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Probing the 4-3-2 Edge Template in Hex

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Abstract. For the game of Hex, we find conditions under which moves into a 4-3-2 edge template are provably inferior.

1 Introduction

Hex, the two-player board game invented independently by Piet Hein [8] and John Nash [4, 11, 12] in the 1940s, is played on a four-sided grid of hexagonal cells. In alternating turns, each player colors an uncolored, or empty, cell with her color (or, if each player has a set of colored stones, by placing a stone of her color on an empty cell). A player wins by connecting her two sides via a set of cells that have her[1] color, as shown in Fig. 1. For more on Hex, see Ryan Hayward and Jack van Rijswijck's paper, Thomas Maarup's webpage, Jack van Rijswijck's webpage, or Cameron Browne's book [3, 6, 10, 13].

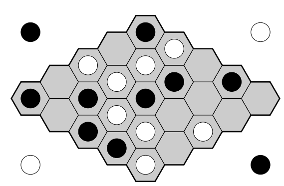

Fig. 1. A Hex board state with a winning White connection.

Given a player P (in this paper, B for Black or W for White) and an empty cell c of a board state S, $S + P[c]$ denotes the board state obtained from S by P-coloring c, namely, by coloring c with P's color. See Fig. 2. We denote the opponent of P by $\overline{P}$.

A *game state* $P(S)$ specifies a board position S and the player to move P. With respect to a player P and game states $Q(S_1)$ and $Q(S_2)$, where Q can be P or $\overline{P}$, we write $Q(S_1) \geq_P Q(S_2)$ if $Q(S_1)$ is at least as good for P as $Q(S_2)$ in the following sense: P has a winning strategy for $Q(S_1)$ if P has a winning strategy for $Q(S_2)$. In this case, we say that $Q(S_1)$ P-*dominates* $Q(S_2)$.

[1] For brevity, we use 'she' and 'her' whenever 'he or she' or 'her and his' are meant.

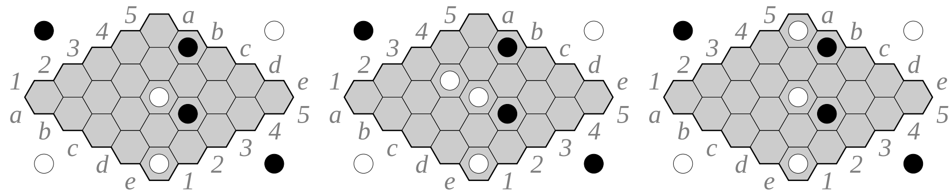

Fig. 2. State S (left), state $T = S + W[b3]$, and state $U = S + W[a5]$.

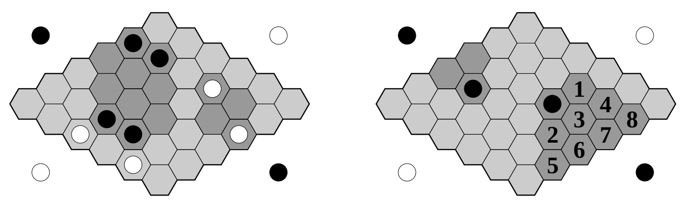

Fig. 3. Two virtual connections (left). A Black edge bridge and 4-3-2 (right).

Consider for example states T and U in Fig. 2. As the reader can check, Black has a winning move in T but no winning move in U, so $B(T) \geq_B B(U)$. Draws are not possible in Hex, so White has a second-player winning strategy for U but not for T, so $B(U) \geq_W B(T)$.

We extend this terminology as follows: with respect to a player P and board states S_1 and S_2, we write $S_1 \geq_P S_2$ if $P(S_1) \geq_P P(S_2)$ and $\overline{P}(S_1) \geq_P \overline{P}(S_2)$. In this case, we say that S_1 *P-dominates* S_2.

With respect to a game state $P(S)$, an empty cell c_1 is *P-inferior* to an empty cell c_2 if c_2 is a P-winning move or c_1 is a P-losing move (equivalently, $\overline{P}(S + P[c_2]) \geq_P \overline{P}(S + P[c_1])$). In this case, we say that c_2 *P-dominates* c_1. Note that domination of game states, board states, and cells is reflexive and transitive.

