Discs in Unbreakable Graphs*

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Abstract. We show that every vertex in an unbreakable graph is in a disc, where a disc is a chordless cycle, or the complement of a chordless cycle, with at least five vertices. A corollary is that every vertex in a minimal imperfect graph is in a disc.

Key Words. perfect, minimal imperfect, Berge, star cutset, unbreakable, disc, weakly chordal, weakly triangulated

Claude Berge [B] introduced the notion of a *perfect graph* as a graph for which, for every (vertex) induced subgraph, the chromatic number is equal to the size of a largest clique. A *disc* is a chordless cycle, or the complement of a chordless cycle, with at least five vertices. Berge and Alain Ghouila-Houri observed that every odd disc is minimal imperfect; this led Berge to pose the following *Strong Perfect Graph Conjecture* (see [BC] for more background).

SPG Conjecture. *A graph is minimal imperfect if and only if it is an odd disc.*

A graph with no odd disc is usually called a *Berge graph.* In light of the Berge/Ghouila-Houri observation, to prove the SPGC it is sufficient to prove that Berge graphs are perfect.

Vašek Chvátal [C] introduced the notion of a *star cutset* as a non-empty subset C of the vertices of a graph G, for which $G - C$ is disconnected and some vertex of C is adjacent to all other vertices of C. A graph is *unbreakable* if it has at least three vertices and neither the graph nor its complement has a star cutset. Chvátal proved the following.

Theorem [C]. *Every minimal imperfect graph is unbreakable.*

Hence unbreakable graphs are of interest for the following reason: to prove the SPGC it is sufficient to prove that unbreakable Berge graphs are perfect.

We [H] introduced the notion of a *weakly triangulated graph* as a graph with no discs, and proved the (contrapositive of) the following theorem. This together with

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Chvátal's theorem implies that every weakly triangulated graph is perfect.

Theorem [H]. *Every unbreakable graph contains some disc.*

The purpose of this paper is to establish the following theorem, which is a strengthening of the previous theorem.

Theorem 1. *In an unbreakable graph, every vertex is in a disc.*

Corollary. *In a minimal imperfect graph, every vertex is in a disc.*

Before proving the theorem, we state and prove a lemma; to state the lemma, we need two standard definitions. Given a vertex v in a graph G, the *neighbourhood* $N_G(v)$ and the *non-neighbourhood* $M_G(v)$ denote the vertices of $G - v$ that are respectively adjacent and non-adjacent to v. We shall write $N(v)$ and $M(v)$ when the graph G is clear from the context. Also, in the proofs we shall use the terms *sees* and *misses* in place of *is adjacent to* and *is non-adjacent to* respectively.

Lemma 1. *Let v be a vertex in a graph G, and suppose that*

- *v is on no disc,*
- *$N(v)$ and $M(v)$ are non-empty,*
- *every vertex of $N(v)$ is adjacent to some vertex of $M(v)$, and*
- *every vertex of $M(v)$ is non-adjacent to some vertex of $N(v)$.*

Then either G or $\overline{G}$ has a star cutset.

Proof. Let Q be the set of vertices of $N(v)$ that miss at least one vertex of $M(v)$; since $M(v)$ is non-empty and each vertex of $M(v)$ misses some vertex of $N(v)$, the set Q is non-empty. For each vertex q of Q, let A_q be a smallest (namely, with fewest vertices) non-empty component of $G[M(q) \cap M(v)]$. Let q_0 be a vertex for which A_q is smallest, over all vertices q in Q. Let $A = A_{q_0}$.

Case 1. *Every vertex of $M(q_0) \cap N(v)$ misses every vertex of A.*
Since A is a component of $G[M(q_0) \cap M(v)]$, no vertex of A sees any vertex in $(M(q_0) \cap M(v)) - A$; by Case 1 assumption, no vertex of A sees any vertex in $M(q_0) \cap N(v)$. Thus no vertex of A sees q_0 or any vertex of $M(q_0) - A$. Let $C = q_0 \cup N(q_0) - v$; now A is a component of $G - C$, and C is a star cutset of G that separates A from the component of $G - C$ containing v. Thus the lemma holds in Case 1.

