \;\Leftrightarrow \begin{cases} r_x > r_{y^*}, \text{ for all } y^* \in M_y \\ \qquad\qquad or \\ r_y > r_{x^*}, \text{ for all } x^* \in M_x. \end{cases}$$

The following theorem is a key result of this section.

THEOREM 7.5. *A graph is a fixed distance NeST graph if and only if it has a radius-only representation.*

Proof. ($\Rightarrow$) Let G have a tolerance-free star NeST representation (T, S). We will assume, without loss of generality, that all neighborhood subtree intersections are nonempty. Let $R = \{r_x : x \in V(G)\}$.

We wish to show that

$$xy \in E(G) \Leftrightarrow \begin{cases} r_x > r_{y^*}, \text{ for all } y^* \in M_y \\ \qquad\qquad or \\ r_y > r_{x^*}, \text{ for all } x^* \in M_x, \end{cases}$$

thereby proving that R is a radius-only representation of G.

Suppose that $xy \in E(G)$. We have $|T_{xy}| > |T_{x\dot{x}}|$ or $|T_{xy}| > |T_{y\dot{y}}|$; assume the former without loss of generality. Observe that since neighborhood subtree centers are equidistant we have that

$$|T_{ab}| > |T_{ac}| \Rightarrow r_b > r_c.$$

As a consequence, we have $r_y > r_{\dot{x}}$.

If $M_x = \emptyset$ then $r_y > r_{x^*}$, for all $x^* \in M_x$, is trivially true. Let $z \in M_x$. If $|T_{x\dot{x}}| > |T_{xz}|$ then $r_{\dot{x}} > r_z$ and, by transitivity, $r_y > r_z$. Otherwise, $|T_{x\dot{x}}| = |T_{xz}|$ and so $r_y > r_{\dot{x}}$ implies $r_y > r_z$. Hence, we have shown that $r_y > r_{x^*}$, for all $x^* \in M_x$.

Suppose that $xy \notin E(G)$. We then have $x \in M_y$ and $y \in M_x$. Hence, $r_x \leq r_{y^*}$, for some $y^* \in M_y$, is true simply because $r_x \leq r_x$. Similarly, $r_y \leq r_{x^*}$, for some $x^* \in M_x$, is true.

($\Leftarrow$) Let G have a radius-only representation R'. We may assume that $r'_x \neq r'_y$ for all $x \neq y$. For simplicity, suppose $V(G) = \{1, \ldots, |V(G)|\}$ and R' is indexed such that

$r'_x > r'_y \Leftrightarrow x > y$. Pick some $m > 0$ and assign values to the radii $\{r_x : x \in V(G)\}$ by the formula $r_k = m + k(\frac{m}{|V(G)|})$. Hence, $r'_x > r'_y \Leftrightarrow r_x > r_y$.

Let $\mathcal{T}$ be a star graph with $|V(G)|$ edges. Let T be a tree embedding of $\mathcal{T}$ where the edges of $\mathcal{T}$ are mapped to line segments of length $3m$. Place all neighborhood subtree centers a distance m from the origin of T such that no two neighborhood subtrees are located at the same point of T. Hence, (T, S) is a tolerance-free star NeST representation. Observe that by our assignment of values to the neighborhood subtree radii we have $|T_{xy}| = r_x + r_y - 2m$, for all $x, y \in V(G)$. We also have that $r_x > r_y \Leftrightarrow |T_{xz}| > |T_{yz}|$, for all $x, y, z \in V(G)$.

We will show that (T, S) is a tolerance-free, star NeST representation of G.

Suppose that $xy \in E(G)$. We have that $r_x > r_{y^*}$, for all $y^* \in M_y$ or $r_y > r_{x^*}$, for all $x^* \in M_x$; assume the former without loss of generality. It follows that $|T_{xy}| > |T_{yy^*}|$, for all $y^* \in M_y$. In particular, we have $|T_{xy}| > |T_{y\dot{y}}|$.

Suppose that $xy \notin E(G)$. We have that $r_x \leq r_{y^*}$, for some $y^* \in M_y$, and $r_y \leq r_{x^*}$, for some $x^* \in M_x$. It follows that $|T_{xy}| \leq |T_{yy^*}|$, for some $y^* \in M_y$, and $|T_{xy}| \leq |T_{xx^*}|$, for some $x^* \in M_x$. However, $|T_{x\dot{x}}| \geq |T_{xx^*}|$, for all $x^* \in M_x$, and $|T_{y\dot{y}}| \geq |T_{yy^*}|$, for all $y^* \in M_y$. We conclude that $|T_{xy}| \leq |T_{y\dot{y}}|$ and $|T_{xy}| \leq |T_{x\dot{x}}|$.