For example, let S be as shown in Fig. 2 with White to move. In S, b3 loses for White since Black has a winning move in T, and a5 wins for White since Black has no winning move in U. Thus for S, b3 is White-inferior to a5 (equivalenty, a5 White-dominates b3).

We write $S \equiv_P T$ if $S \geq_P T$ and $T \geq_P S$. Draws are not possible in Hex, so $S \equiv_P T$ if and only if $S \equiv_{\overline{P}} T$, so we write $\equiv$ in place of $\equiv_P$.

In the search for a winning move, an inferior cell can be pruned from consideration as long as some cell that dominates it is considered. With respect to a board state and a player P, a subset V of the set of empty cells U is *P-inferior* if each cell in V is P-inferior to some cell of $U - V$.

With respect to a board state and a player P, a *virtual connection* is a subgame in which P has a second-player strategy to connect a specified pair of cell sets; thus P can connect the two sets even if $\overline{P}$ has the first move. We say that the cell sets are *virtually connected*, and refer to the empty cells of the virtual connection as its *carrier*. The left diagram in Fig. 3 shows two virtual connections. The smaller virtual connection, with a two-cell carrier, is often called a *bridge*.

A virtual connection between a set of P-colored cells and one of P's board edges is an *edge template* for P. Two examples are the *edge bridge* and the *edge 4-3-2*, shown in Fig. 3. For more templates, see David King's webpage [9].

Throughout this paper, we refer to an edge 4-3-2 simply as a *4-3-2*, and we refer to a 4-3-2's eight carrier cells by the labels used in Fig. 3. Note that a 4-3-2 is indeed a virtual connection: if White plays at any of {2,5,6}, Black can reply at 4; if White plays at any of {1,3,4,7,8}, Black can reply at 2. The reader can check that a 4-3-2's carrier is minimal: if any of the eight cells belongs to the opponent, the player no longer has a virtual connection.

With respect to a particular virtual connection of a player, a *probe* is a move by the opponent to a carrier cell; all other opponent moves are *external*. In this paper, we explore this question: when are probes of a Black 4-3-2 inferior?

2 Dead, vulnerable, captured, and capture-dominated

For a board state and a player P, a set of empty cells C is a *P-connector* if P-coloring its cells yields a winning connection; the set is *minimal* if no proper subset is a P-connector. An empty cell is *dead* if it is not on any minimal P-connector. See Fig. 4.

Note that each dead cell is Q-inferior to all other empty cells for both players Q; also, coloring a dead cell an arbitrary color does not change a game state's win/loss value. An empty cell is *P-vulnerable* if some $\overline{P}$-move makes it dead; the cell of this move is a *killer* of the vulnerable cell. Thus, in the search for a P-winning move, dead and P-vulnerable cells can be pruned from consideration.

A set of cells C is *P-captured* if P has a second-player strategy that makes each cell in the set dead or P's color. Since the color of dead cells does not matter, C can be P-colored without changing the value of the board position. For example, the carrier of a Black edge bridge is Black-captured since, for each of the two carrier cells, the cell can be killed by a Black reply at the other carrier cell [5]. An empty cell is *P-capture-dominated*[2] by another empty cell if playing the latter P-captures the former.

[2] Previous papers on dead cell analysis refer to this simply as *domination* [2, 7]. In this paper, we use the term domination in a more general sense.

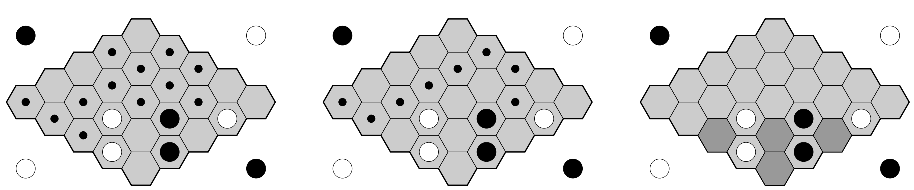

Fig. 4. A Black-connector, a minimal Black-connector, and dead cells.

