THE OPTIMAL CROSSING-FREE HAMILTON CYCLE PROBLEM FOR PLANAR DRAWINGS OF THE COMPLETE GRAPH

by

Ryan B. Hayward

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Dedicated in memory of

Albert George Homer Hayward

January 13, 1888 to August 17, 1970

und zur Erinnerung an

Frederick Geoffery Banerd

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ABSTRACT

When a graph is drawn in the plane, the resulting drawing may have pairs of crossing arcs, called crossings. A Hamilton cycle of a graph is a cycle that visits every vertex. A drawing in the rectilinear plane is a drawing all of whose arcs are straight line segments. We consider the problem of determining $\phi(n)$ and $\bar{\phi}(n)$, the maximum number of crossing-free Hamilton cycles (cfhc's) of any drawing of K_n (the complete graph on n vertices) in the Euclidean and rectilinear planes respectively.

We present a survey of all papers (some not in the open literature) on this very recent problem. Also, for the first time several different generalized drawings, or constructions, of K_n are collected together. Some of the drawings have never before appeared in the open literature, e.g. David Singer's rectilinear drawing of K_{10} with only 62 crossings. For small n, the number of cfhc's of these constructions of K_n are counted with a computer program. Lower bounds for $\phi(n)$ and $\bar{\phi}(n)$ for small n are consequently established.

Unfortunately, no new exact values of $\phi(n)$ or $\bar{\phi}(n)$ are found. However, by developing a recursive geometric counting argument due to Selim Akl, we improve the best known lower bound for $\bar{\phi}(n)$ from (asymptotically) $c * 2.270719168^n$ to $k * 3.191847754^n$, where c and k are constants. A feature of our technique is that we are able to use computer-generated data in our counting argument. In fact, our exposition is such that with access to greater computer resources, our lower bound for $\bar{\phi}(n)$ could be easily improved.

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CONTENTS

FIGURES

TABLES

1. THE OPTIMAL CROSSING-FREE HAMILTON CYCLE PROBLEM

1.1 Introduction

In this chapter we define and survey the optimal crossing-free Hamilton cycle problem, i.e. the problem of determining the maximum number of crossing-free Hamilton cycles of any planar drawing of K_n, the complete graph on n vertices.

The first section consists of definitions. Those readers familiar with graph theory can skip most of the first page. Towards the end of the section are a few comments concerning the uniformity of definitions in the literature. The second section is a survey of the optimal cfhc problem. As this problem is related to the crossing number problem, i.e. the problem of determining the minimum number of crossings of any planar drawing of K_n, some references to the crossing number problem are also mentioned.

1.2 Definitions

A **graph** G(V,E) is a nonempty set V and a possibly empty set E of unordered pairs of not necessarily distinct elements of V. Elements of V are called **vertices** of G, and elements of E (if any) are **edges**. Consider an edge e = (v,w) , where v and w are vertices of G. Then e is a **loop** if v = w ; else e is a **link**. The edge e is said to be **incident** with, or on, vertices v and w; likewise v and w are each **incident** with, or on, the edge e. Two edges incident with a common vertex are called **adjacent**; similarly, two vertices on a common edge are **adjacent** (i.e, two vertices v and w of graph G are adjacent if and only if there is an edge (v,w) of G.) If there are distinct vertices v and w of G such that there is more than one edge (v,w), then G is said to have **multiple edges**. A graph with no loops or multiple edges is said to be **simple**. The **complete graph** on n vertices, K_n , is the simple graph with an edge on every pair of distinct vertices. Note that K_n has $\binom{n}{2}$ edges. A **cycle** of a graph G is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ where vertices corresponding to distinct indices are distinct, and where (v_i, v_j) is an edge of G, for $0 < i < k$, as is (v_1, v_k). A **Hamilton cycle** of a graph G is a cycle that includes all vertices of G. Not all graphs have cycles (for instance, trees have no cycles, and frogs have no ears).

An **embedding** of a graph G in a surface S is a mapping of the vertices of G to distinct points of S, called **nodes**, and the edges of G to disjoint open smooth curves of S, called **arcs**, such that:

1) no arc contains a node;

2) the arc corresponding to an edge (v,w) joins the nodes corresponding to the vertices v and w.

Note that in an embedding, all arcs are disjoint; that is, no two arcs intersect. Also note that, with respect to a given surface, it may not be possible to embed certain graphs. Consider, for example, Kuratowski's classic result that K_5 cannot be embedded in the Euclidean plane. See **[BM]** for an introduction to the theory of graphs.