Hence, we have constructed a tolerance-free, star NeST representation of G. □

THEOREM 7.6. *If G is a fixed distance NeST graph then G is a proper NeST graph.*

Proof. In the proof of Theorem 7.5, we constructed a tolerance-free, star NeST representation of a fixed distance NeST graph where all neighborhood subtree centers were a distance $2m$ apart. However, the neighborhood subtree radii had respective lower and upper bounds of m and $2m$. Hence, no neighborhood subtree properly contained another neighborhood subtree. Thus the constructed tolerance-free, star NeST representation is proper. □

Monma, Reed and Trotter ask whether threshold tolerance graphs can be characterized as intersection graphs of subtrees in a tree [MRT]. In response, we shall show that threshold tolerance graphs are exactly fixed distance NeST graphs.

DEFINITION 7.7. *G is a* threshold tolerance graph *if there is a pair of sets of numbers, (A, B), where $A = \{a_x : x \in V(G)\}$ and $B = \{b_x : x \in V(G)\}$, such that the following edge condition holds:*

$$xy \in E(G) \quad \Leftrightarrow \quad \begin{cases} a_x > b_y \\ \quad or \\ a_y > b_x. \end{cases}$$

We call (A, B) a *threshold tolerance representation* of G.

We now use a theorem due to Mike Saks (see Theorem 2.5 in [MRT]), which can be restated as follows:

THEOREM [Saks] *A graph G is a threshold tolerance graph if and only if there exists a total order $>$ of $V(G)$ such that*

$$xy \in E(G) \;\; \Leftrightarrow \begin{cases} x > M_y \\ \quad or \\ y > M_x, \end{cases}$$

where $x > M_y$ means that $x > z$, for all $z \in M_y$, similarly for $y > M_x$.

THEOREM 7.8. *G is a fixed distance NeST graph if and only if G is a threshold tolerance graph.*

Proof. By Saks' theorem, it follows that G is threshold tolerance if and only if G admits a radius-only representation. The result follows by Theorem 7.5. □

In [MRT] a polynomial time recognition algorithm is presented for threshold tolerance graphs. This together with Theorem 7.8 implies polynomial recognition of fixed distance NeST graphs.

8. Fixed Tolerance NeST Graphs. The final class of graphs we examine is the class of fixed tolerance NeST graphs, introduced by Bibelnieks and Dearing, and referred to by them as *constant NeST graphs* [BD]. Not surprisingly, the relationship between fixed tolerance NeST graphs and NeST graphs is analogous to the relationship between interval graphs and interval tolerance graphs (see figure 1.1). The consideration of this relationship leads to a characterization of fixed tolerance NeST graphs.

DEFINITION 8.1. *A graph G is a* fixed tolerance NeST graph *if there exists a NeST representation of G where all tolerances are equal. Such a representation is called a* fixed tolerance NeST representation *of G.*

Before reaching our characterization we present a definition and two theorems.

DEFINITION 8.2. *[T] A graph G is an* intersection graph of neighborhood subtrees on a tree*, or simply a* neighborhood subtree intersection graph*, if there exists a neighborhood subtree graph representation (T, S) in which T is an embedded tree and S is a collection of neighborhood subtrees in T, such that the following edge condition holds:*

$$xy \in E(G) \;\; \Leftrightarrow \;\; T_{xy} \neq \emptyset.$$

Bibelnieks and Dearing proved the following theorem which characterizes fixed tolerance NeST graphs as those graphs with neighborhood subtree intersection representations. Hence, such a model is generalized by the incorporation of tolerances, just as interval representations are generalized by interval tolerance representations.

THEOREM 8.3. *[BD] The class of fixed tolerance NeST graphs is exactly the class of neighborhood subtree intersection graphs.*

THEOREM 8.4. *If G is a fixed tolerance NeST graph then there is a neighborhood subtree intersection representation of G in which all neighborhood subtree radii are equal and no neighborhood subtree is properly contained within another.*

Proof. Let (T, S) be a neighborhood subtree intersection representation of the fixed tolerance NeST graph G. Such a representation exists by Theorem 8.3.