Note that *vulnerable*, *captured*, and *dominated* are defined with respect to a player; by contrast, *dead* is not. See *Hex and Combinatorics* [6] or *Dead Cell Analysis in Hex and the Shannon Game* [2] for more on inferior cell analysis.

3 A conjecture

As noted previously, the carrier cells of a Black edge bridge are Black-captured, and so White-inferior to all empty cells. For a 4-3-2, things are not so simple.

As shown in Fig. 5, probes 1,2,4 can each be the unique winning move. Also, as shown in Fig. 6, probes 3,5 can win when probes 1,2,4 do not; however, in the example shown there is also a winning external move, and probes 3,5 merely delay an eventual external winning move. We know of no game state in which one of the probes 3,5,6,7,8 is the unique winning move, nor of a game state in which one of the probes 6,7,8 wins but probes 1,2,4 all lose. Probes 1,2,4 seem generally to be stronger than the others, so we conjecture the following:

Conjecture 1. Probes 3,5,6,7,8 of a Black 4-3-2 are White-inferior.

Thus, for a player P and a particular $\overline{P}$-4-3-2, we conjecture that if P has a winning move, then there is some P-winning move that is not one of the five probes 3,5,6,7,8. In the rest of this paper we find conditions under which the conjecture holds.

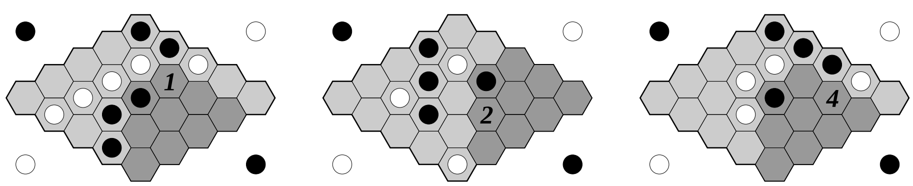

Fig. 5. Only White winning moves: probe 1, probe 2, probe 4.

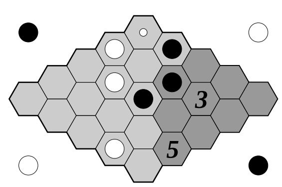

Fig. 6. Only White winning moves: probe 3, probe 5, or the dotted cell.

4 Black maintains the 4-3-2

In Hex, maintaining a particular 4-3-2 is often critical; in such cases, if the opponent ever probes that 4-3-2, the player immediately replies by restoring the virtual connection. Under these conditions, described in the following theorem, our conjecture holds (except possibly for probe 5, whose status we do not know).

Theorem 1. *Consider a game state with a Black 4-3-2 and White to move. Assume that Black responds to a White probe of this 4-3-2 by restoring the virtual connection. Then each White probe in* $\{3,6,7,8\}$ *is White-inferior.*

To prove the theorem, we will show that it is better for White to probe in $\{1,2,4\}$ than in $\{3,6,7,8\}$. To begin, consider possible Black responses to White probes 1,2,4. Against White 1, every other carrier cell maintains the virtual connection; however, Black 2 captures {3,5,6,7}, so {3,5,6,7} are Black-dominated by 2 and need not be considered as Black responses; similarly, Black 4 captures {7,8}. Thus, we may assume: after White 1, Black replies at one of {2,4}; after White 2, Black replies at one of {3,4}; after White 4, Black replies at 2.

We shall show that if White probes at any of {3,6,7,8}, then Black has a response that maintains the 4-3-2 and results in a state where at least one of the following holds: the state is Black-dominated by both states that result after White probes at 1 and Black replies in {2,4}; the state is Black-dominated by both states that result after White probes at 2 and Black replies in {3,4}; the state is Black-dominated by the state that results after White probes at 4 and Black replies at 2.

Our proof of Theorem 1 uses three kinds of arguments. The first two deal with particular forms of domination, which we call path-domination and neighborhood-domination. The third deals directly with strategies. Before presenting the proof, we give some definitions and lemmas.

For a player P and a board state with empty cells c_1 and c_2, we say that c_2 *path-dominates* c_1 if every minimal P-connector that contains c_1 also contains c_2. As the following lemma shows, path-domination implies domination.