Case 2. *Some vertex r of $M(q_0) \cap N(v)$ sees some vertex a^* of A.*
Extend the definition of *neighbourhood* to include sets of vertices, by defining $N(S) = (\bigcup_{v \in S} N(v)) - S$ for any subset S of vertices; in other words, the neighbourhood of S is defined as all vertices not in S that are adjacent to at least one vertex of S.

Let $B = N(A) \cap M(v)$. Since A is a component of $G[M(q_0) \cap M(v)]$, no vertex of A sees any vertex in $(M(q_0) \cap M(v)) - A$, so $N(A) \cap M(v) = B \subseteq N(q_0) \cap M(v)$, and so $B \subseteq N(q_0)$.

Claim 1. *Vertex r sees every vertex of B.*
To prove this claim, argue by contradiction: suppose that r misses some vertex b in B. Since $B \subseteq N(A)$ and $G[A]$ is connected, it follows that $G[b \cup A]$ is connected. Thus, since r sees at least one vertex of A and misses b, there is a path in $G[b \cup A]$ from b to a vertex that sees r. Let $(b, a_1, a_2, \ldots, a_t)$ be any shortest such path; possibly $t = 1$. Since $B \subseteq N(q_0)$, q_0 sees b. Thus $(v, q_0, b, a_1, a_2, \ldots, a_t, r)$ is a chordless cycle with at least five vertices, contradicting the assumption that v is on no disc. This completes the proof of Claim 1.

Claim 2. *Vertex r sees every vertex of A.*
Again, argue by contradiction: suppose that r misses some vertex a' of A. Then r must be in Q; say $r = q_1$. Let A' be the component of $G[M(r) \cap M(v)]$ that contains a'. Since r sees all of $B = N(A) \cap M(v)$, A' is a subset of A; since r sees a^*, A' is a proper subset of A. But then $|A_r = A_{q_1}| \leq |A'| < |A = A_{q_0}|$, contradicting our choice of q_0. Thus Claim 2 holds.

We now shift our attention from G to $\overline{G}$. By repeating arguments similar to those made so far, we shall find a star cutset in $\overline{G}$. For each vertex a of A, let Z_a be the component of $\overline{G}[M_{\overline{G}}(a) \cap M_{\overline{G}}(v)]$ which contains r. Z_a is well defined, since r is in $M_{\overline{G}}(a) \cap M_{\overline{G}}(v)$ for all a in A. Let a_0 be a vertex in A for which Z_a is smallest. Let $Z = Z_{a_0}$.

Let $Y = N_{\overline{G}}(Z) \cap M_{\overline{G}}(v)$. Since Z is a component of $\overline{G}[M_{\overline{G}}(a_0) \cap M_{\overline{G}}(v)]$, no vertex of Z sees any vertex in $(M_{\overline{G}}(a_0) \cap M_{\overline{G}}(v)) - Z$, so $Y \subseteq N_{\overline{G}}(a_0) \cap M_{\overline{G}}(v)$, so $Y \subseteq N_{\overline{G}}(a_0)$.

Claim 3. *In $\overline{G}$, every vertex of $M_{\overline{G}}(a_0) \cap N_{\overline{G}}(v)$ misses every vertex of Z.*
Argue by contradiction: suppose that in $\overline{G}$, some vertex d in $M_{\overline{G}}(a_0) \cap N_{\overline{G}}(v)$ sees some vertex of Z. Then in $\overline{G}$, d sees all Y (otherwise, arguing as in Claim 1, we conclude that v is on a chordless cycle $(v, a_0, y, z_1, z_2, \ldots, z_t, d)$ in $\overline{G}$ with at least five vertices, contradiction).

Next, observe that d is in $M_G(v) \cap N_G(a_0)$, so d is in $A \cup B$. Also, q_0 is in $N_{\overline{G}}(a_0) \cap N_{\overline{G}}(r)$, so q_0 is in Y, so d is in $N_{\overline{G}}(q_0)$. Since $B \subseteq N_G(q_0)$, d is not in B, so d is in $A \cup B - B = A$. Say $d = a_1$.