We may assume that all neighborhood subtrees in S are standard neighborhood subtrees by application of Lemma 3.4. Hence, all neighborhood subtrees are maximal.

We define a new neighborhood subtree intersection representation (T', S'), where $r = max\{r_x : x \in V(G)\}$, as follows:

- $T' = (\bigcup_{x \in V(G)} L_x) \cup T$ where, for each $x \in V(G)$, L_x is a line segment, arbitrarily long, attached to T at c_x,
- for all $x \in V(G)$, c'_x is located on L_x such that $d(c_x, c'_x) = r - r_x$ and
- $r_x = r$, for all $x \in V(G)$.

Observe that $T'_x \cap T = T_x$, for all $x \in V(G)$. It follows that (T', S') is also a neighborhood subtree intersection representation of G. Furthermore, all neighborhood subtree radii are equal. Finally, it is easily seen that all neighborhood subtrees in S' are maximal, and so, all neighborhood subtrees share the same diameter. This is sufficient to conclude that no neighborhood subtree is properly contained within another neighborhood subtree. □

THEOREM 8.5. *G is a fixed tolerance NeST graph if and only if there is a G-tree (T, S), and a positive constant k, such that*

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq k,$$

where $S = \{c_z : z \in V(G)\}$

Proof. ($\Rightarrow$) Let G have a neighborhood subtree intersection representation (T, S) where all neighborhood subtree radii are equal and no neighborhood subtree is properly contained within another. Lemma 6.3 informs us that, for any neighborhood subtree intersection T_{xy}:

$$|T_{xy}| = \begin{cases} r_x + r_y - d(c_x, c_y), & \text{if } T_{xy} \neq \emptyset \\ 0, & \text{if } T_{xy} = \emptyset. \end{cases}$$

Hence, the edge condition $xy \in E(G) \Leftrightarrow T_{xy} \neq \emptyset$ can be refined to

$$xy \in E(G) \Leftrightarrow r_x + r_y - d(c_x, c_y) \geq 0$$

and since $r_x = r_y = r$ we have

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq 2r.$$

Hence, the G-tree (T, S'), where $S' = \{c_z : z \in V(G)\}$, along with $k = 2r$, satisfies the condition

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq k.$$

($\Leftarrow$) Let (T, S') be a G-tree, where $S' = \{c_z : z \in V(G)\}$, such that

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq k,$$

for some $k > 0$.

We form a neighborhood subtree intersection representation (T, S) by letting, for all $x \in V(G)$, $T_x = T(c_x, r)$ where $r = k/2$. It follows that

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) \leq 2r$$

and so

$$xy \in E(G) \Leftrightarrow T_{xy} \neq \emptyset.$$

Hence, (T, S) is a neighborhood subtree intersection representation of G and so G is a fixed tolerance NeST graph. □

Lemma 8.5 may be restated in the following way: G is a fixed tolerance NeST graph if and only if there exists an embedded tree and a placement of the vertices of G in the embedded tree such that x and y are neighbors in G if and only if they are "close" in the embedded tree.

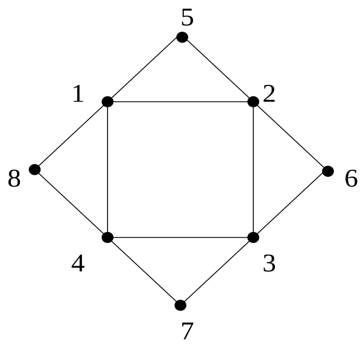

FIG. 4. *The sunflower graph.*

9. The Sunflower Graph. Bibelnieks and Dearing [BD] proved that NeST graphs are weakly triangulated, and so perfect via a result in [Hy]. They further asked whether this inclusion is proper. In this section we show that this is indeed the case, by proving that the graph in figure 4, which we call the *sunflower graph*, is weakly triangulated but not neighborhood subtree tolerance.

DEFINITION 9.1. *A graph isomorphic to* $(\{1,2,3,4\},\{\{1,2\},\{3,4\}\})$ *is called a* $2K_2$.

The $2K_2$ structure of a proper NeST graph, along with our characterization of proper NeST graph's, results in the following structural property:

LEMMA 9.2. *Suppose* G *is a proper NeST graph with* G*-tree representation* (T,S). *Let* $W=\{1,2,3,4\}\subseteq V(G)$ *and* G_W *be a* $2K_2$, *where* $12, 34 \in E(G)$. *It follows that* $P(c_1,c_2)\cap P(c_3,c_4)=\emptyset$ *where* $S=\{c_z : z\in V(G)\}$.