A **drawing** of a graph is defined exactly as is an embedding, except that it is not required that the edges of G be mapped to disjoint arcs. Again, images of vertices are called **nodes**, and images of edges are called **arcs**. An arc and a node of a drawing D of a graph G are **incident** if and only if their respective pre-images in G are incident; similarly, two arcs (respectively nodes) of a drawing D of a graph G are **adjacent** if and only if their pre-images are adjacent in G.

A drawing is **good** if:

3) adjacent arcs do not intersect (in the surface S, of course);

4) any two arcs intersect in at most one point of S;

5) no three arcs intersect in a common point of S;

6) any two arcs that intersect are not tangent at their point of intersection.

A drawing D of a graph G is defined as **bad** not if it is drawn by someone else, but if it is not good. Figure 1 gives those configurations that are prohibited by conditions 3) to 6), and shows how they can be removed.

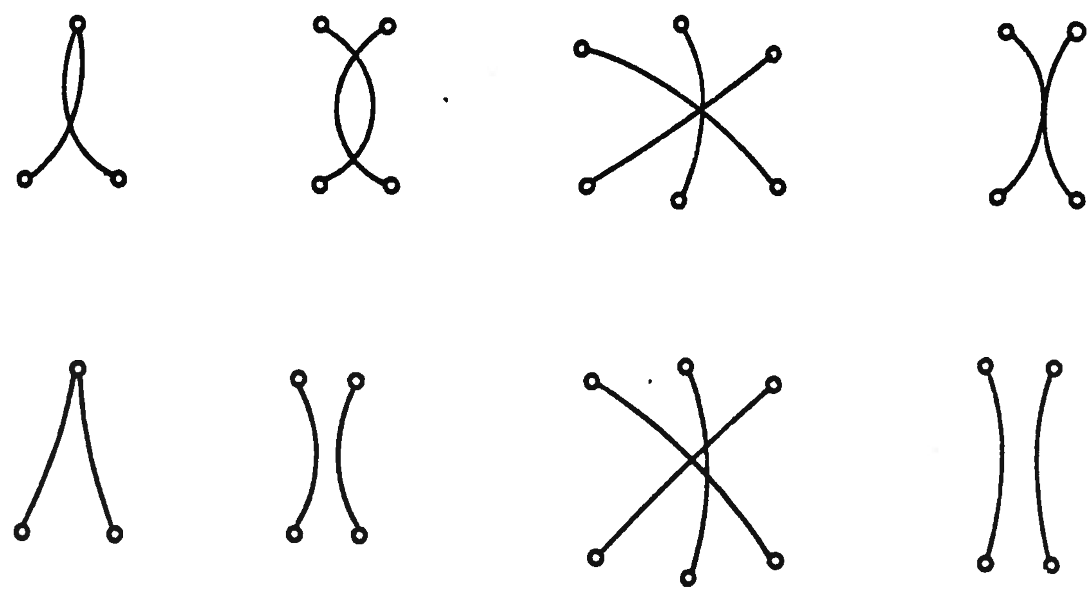


Figure 1. Making drawings good [E]

A **crossing** is a set of two intersecting arcs. The **responsibility of an arc** is the number of arcs with which it intersects, i.e. the total number of crossings on that arc. The **responsibility** of a **node** is the sum of the responsibilities of all incident arcs. The **crossing number** **x(D(G,S)) of a drawing** D(G,S) of a graph G in a surface S is the number of crossings in D. [We will frequently abbreviate the drawing D(G,S) as D.] The **crossing number** **ν(G,S) of a graph** G in a surface S is the minimum number of crossings of any drawing of G in S. A drawing D(G,S) is **crossing number optimal**, or **x-optimal**, if x(D(G,S)) = ν(G,S).

$\nu(n)$ is the crossing number of K_n in the Euclidean plane, i.e. $\nu(n) = \nu(K_n,E)$, where E is the Euclidean plane. Planar drawings in which arcs are restricted to straight line segments will be referred to as **drawings in the rectilinear plane.** Let **$\bar{\nu}(n)$** be the rectilinear crossing number, i.e. the minimum number of crossings of any drawing of K_n in the rectilinear plane.

Two drawings D(G,S) and D′(G,S) of a graph G in a surface S are **isomorphic** if and only if there is a bijection of the nodes that preserves both incidence and crossings of arcs. A **crossing-free Hamilton cycle** of a drawing D(G,S) is the image of the edges of a Hamilton cycle of G, such that no two of the resulting arcs intersect. The number of crossing-free Hamilton cycles (cfhc′s) of the drawing D is **cfhc(D).** The maximum number of crossing-free Hamilton cycles of all drawings of G in the surface S is **ϕ(G,S),** also called the **cfhc number** of G in S. A drawing D is **cfhc-optimal** if cfhc(D) = ϕ(G,S). **ϕ(n)** is $\phi(K_n,E)$, and **$\bar{\phi}$(n)** is the maximum number of cfhc′s of any drawing of K_n in the rectilinear plane.