Now in $\overline{G}$, d sees every vertex of Z (otherwise, arguing as in Claim 2, we conclude that $|Z_{a_1}| < |Z_{a_0}|$, contradicting the choice of a_0). Since r is in Z, it follows that d is in $N_{\overline{G}}(r)$, contradicting the fact that d is in A and so in $N_G(r)$. Thus Claim 3 holds.

Finally, arguing as in Case 1, Claim 3 implies that $C = a_0 \cup N_{\overline{G}}(a_0) - v$ is a star cutset of $\overline{G}$, separating Z from the component of $\overline{G} - C$ containing v. This concludes the proof of the lemma. $\square$

Proof of Theorem 1. Let G be a graph with a vertex v on no disc. We shall show that G is not unbreakable. If G has fewer than three vertices, we are done, so assume G has at least three vertices. We shall show that G or $\overline{G}$ has a star cutset.

If $N(v)$ is empty, then any vertex of $M(v)$ is a star cutset of G; if $M(v)$ is empty, then any vertex of $N(v)$ is a star cutset of $\overline{G}$. Suppose then that $M(v)$ and $N(v)$ are both non-empty. If some vertex q of $N(v)$ sees no vertex of $M(v)$, then $v \cup N(v) - q$ is a star cutset of G. If some vertex p of $M(v)$ sees all vertices of $N(v)$, then $v \cup N_{\overline{G}}(v) - p$ is a star cutset of $\overline{G}$.

Suppose then that every vertex of $N(v)$ sees some vertex of $M(v)$, and every vertex of $M(v)$ misses some vertex of $N(v)$. Now the hypotheses of Lemma 1 hold, and G or $\overline{G}$ has a star cutset. $\square$

In closing, we point out that a shorter proof of the Corollary can be obtained by proving an analogue of Theorem 1 that holds only for Berge graphs. The following lemma can be used to prove such a theorem.

Lemma 2. *Let v be a vertex of a Berge graph G, and suppose that*

- *v is on no disc,*
- *P and Q are non-empty subsets of $M(v)$ and $N(v)$ respectively, and*
- *$G[P]$ and $\overline{G}[Q]$ are connected.*

Then either

- *some vertex in Q is adjacent to no vertex in P, or*
- *some vertex in P is adjacent to every vertex in Q.*

Proof. If some vertex in Q sees no vertex in P we are done, so assume that

$$\textit{every vertex in } Q \textit{ sees some vertex in } P. \qquad (*)$$

We shall show that some vertex in P sees every vertex in Q.

Case 1. *$G[P]$ and $\overline{G}[Q]$ are each a chordless path.*
Argue by induction on $|P|$ and $|Q|$. If $|P| = 1$ or $|Q| = 1$ the desired conclusion follows from $(*)$. Suppose then that $|P| = k \geq 2$ and $|Q| = t \geq 2$ and that the lemma holds in Case 1 for smaller values of k or t. Let $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_t)$ be the chordless paths in G and $\overline{G}$ induced by P and Q respectively.

Observe that $(*)$ together with the inductive assumption implies that some vertex p^+ of P sees every vertex of $Q' = Q - \{q_1\}$. If p^+ also sees q_1 then we are done, so suppose p^+ misses q_1. By $(*)$ there is some vertex in P that sees q_1; let P' be a shortest subpath of P from p^+ to a vertex p^- that sees q_1. If P' is a proper subpath of P then we are done by induction on k, since every vertex of Q sees some vertex of P'. So assume that $P' = P$. By relabelling the vertices of P if necessary, we may assume that $p^+ = p_1$ and $p^- = p_k$. Thus

$$p_1 \textit{ misses } q_1 \textit{ and sees each of } \{q_2, \ldots, q_t\}, \textit{ and}$$
$$q_1 \textit{ sees } p_k \textit{ and misses each of } \{p_{k-1}, \ldots, p_1\}.$$