Proof. Since (T,S) is a G-tree representation of G, the following edge condition is satisfied, for all $x,y\in V(G)$ and $c_x, c_y\in S$:

$$xy\in E(G)\Leftrightarrow d(c_x,c_y)<min\{d(c_x,c_{\dot{x}}),d(c_y,c_{\dot{y}})\}.$$

Since 12 and 34 are edges we have that

$$\begin{gathered}(d(c_1,c_2)<d(c_1,c_3)\ \ and\ \ d(c_1,c_2)<d(c_1,c_4))\ \ or\\ (d(c_1,c_2)<d(c_2,c_3)\ \ and\ \ d(c_1,c_2)<d(c_2,c_4))\end{gathered}\tag{2}$$

and

$$\begin{gathered}(d(c_3,c_4)<d(c_1,c_3)\ \ and\ \ d(c_3,c_4)<d(c_2,c_3))\ \ or\\ (d(c_3,c_4)<d(c_1,c_4)\ \ and\ \ d(c_3,c_4)<d(c_2,c_4)).\end{gathered}\tag{3}$$

Without loss of generality, assume that

$$d(c_1,c_2)<d(c_1,c_3)\ \ and\ \ d(c_1,c_2)<d(c_1,c_4).\tag{4}$$

Observe that the two clauses of (3) are symmetric about vertices 3 and 4. Hence, without loss of generality, assume we have $d(c_1,c_2)<d(c_1,c_3)$, $d(c_1,c_2)<d(c_1,c_4)$, $d(c_3,c_4)<d(c_1,c_3)$ and $d(c_3,c_4)<d(c_2,c_3)$.

Consider the path $P(c_2,c_4)$ in T. Let m_{24} denote the midpoint of this path. Let p be the unique intersection point of $P(c_1,c_2)$, $P(c_1,c_4)$ and $P(c_2,c_4)$. Similarly, let q be the unique intersection point of $P(c_2,c_3)$, $P(c_3,c_4)$ and $P(c_2,c_4)$. Since $d(c_1,c_2)<d(c_1,c_4)$, p is closer to c_2 than to c_4. Since $d(c_3,c_4)<d(c_2,c_3)$, q is closer to c_4 then to c_2. Hence, p lies on $P(c_2,m_{24})$ and q lies on $P(c_4,m_{24})$. Furthermore, $P(p,q)>0$. This situation is depicted in figure 5.

It is easily seen that $P(c_1,c_2)\cap P(c_3,c_4)=\emptyset$ in T. □

LEMMA 9.3. *If* G *is a proper NeST graph then there exists a* G*-tree representation* (T,S) *of* G *where* T *is a tree embedding of a tree with no vertices of degree greater than 3.*

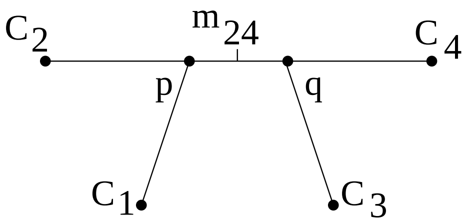

FIG. 5. $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$

Proof. We give a sketch of the proof only; a more detailed proof can be easily derived. Let G have a G-tree representation (T, S) and let T be a tree embedding of $\mathcal{T}$. Let $v \in V(\mathcal{T})$ have degree $m > 3$. We will describe a process by which the degree of v can be reduced to 3. This process can then be applied to all vertices of $\mathcal{T}$, of degree greater than 3, to achieve the desired result.

Let f be the mapping from $\mathcal{T}$ to T. Hence, $f(v)$ is the (unique) intersection of the m curves $f(vv_1)$, $f(vv_2)$, ..., $f(vv_m)$ where $v_1, v_2 \ldots, v_m$ are the neighbors of v in $\mathcal{T}$. We perturb T to obtain a new embedded tree T' by removing the curves $f(vv_m)$ and $f(vv_{m-1})$, replacing them with the curves $f(pv)$, $f(pv_{m-1})$ and $f(pv_m)$ such that

- $f(pv) \cup f(pv_{m-1}) = f(vv_{m-1})$,
- $|f(pv)| = \epsilon$ (specified below) and
- $|f(pv_m)| = |f(vv_m)|$.