Lemma 1. *For a player* P *and empty cells* c_1, c_2 *of a board state* S*, assume that* c_2 *path-dominates* c_1*. Then* $S + P[c_2] \geq_P S + P[c_1]$*.*

Proof. A P-state is a state in which it is P's turn to move. We prove that $S + P[c_2]$ is P-winning whenever $S + P[c_1]$ is P-winning. Thus, assume P has a winning strategy tree T_1 for $S + P[c_1]$. By definition, T_1 considers all possible $\overline{P}$-continuations for all $\overline{P}$-states and specifies a unique P-winning response in each P-state. Without loss of generality, assume that T_1 continues play until the board is completely filled, namely, it does not stop when a winning path is formed. Thus, all leaves in T_1 appear at the same depth and contain a P-winning path.

Construct a strategy tree T_2 by replacing each occurrence of c_2 in T_1 with c_1. We claim that T_2 is a P-winning strategy tree for $S + P[c_2]$.

First, note that in T_2 the board is played until filled, and that all legal moves for $\overline{P}$ are considered at each stage. Furthermore, a unique P-response is given in each P-state. Thus, T_2 is a valid strategy tree. It remains only to show that each leaf of T_2 has a P-connector.

By contradiction, assume that some leaf L_2 in T_2 has no P-connector. Consider the corresponding leaf L_1 in T_1, attained via the same sequence of moves

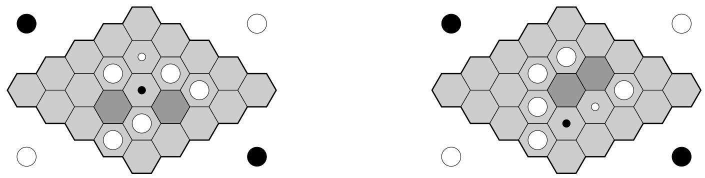

Fig. 7. Killing Black-vulnerable cells without path-domination. The White-dotted cell kills the Black-dotted cell because of White-captured cells that include the shaded cells.

with c_1 replaced by c_2. Since L_1 has a P-connector, this connector must use cell c_1, as it is the only cell that can be claimed by P in L_1 and not claimed by P in L_2. However, c_1 is claimed by $\overline{P}$ in L_2, so c_2 is claimed by $\overline{P}$ in L_1. This is a contradiction, as our P-connector in L_1 requires c_2 as well as c_1. Thus, each leaf in T_2 is P-winning. □

If c_2 P-path-dominates c_1, then c_1 is P-vulnerable to c_2. As Fig. 7 shows, the converse does not always hold; it may be that cells captured by the killer are needed to block all minimal connectors.

Lemma 1 yields the following corollary:

Corollary 1. *Let S be a Hex state with empty cells c_1, c_2 such that c_2 P-path-dominates c_1, and c_1 P-path-dominates c_2. Then $S + P[c_1] \equiv S + P[c_2]$.*

Proof. By Lemma 1, $S + P[c_1] \geq_P S + P[c_2]$ and $S + P[c_2] \geq_P S + P[c_1]$. □

Using Lemma 1 and Corollary 1, as well as capturing cells near the edge, we can determine many domination and equivalence relationships between states obtained via the exchange of two moves within the 4-3-2 carrier. We summarize these relationships in Fig. 8, and present two of their proofs as Lemmas 2 and 3. The omitted proofs are similar.

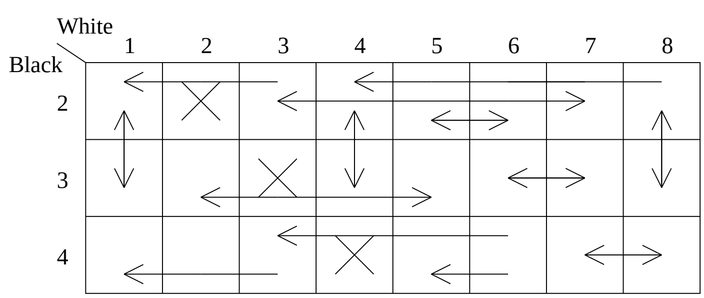

Fig. 8. Some White-domination relations among exchange states. Each arc points from a state to a White-dominating state. Bi-directional arcs indicate equivalent states. X indicates an impossible exchange state. Arcs which follow by transitivity are not shown.