By $(*)$ and the inductive assumption, some vertex of P sees every vertex of $Q'' = Q - \{q_t\}$. Since each vertex of $\{p_1, \ldots, p_{k-1}\}$ misses q_1, it follows that

$$p_k \textit{ sees each of } \{q_1, \ldots, q_{t-1}\}.$$

To complete the proof of Case 1, we shall show that p_k sees q_t, and so all of Q. Argue by contradiction: assume that

$$p_k \textit{ misses } q_t.$$

By the inductive assumption with $P' = P - \{p_1\}$ in place of P, it follows that some vertex of Q sees no vertex of P', or some vertex of P' sees every vertex of Q. The latter is not possible, since each vertex of $P' - \{p_k\}$ misses q_1 and p_k misses q_t. Thus the former holds and since p_k sees each vertex of $Q - \{q_t\}$ it follows that

$$q_t \textit{ sees no vertex of } \{p_2, \ldots, p_k\}.$$

If $t = 2$ then $\{v, q_t, p_1, \ldots, p_k, q_1\}$ induces a chordless cycle in G, contradicting the assumption that v is on no disc. If $k = 2$ then $\{v, p_1, q_1, \ldots, q_t, p_k\}$ induces a chordless cycle in $\overline{G}$, contradicting the assumption that v is on no disc. So assume that $t \geq 3$ and $k \geq 3$.

Let $Q'' = \{q_2, \ldots, q_{t-1}\}$ and $P'' = \{p_2, \ldots, p_{k-1}\}$. By inductive assumption,

$$\textit{some vertex of } Q'' \textit{ sees no vertex of } P'', \textit{ or}$$
$$\textit{some vertex of } P'' \textit{ sees every vertex of } Q''.$$

Suppose that the former holds. Pick any q_s in Q'' that misses all of P''. Then $\{p_1, \ldots, p_k, q_s\}$ and $\{p_1, \ldots, p_k, q_1, q_t\}$ each induce a chordless cycle in G, of sizes $k+1$ and $k+2$ respectively, contradicting the assumption that G is Berge.

Suppose that the latter holds. Pick any p_j in P'' that sees all of Q''. Then $\{q_1, \ldots, q_t, p_j\}$ and $\{q_1, \ldots, q_t, p_1, p_k\}$ each induce a chordless cycle in $\overline{G}$, of sizes $k+1$ and $k+2$ respectively, contradicting the assumption that G is Berge. This concludes the proof of Case 1.

Case 2. *$G[P]$ and $\overline{G}[Q]$ are arbitrary connected graphs.*
Assuming $(*)$, we wish to show that some vertex of P sees every vertex of Q. The proof is by induction on $|Q|$. If $|Q| = 1$ the desired conclusion follows from $(*)$. Assume then that $t = |Q| \geq 2$. Let q_1 be an end vertex of any longest chordless path of $\overline{G}[Q]$. Since $\overline{G}[Q - \{q_1\}]$ is connected, it follows by inductive assumption that some vertex p^+ in P sees every vertex of $Q - \{q_1\}$. If p^+ sees q_1 we are done, so suppose

that p^+ misses q_1.

Let $P' = (p_1, \ldots, p_k)$ be a shortest chordless path in $G[P]$ from $p^+ = p_1$ to a vertex p_k that sees q_1; the existence of p_k is guaranteed by $(*)$. To conclude, we show that p_k sees every vertex of Q. To see why, let q_r be any vertex of $Q - \{q_1\}$. Since $\overline{G}[Q]$ is connected, there is a chordless path $Q' = (q_1, \ldots, q_r)$ in $\overline{G}[Q]$. By Case 1, some vertex of Q' sees no vertex of P' or some vertex of P' sees every vertex of Q'. The former cannot hold, since q_1 sees p_k and every vertex of $Q' - \{q_1\}$ sees p_1. Thus the latter holds. Since every vertex of $\{p_1, \ldots, p_{k-1}\}$ misses q_1, it must be p_k that sees all of Q'. In particular, p_k sees q_r. This concludes the proof of Case 2, and of the lemma. $\square$

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