Observe that T' is no longer a tree embedding of $\mathcal{T}$. Rather, it is a tree embedding of the tree $\mathcal{T}'$ where $V(\mathcal{T}') = V(\mathcal{T}) \cup \{w\}$, $E(\mathcal{T}') = (E(\mathcal{T}) \cup \{wv, wv_{m-1}, wv\}) - \{vv_{m-1}, vv_m\}$ and $p = f(w)$. Furthermore, the degree of v in $\mathcal{T}'$ is less than the degree of v in $\mathcal{T}$ by 1 (at the cost of creating a degree 3 vertex w).

It remains to be shown that we can chose ϵ so that (T', S) is a G-tree representation of G. We must verify that the edge condition

$$xy \in E(G) \Leftrightarrow d(c_x, c_y) < min\{d(c_x, c_{\dot{x}}), d(c_y, c_{\dot{y}})\}$$

is satisfied by (T', S).

Observe that, for all $x, y \in V(G)$,

$$|d'(x, y) - d(x, y)| \leq \epsilon,$$

where d' measure pathlength in T'.

Let $\epsilon = min\{d(c_x, c_{\dot{x}}) - d(c_x, c_y) : d(c_x, c_{\dot{x}}) > d(c_x, c_y),\ xy \in E(G)\}/3$.

Suppose $xy \in E(G)$. It follows that $d(c_x, c_y) < d(c_x, c_{\dot{x}})$ or $d(c_x, c_y) < d(c_y, c_{\dot{y}})$; assume the former. By definition of ϵ, $d(c_x, c_{\dot{x}}) - d(c_x, c_y) \geq 3\epsilon$ which implies that $d(c_x, c_y) + \epsilon < d(c_x, c_{\dot{x}}) - \epsilon$. Hence, $d'(c_x, c_y) < d'(c_x, c_{\dot{x}})$.

Suppose $xy \notin E(G)$. By definition of $\dot{x}$, $d(c_x, c_{\dot{x}}) \leq d(c_x, c_z)$, for all $z \in M_x$. In particular, $d(c_x, c_{\dot{x}}) \leq d(c_x, c_y)$. Similarly, $d(c_y, c_{\dot{y}}) \leq d(c_x, c_y)$. □

COROLLARY 9.4. *If G is a proper NeST graph with G-tree representation (T, S), where T is a tree embedding of $\mathcal{T}$ and $\mathcal{T}$ has no vertices of degree greater than 3 then, for any $W = \{1, 2, 3, 4\} \subseteq V(G)$, exactly one of the following is true:*

$$P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$$

or

$$P(c_1, c_3) \cap P(c_2, c_4) = \emptyset$$

or

$$P(c_1, c_4) \cap P(c_2, c_3) = \emptyset.$$

Proof. Suppose at least two of the above three equalities are true. Without loss of generality, suppose $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$ and $P(c_1, c_3) \cap P(c_2, c_4) = \emptyset$. Since c_1 and c_3 are connected in T, there must be a unique path P in T from $P(c_1, c_2)$ to $P(c_3, c_4)$. Hence, $P \subseteq P(c_1, c_3)$. However, it must be that $P \subseteq P(c_2, c_4)$ which implies that $P(c_1, c_3) \cap P(c_2, c_4) \neq \emptyset$, which contradicts our assumption.

Suppose none of the above equalities hold. Consider the unique point p which is the intersection of $P(c_1, c_2)$, $P(c_1, c_3)$ and $P(c_2, c_3)$. If $p \notin P(c_3, c_4)$ then $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$, which is a contradiction. Similarly, if $p \notin P(c_2, c_4)$ then $P(c_1, c_3) \cap P(c_2, c_4) = \emptyset$ and if $p \notin P(c_1, c_4)$ then $P(c_1, c_4) \cap P(c_2, c_3) = \emptyset$. It must be that $p \in \cap_{x,y \in W} P(c_x, c_y)$. It follows that $p = f(v)$ for some $v \in V(\mathcal{T})$ where $deg(v) \geq 4$ and f is the mapping from $\mathcal{T}$ to T. This contradicts the supposition that $\mathcal{T}$ has no vertices of degree greater than 3. □

For brevity, we will denote $P(c_1, c_2) \cap P(c_3, c_4) = \emptyset$, in some embedded tree, by $12 \bullet 34$.