Lemma 2. $S + W[2] + B[3] \equiv S + W[5] + B[3]$.

Proof. $B[3]$ forms an edge bridge, so cells 6 and 7 can be filled-in for Black without changing the value of $S + B[3]$. It can then be seen that all minimal White-connectors that use cell 2 require cell 5, and vice-versa. Thus the result follows from Corollary 1. □

Lemma 3. $S + W[1] + B[4] \geq_W S + W[3] + B[4]$.

Proof. $B[4]$ forms an edge bridge, so cells 7 and 8 can be filled-in for Black without changing the value of $S + B[4]$. It can then be seen that all minimal White-connectors that use cell 3 require cell 1. Now use Lemma 1. □

The P-neighborhood of a cell is the set of all neighbors that are empty or P-colored. A cell c_1 P-dominates a cell c_2 when c_1's P-neighborhood contains c_2's P-neighborhood; in this case, we say that c_1 *P-neighbor-dominates* c_2. Neighbourhood-domination implies domination, so we have the following:

Lemma 4. $S + W[4] + B[2] \geq_W S + W[3] + B[2]$.

Proof. In state $S + B[2]$, cells 5 and 6 are Black-captured, so cell 4 White-neighbor-dominates cell 3. □

Lemma 5. $S + W[2] + B[4] \geq_W S + W[6] + B[4]$.

Proof. In state $S + B[4]$, cells 7 and 8 are Black-captured, so cell 2 White-neighbor-dominates cell 6. □

To prove our final lemma, we explicitly construct a second-player strategy for Black on the 4-3-2 carrier.

Lemma 6. $S + W[1] + B[2] \geq_W S + W[6] + B[4]$.

Proof. In state $S + W[6] + B[4]$, Black adopts the following pairing strategy: if White ever occupies one of {2,5}, Black immediately takes the other; Black does this also with {1,3}. Note that cells 7 and 8 are already filled-in for Black due to the edge bridge from $B[4]$. We will show that this pairing strategy always results in a position White-dominated by $S + W[1] + B[2]$.

Note that this pairing strategy maintains the 4-3-2 virtual connection. Thus, via the carrier, White cannot connect cell 1 to either cell 2 or cell 5. Since the pairing strategy prevents White from claiming both cell 2 and cell 5, then the outcome will be that neither is on any minimal White-connector. Thus cells 2 and 5 are captured by Black via this strategy, so White cannot benefit from claiming cell 3, as it is not on any minimal connector. So, without loss of generality we assume White claims cell 1 and Black claims cell 3. But then the outcome of this strategy will be equivalent to $S + W[1] + B[2] + B[3] + B[4] + B[5] + B[6] + B[7] + B[8]$, which is White-dominated by $S + W[1] + B[2]$. Thus, regardless of White's strategy in state $S + W[6] + B[4]$, Black can ensure an outcome that is White-dominated by $S + W[1] + B[2]$. □

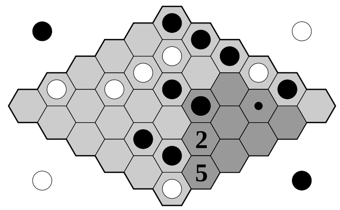

Fig. 9. $S + W[2] + B[4] \not\geq_W S + W[5] + B[4]$.

We now prove Theorem 1.

Proof. As mentioned earlier, our assumptions imply that White 4 loses to Black 2. By Lemma 1 and neighbor-domination, $S + W[4] + B[2]$ White-dominates $S + W[3] + B[2]$, $S + W[7] + B[2]$, and $S + W[8] + B[2]$. Thus, White probes $3, 7, 8$ also lose to Black 2.

Likewise, White 1 loses to Black 2 or Black 4. By Lemma 6, $S + W[1] + B[2] \geq_W S + W[6] + B[4]$; by Lemma 1, $S + W[1] + B[4] \geq_W S + W[6] + B[4]$. Thus, regardless of which move defeats White 1, White 6 loses to Black 4. □

Under the hypothesis of Theorem 1, we conjecture that probe 5 is also White-inferior. Our arguments seem unlikely to resolve this, as it is not true for all states S that $S + W[2] + B[4] \geq_W S + W[5] + B[4]$. See Fig. 9.

