Generating Weakly Triangulated Graphs*

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Abstract. We show that a graph is weakly triangulated, or weakly chordal, if and only if it can be generated by starting with a graph with no edges, and repeatedly adding an edge, so that the new edge is not the middle edge of any chordless path with four vertices.

This is a corollary of results due to Sritharan and Spinrad, and Hayward, Hoàng and Maffray, and a natural analogue of a theorem due to Fulkerson and Gross, which states that a graph is triangulated, or chordal, if and only if it can be generated by starting with a graph with no vertices, and repeatedly adding a vertex, so that the new vertex is not the middle vertex of any chordless path with three vertices.

Our result answers the question of whether there exists a composition scheme that generates exactly the class of weakly triangulated graphs.

Key Words. weakly triangulated, weakly chordal, triangulated, chordal, graph composition scheme, graph generation scheme, chordless path, P_4, two-pair

A graph is *triangulated*, or *chordal*, if it does not contain a chordless cycle with four or more vertices. The following well-known result of Fulkerson and Gross, via a theorem of Dirac, gives a way to build up triangulated graphs from the empty graph:

Theorem 1. **[FG] [D]** *A graph is triangulated if and only if it can be generated in the following manner:*
- *Start with a graph G_0 with no vertices.*
- *Repeatedly add a vertex v_j to G_{j-1} to create the graph G_j, such that v_j is not the middle vertex of any P_3 of G_j.*

A graph is *weakly triangulated*, or *weakly chordal*, if neither the graph nor its complement contains a chordless cycle with five or more vertices. The class of weakly triangulated graphs, introduced in [H], contains the class of triangulated graphs, and also the class of complements of triangulated graphs. Hayward, Hoàng and Maffray showed:

Theorem 2. **[HHM]** *Every weakly triangulated graph either is a clique or has a pair of vertices $\{v, w\}$, such that the removal of their common neighbors leaves them in different components.*

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Such a pair of vertices $\{v, w\}$ is called a *two-pair*, from the fact that any chordless path between them has exactly two edges. Sritharan and Spinrad showed:

Theorem 3. [**SS**] *Let G be an arbitrary graph with two-pair $\{x, y\}$. Then the graph obtained by adding the edge (x, y) to G is weakly triangulated if and only if G is weakly triangulated.*

In this note we obtain, as a corollary of Theorems 2 and 3, an analogue of Theorem 1 for weakly triangulated graphs. This answers the question of whether there exists a well-defined composition scheme that generates exactly the class of weakly triangulated graphs.

Theorem 4. *A graph is weakly triangulated if and only if it can be generated in the following manner:*
- *Start with a graph G_0 with no edges.*
- *Repeatedly add an edge e_j to G_{j-1} to create the graph G_j, such that e_j is not the middle edge of any P_4 of G_j.*

Proof of Theorem 4. A P_4 is a chordless path with four vertices. Call an edge of a graph *peripheral* if it is not the middle edge of any P_4. Call a total order $(e_1, e_2, \ldots, e_m)$ of the m edges of a graph $G = (V, E)$ a *peripheral edge order* if, for $1 \leq j \leq m$, the edge e_j is peripheral in the graph $G_j = (V, E_j)$, where $E_j = \{e_1, e_2, \ldots, e_j\}$. The theorem is equivalent to the following.

(1) *A graph is weakly triangulated if and only if it admits a peripheral edge order.*

Theorems 2 and 3 yield a recognition algorithm for weakly triangulated graphs [SS]: start with an arbitrary graph; while there is some two-pair, add an edge between the two-pair. The original graph is weakly triangulated if and only if the final graph is a clique.

Call a total order $(e_1, e_2, \ldots, e_t)$ of the t non-edges of a graph $G = (V, E)$ a *two-pair non-edge order* if, for $1 \leq j \leq t$, the non-edge e_j is a two-pair in the graph $G_j = (V, \overline{E_j})$, where $\overline{E_j} = \{(x, y) \mid x, y \in V, x \neq y, \text{ and } \{x, y\} \notin \{e_1, e_2, \ldots, e_j\}\}$. By ordering two-pairs in the reverse order in which they are encountered in the algorithm of Sritharan and Spinrad, we obtain the following.

(2) *A graph is weakly triangulated if and only if it admits a two-pair non-edge order.*

To complete the proof of our theorem, we shall prove that (1) and (2) are equivalent, via the following result.

$$\text{(3)} \qquad \begin{array}{c} (e_1, e_2, \ldots, e_m) \textit{ is a peripheral edge order of the graph } G \\ \textit{if and only if} \\ (e_1, e_2, \ldots, e_m) \textit{ is a two-pair non-edge order of the graph } \overline{G}. \end{array}$$

($\Leftarrow$) This follows immediately by observing that if $\{x, y\}$ is a two-pair in an arbitrary graph $H = \overline{G}$, then (x, y) must be a peripheral edge of $\overline{H} = G$. To see this, argue the contrapositive: if (x, y) is not peripheral in $\overline{H}$, then x and y are the end vertices of some P_4 in H, and there is a chordless xy-path in H with three edges.

($\Rightarrow$) This is not immediate, as the converse of the preceding observation is in general not true: (x, y) may be peripheral in $\overline{H}$ when $\{x, y\}$ is not a two-pair in H. We need to use induction on the number of edges. Let $(e_1, \ldots, e_m)$ be a peripheral edge order of G, and assume that ($\Rightarrow$) holds for any number of edges less than m.

Since $(e_1, \ldots, e_{m-1})$ is a peripheral edge order of $G_{m-1} = (V, E_{m-1})$, the inductive hypothesis implies that $(e_1, \ldots, e_{m-1})$ is a two-pair non-edge order of $\overline{G}_{m-1}$. By (2) it follows that $\overline{G}_{m-1}$ is weakly triangulated. This together with the fact that e_m is peripheral in $G_m = G$ implies that e_m is a two-pair in $\overline{G}_m = \overline{G}$.

To see why this last conclusion holds, let S be the vertex set of any chordless xy-path P in $\overline{G}_m$. Observe that P has at most three edges, for otherwise S induces a chordless cycle in $\overline{G}_{m-1}$ with at least five vertices, contradicting the assumption that $\overline{G}_{m-1}$ is weakly triangulated. Also, P cannot have exactly three edges, for in this case S induces a P_4 in $\overline{G}_m$ and G_m, contradicting the assumption that e_m is peripheral in G_m. Finally, P cannot have exactly one edge, since e_m is an edge of G_m and thus a non-edge of $\overline{G}_m$. It follows that e_m is a two-pair in $\overline{G}_m$. This concludes the proof of (3) and of the theorem. □

As a parenthetic comment, observe that the property of having a peripheral edge is weaker than the property of admitting a peripheral edge order, and holds for a larger class of graphs than weakly triangulated graphs. Theorem 5 was conjectured by Sritharan and proved recently by Chvátal and Rusu.

Theorem 5. [CR] *Let G be any graph with no chordless cycle with five or more vertices. Then either G has no edges, or G has some edge that is not the middle edge of any P_4.*

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