LEMMA 9.5. *The sunflower graph is not a proper NeST graph.*

Proof. Let G be the sunflower graph and assume that G has a G-tree representation (T, S) where T is a tree embedding of a tree $\mathcal{T}$ with no vertices of degree greater than 3. Before performing a case by case analysis we make two observations. Firstly, Corollary 9.4 informs us that exactly one of $12 \bullet 34$, $13 \bullet 24$ and $14 \bullet 23$ is true in T. Observe that, with respect to G, the first and last of these three cases are symmetric. Hence, we address the first two cases only in our analysis.

Secondly, observe that there are eight $2K_2$'s in G. These are the subgraphs induced by the vertex sets $\{1, 3, 5, 7\}$, $\{1, 3, 5, 6\}$, $\{1, 3, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{2, 4, 5, 8\}$,

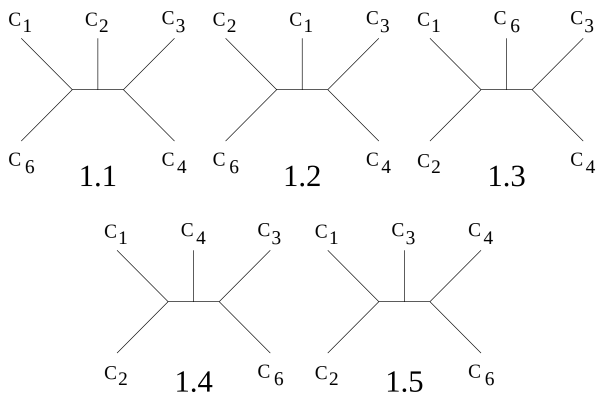

FIG. 6. *Orientations of* H_1

$\{2,4,5,7\}$, $\{2,4,6,8\}$ and $\{2,4,6,7\}$. By Lemma 9.2, we can conclude that:

(5) $15\bullet37$
(6) $15\bullet36$
(7) $18\bullet37$
(8) $18\bullet36$
(9) $25\bullet48$
(10) $25\bullet47$
(11) $26\bullet48$
(12) $26\bullet47$

We now begin a case by case analysis:
Case 1: Suppose that $12\bullet34$. Let H_1 be the subtree of T formed by the union of all pairwise paths between the neighborhood centers associated with vertices $1,2,3,4$ and 6:

$$H_1 = \bigcup_{x,y\in\{1,2,3,4,6\}} P(c_x, c_y)$$

H_1 has five nonisomorphic structures which appear in figure 6.

Observe that structures 1.1 and 1.5, as well as 1.2 and 1.4, are symmetric with respect to G. Hence, we will address structures 1.1, 1.2 and 1.3 only. Suppose that H_1 is oriented as in 1.1. (9) implies that we have $28\bullet46$. However, this contradicts (12) by Corollary 9.4. Suppose that H_1 is oriented as in 1.2. (7) and (10) imply $16\bullet38$. This contradicts (9) by Corollary 9.4. Finally, suppose that H_1 is oriented as in 1.3. (9) implies $28\bullet46$. This contradicts (12) by Corollary 9.4. Thus, $12\bullet34$ is not possible.

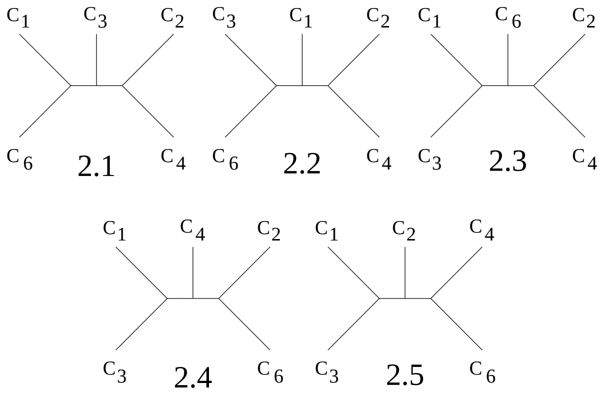

FIG. 7. *Orientations of H_2*

Case 2: Suppose that 13•24. Let H_2 be the subtree of T formed by the union of all pairwise paths between the neighborhood centers associated with vertices $1, 2, 3, 4$ and 6. H_2 has five nonisomorphic structures which appear in figure 7.