Any state S in which probe 5 is not White-inferior must satisfy the following conditions: $S' = S + W[6] + B[2] \equiv S + W[5] + B[2]$ (by Corollary 1), so S' wins for White; also, $S + W[2] + B[4]$ loses for White, while $S + W[2] + B[3]$ wins for White (by Lemma 2).

5 Unconditional Pruning of the 4-3-2

Other than not knowing the status of probing at 5, we have so far confirmed our conjecture under the added assumption that Black maintains the 4-3-2. In this section we establish two theorems that apply without making this added assumption.

Theorem 2 applies to a 4-3-2 that lies in an acute corner of the Hex board. Theorem 3 applies to a state which, if it loses for White, implies that seven of the eight 4-3-2 probes also lose.

A Black 4-3-2 can be aligned into an acute corner of the Hex board in two ways, as shown in Fig. 10. When probing such 4-3-2s, the bordering White edge makes capturing easier, yielding the following results.

Lemma 7. *For a Black 4-3-2 as shown in Fig. 10(left), the set of probes* $\{2, 3, 5\}$ *is White-inferior.*

Proof. (sketch) Probes 2 and 5 are capture-dominated by probe 6. Probe 3 can be pruned as follows. First show that $S + W[4] \geq_W S + W[4] + B[1] \equiv S + W[4] +$

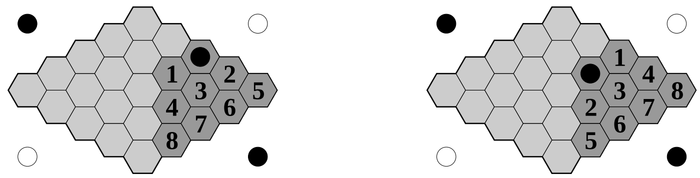

Fig. 10. Acute corner 4-3-2s.

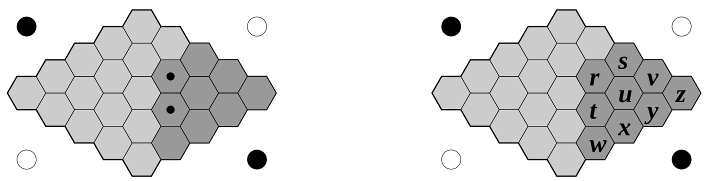

Fig. 11. Dotted cells Black-dominate undotted shaded cells in the acute corner (left). Labels used in the proof of Theorem 2 (right).

$W[3] + B[1]$. It can then be shown that the White probe at 3 is reversible[3] to a Black response at 1, namely that $S \geq_W S + W[3] + B[1]$. From this the desired conclusion follows. We omit the details. □

Lemma 8. *For a Black 4-3-2 as shown in Fig. 10(right), the set of probes {1, 3, 4, 5, 6, 7, 8} is White-inferior.*

Proof. Given any White probe in {1,3,4,5,6,7,8}, Black can respond at cell 2 and capture all cells in the 4-3-2 carrier by maintaining the 4-3-2. □

Theorem 2. *Let S be a Hex state with the nine cells of a potential acute corner Black 4-3-2 all empty, as in Fig. 11. Then each of the seven undotted cells is Black-dominated by at least one dotted cell.*

Proof. (sketch) Let S be a board state in which the nine cells of an acute corner Black 4-3-2, labelled $r, \ldots, z$ as in Fig. 11, are all empty. We want to show that each cell in the carrier is Black-dominated by at least one of r, t.

Cell t Black-capture-dominates cells u, v, w, x, y, z. The argument that cell r Black-dominates cell s is more complex, as follows. Let $S_r = S + B[r]$ and $S_s = S + B[s]$; we want to show that S_r Black-dominates S_s.

First assume that from S_s or S_r, Black is next to move into the carrier. In S_r, a Black move to t Black-captures all other carrier cells, so $S_r + B[t] \geq_B S_s + B[\beta]$ for every possible β in the carrier, so we are done in this case.