Observe that structures 2.1 and 2.5, as well as 2.2 and 2.4, are symmetric with respect to G. Hence we will address structures 2.1, 2.2 and 2.3 only. Suppose that H_2 is oriented as in 2.1. (9) implies 24•68. However, this contradicts (12) by Corollary 9.4. Suppose that H_2 is oriented as in 2.2. (12) and (8) imply 24•67. This contradicts (12) by Corollary 9.4. Finally, suppose that H_2 is oriented as in 2.3. (9) implies 24•68. This contradicts (12) by Corollary 9.4. Thus, 13•24 is not possible.

Since both cases result in contradictions, G is not a proper NeST graph. □

LEMMA 9.6. *If G is a NeST graph, but not a proper NeST graph, then there exists $x, y \in V(G)$ such that $xy \notin E(G)$ and $N_x \subseteq N_y$.*

Proof. Let G be a NeST graph but not a proper NeST graph. Let G have a tolerance-free NeST representation (T, S). Define, for each $x \in V(G)$, $P_x = \{z \in V(G) : T_x \subset T_z\}$. Since (T, S) is not a proper NeST representation it follows that, for some $x \in V(G)$, $P_x \neq \emptyset$.

Part 1: Suppose that, for all x where $P_x \neq \emptyset$, we have $P_x \subseteq N_x$. Suppose $P_w \neq \emptyset$. We derive a new tolerance-free NeST representation (T', S') for G as follows:

- $T' = T \cup L$ where L is an arbitrarily long line segment attached to T at c_w
- $S' = S$, except that c'_w is located on L such that $d(c_w, c'_w) = \delta$ where $\delta > max\{r_z - d(c_w, c_z) : z \in P_w\}$ and $r'_w = r_w + \delta$.

Observe that for $y \notin P_w$, $|T'_{wy}| = |T_{wy}|$. For $y \in P_w$, $|T'_{wy}| > |T_{wy}|$. Yet $wy \in E(G)$ by assumption. Hence, the edge condition

$$xy \in E(G) \Leftrightarrow |T'_{xy}| > min\{|T_{x\dot{x}}|, |T_{y\dot{y}}|\}$$

is satisfied. Furthermore, in (T', S') we have that $P'_w = \emptyset$. We can repeat this process, for all w such that $P_w \neq \emptyset$, to achieve a proper NeST representation for G. This contradicts the hypothesis that G is not a proper NeST graph. We conclude that there exists $x, y \in V(G)$ such that $y \in P_x$ and $xy \notin E(G)$.

Part 2: Let $x, y \in V(G)$ such that $y \in P_x$ and $xy \notin E(G)$. Let $xz \in E(G)$. We have that $|T_{x\dot{x}}| = |T_{xy}| = |T_x| \geq |T_{xz}|$ which implies $|T_{xz}| > |T_{z\dot{z}}|$. Since $T_x \subset T_y$ we have $|T_{yz}| \geq |T_{xz}| > |T_{z\dot{z}}|$. Hence, $yz \in E(G)$. We conclude that $N_x \subseteq N_y$. □

LEMMA 9.7. *The sunflower graph is not a NeST graph.*

Proof. Let G be the sunflower graph. Suppose that G is a NeST graph. Lemma 9.5 informs us that G is not a proper NeST graph. By Lemma 9.6, there exists $x, y \in V(G)$ such that $xy \notin E(G)$ and $N_x \subset N_y$. A simple inspection of G reveals that this is not true. By contradiction, G is not a NeST graph. □

LEMMA 9.8. *The sunflower graph is weakly triangulated.*

Proof. The sunflower graph does not contain a chordless cycle of size five or more; neither does its complement. By definition, the sunflower graph is weakly triangulated. □

LEMMA 9.9. *The class of NeST graphs is a proper subclass of weakly triangulated graphs.*

Proof. [BD] prove that NeST graphs are weakly triangulated. Lemmas 9.7 and 9.8 show this inclusion is proper. □

10. Conclusions and Open Problems. A fundamental result of this paper is a tolerance-free characterization of NeST graphs. Based on this characterization, we obtain several results concerning NeST graphs and various subclasses.

We consider four subclasses of NeST graphs, namely,

- *fixed diameter NeST graphs*, in which all neighborhood subtrees have the same diameter,
- *proper NeST graphs*, in which no neighborhood subtree is properly contained in another,
- *fixed distance NeST graphs*, in which neighborhood subtree centers are equidistant and
- *fixed tolerance NeST graphs*, in which all neighborhood subtrees have the same tolerance,

The first and third of these are new subclasses.

Fixed diameter graphs are to NeST graphs as unit interval graphs are to interval graphs, and as unit interval tolerance graphs are to interval tolerance graphs. For interval and interval tolerance graphs it has been asked whether the respective proper and unit subclasses are equivalent. We ask the analogous question of NeST graphs and answer in the affirmative. That is, proper NeST graphs are equivalent to unit NeST graphs. Consequently, this yields a characterization of proper NeST graphs

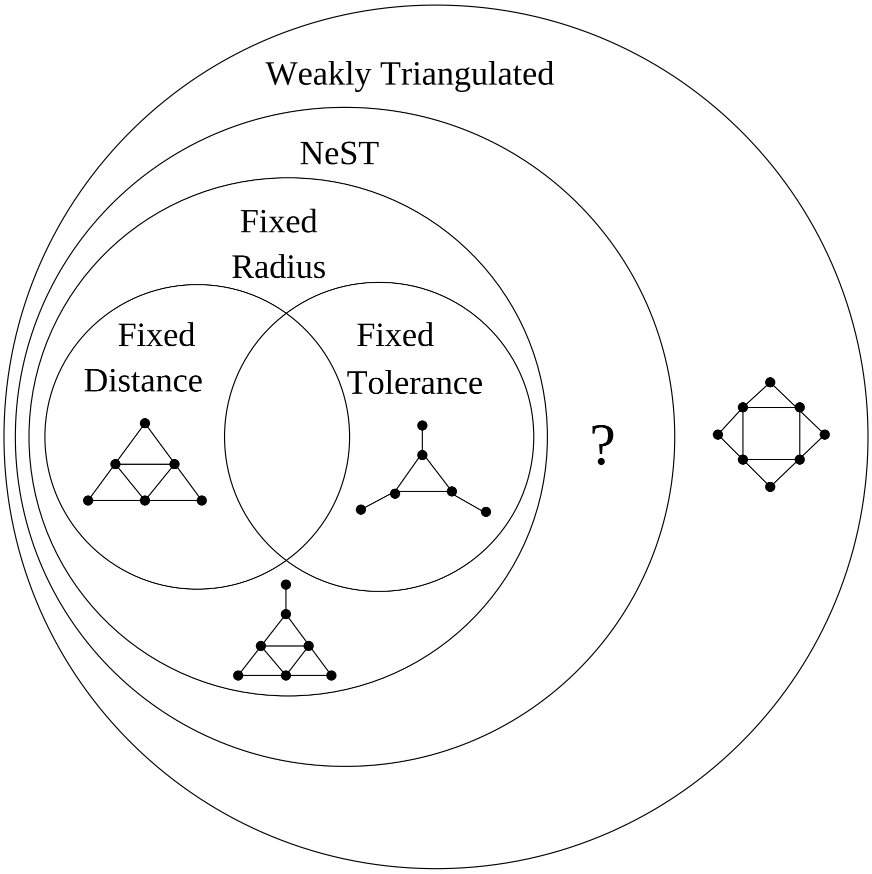

FIG. 8. *Subclasses and superclasses of NeST graphs*

that is both radius-free and tolerance-free.

We obtain a characterization of fixed distance NeST graphs that depends only on the radii of the neighborhood subtrees, namely, the characterization is both tolerance-free and center-free. Via this characterization we show that the class of fixed distance NeST graphs is exactly the class of threshold tolerance graphs. This responds to a question asked by Monma, Reed and Trotter. The equivalence of the two classes implies polynomial recognition of fixed distance NeST graphs by the threshold tolerance graph recognition algorithm presented in [MRT].

Fixed tolerance NeST graphs are exactly the intersection graphs of neighborhood subtrees of a tree [T]. We obtain a characterization of fixed tolerance NeST graphs that depends only on the location of the neighborhood subtree centers within the embedded tree.

Finally, in response to a question of Bibelnieks and Dearing, we prove that NeST graphs are properly included in the class of weakly triangulated graphs.

Some open problems that remain are:

- are all NeST graphs proper NeST graphs?
- are all triangulated graphs NeST graphs?
- what is the recognition complexity of fixed tolerance, proper NeST and NeST graphs?

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December, 1993

Report TR-CS-04-93