Next assume that from S_r or S_s, White is next to move into the carrier. By Lemma 8, from S_r White can do no better than $S_r + W[t]$, so we are done if

[3] A P-move is *reversible* if $\overline{P}$ has a response that leaves $\overline{P}$ in at least as good a position as before the P-move. See *Winning Ways, Volume I* [1].

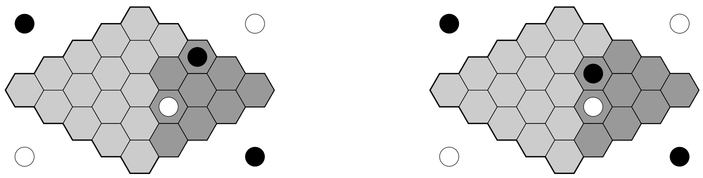

Fig. 12. State $S_s + W[t]$ (left) White-dominates $S_r + W[t]$ (right).

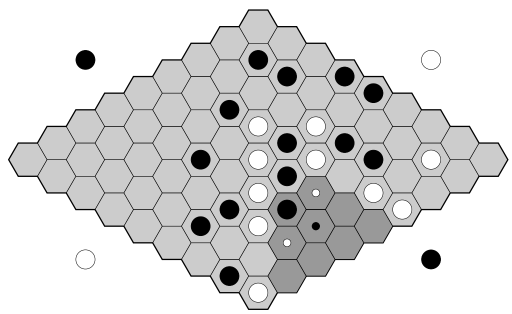

Fig. 13. A state S White-dominated by $S + W[1] + W[2] + B[3]$.

White has some move from S_s that is at least as good, namely if, for some q in the carrier, $S_s + W[q] \geq_W S_r + W[t]$. We can show this for $q = t$; we omit the details. See Fig. 12. □

By Theorem 2, if Black is searching for a winning move and the shaded cells of Fig. 11 are all empty, then Black can ignore the undotted shaded cells.

Next, we consider a result that can be useful when White suspects that probing a particular Black 4-3-2 is futile. We show a state which, if it loses for White, guarantees that seven of the eight probes also lose.

Theorem 3. *Let S be a state with a Black 4-3-2. If $W(S+W[1]+W[2]+B[3])$ is a White loss, then in $W(S)$ each White probe other than 4 loses.*

Proof. $S + W[1] + W[2] + B[3]$ White-dominates both $S + W[1] + B[3]$ and $S+W[2]+B[3]$, so White probes $1, 2$ can be pruned. By Lemma 2, $S+W[2]+B[3]$ is equivalent to $S + W[5] + B[3]$, so White 5 can be pruned. Against White probes 3 or 7, strategy decomposition shows that Black wins by replying at cell 4. By Lemma 1 and Corollary 1 respectively, $S + W[3] + B[4]$ White-dominates $S + W[6] + B[4]$, and $S + W[7] + B[4]$ is equivalent to $S + W[8] + B[4]$. □

In terms of being able to prune probes of a 4-3-2, Theorem 3 is useful only if $S' = S + W[1] + W[2] + B[3]$ loses. Not surprisingly, we gain less information about the probes when S' wins. For example, Fig. 13 shows a state S in which White has a winning move from S' but no winning move from S.

6 Conclusions

We have introduced path-domination and neighborhood-domination, two refinements of domination in Hex, and used these notions to find conditions under which probes of an opponent 4-3-2 edge template are inferior moves that can be ignored in the search for a winning move.

In particular, three of the eight probes can be unique winning moves and so cannot in general be discounted; we conjecture that the other five probes are all inferior. Since 4-3-2s arise frequently in Hex, confirming this conjecture would allow significant pruning in solving game states.

We have confirmed the conjecture in various situations. For example, if the player knows that the opponent's immediate reply to a probe will be to restore the template connection, then four of these five remaining probes are inferior.

External conditions might suggest that all probes of a particular 4-3-2 are losing. We have found a state whose loss implies that seven of the eight probes are losing; establishing this result would allow the seven probes to be ignored.

Also, we have established some domination results that apply when the 4-3-2 lies in an acute corner.

It would be of interest to extend our results to consider the combined maintenance of more than one critical connection, or to automate the inference process so that similar results could be applied to a more general family of virtual connections.

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