1.2.1 Uniformity of definitions

The reader is warned that variety and imprecision abound in the literature with respect to the definitions of drawing, crossing and good drawing. An example of variety: Newborn and Moser refer to what we call crossing-free Hamilton cycles as crossing-free Hamiltonian circuits. An example of imprecision: it is common for authors to fail to distinguish between graphs and drawings.

Geodesics in the Euclidean plane are straight line segments. Some authors refer to the rectilinear crossing number as the geodesic crossing number.

Although we may refer to a node as being "on" an incident arc, we point out that arcs are open curves, and do not actually intersect incident nodes.

Our definition of a good drawing is taken from [BW], except that condition 6) is added from [E]. This is not a major inconsistency in the literature, however, as almost all papers are concerned with optimality, and it is easy to show that if there is a x-optimal drawing D, then there is a good (according to both [BW] and [E]) x-optimal drawing D′, such that all crossings of D are crossings of D′.

It is always possible to eliminate any type 6) crossings from a good [BW] drawing to obtain a good [E] drawing without introducing any new crossings, and without deleting any crossing-free Hamilton cycles.

1.3 A Survey

1.3.1 The optimal cfhc problem

The planar crossing-free Hamilton cycle problem for K_n was posed only very recently by Monroe Newborn and W. O. J. Moser in 1976 **[NM]**. We are aware of only two other papers on this problem in the open literature: that of Selim Akl **[A1]** and that of M. Ajtai, V. Chvátal, M. Newborn and E. Szémerédi **[ACNS]**.

Newborn and Moser set out to determine the cfhc-number of K_n, and also ask what drawings are cfhc-optimal. They considered this problem both in the Euclidean and rectilinear planes. They were able to solve the problem for n = 1 to 6. By counting (with a computer program) cfhc's of many drawings, they also arrived at lower bounds for $\phi(n)$ and $\bar{\phi}(n)$ for n = 7, 8, 9. These results are given later in Table I. The possible connection between cfhc-optimal drawings and x-optimal drawings was noted. To obtain a lower bound for $\bar{\phi}(n)$, they drew the nodes of K_n as [n/3] concentric triangles (where [x] is the greatest integer $\leq$ x), and placed any remaining nodes in the innermost triangle. Call this drawing NM_n. They then used an inductive counting argument to find a lower bound for $cfhc(NM_n)$, and hence the lower bound $\bar{\phi}(n) \geq 0.15 * 10^{[n/3]}$. Note that the set of drawings of E having all arcs as geodesics is a subset of the set of all drawings of E; therefore $\bar{\phi}(n) \leq \phi(n)$.

Using a topological argument (essentially, one can turn in only a finite number of directions a finite number of times in constructing a rectilinear cfhc), they determined the upper bound $\bar{\phi}(n) \leq 2 * 6^{n-2} * [n/2]!$.

In 1979, Akl **[A1]** improved the lower bound of $\bar{\phi}(n)$ by improving the lower bound for $cfhc(NM_n)$. Again, a recursive construction was used. The end result is that for sufficiently large n, $\bar{\phi}(n) > c * 2.270719168^n$, where c is a constant. The true purist who does not like to see theoretical problems tainted with real world applications will likely be dismayed by the following observation of Akl's: " . . . [the cfhc-optimality problem] also arises in connection with various optimization problems in the plane, such as the Euclidean traveling salesman problem." Akl later improved the lower bound slightly [A2], although the asymptotic rate of growth remained 2.270719168^n.

Finally, Ajtai, Chvátal Newborn and Szémerédi [ACNS] confirmed a conjecture of Erdos and Guy involving crossing numbers, by proving that (for sufficiently large m), any planar drawing with n nodes and m arcs has at least $\frac{c\,m^3}{n^2}$ crossings, for a constant c. They then used this result to show that the number of crossing-free planar subdrawings of a drawing with n nodes is no more than 10^{13n}. This then became an upper bound for $\phi(n)$. Their proof was quite elegant, in that the only graph theory needed to show that the former result implies the latter was a very simple counting argument relating crossings, nodes and arcs. This was also the first result to demonstrate a relationship between numbers of crossings and numbers of cfhc's.

1.3.2 Crossing Number Problems

As the cfhc-optimal problem appears related to the crossing number problem, we mention a few references on crossing number problems. Unlike the cfhc-optimal problem, the crossing number problem has been around for 15 years, if not longer, and there is a wealth of literature on the subject.

A very comprehensive bibliography is given in [E] ; for a survey in the public domain, the reader is directed to the expository papers of Guy [G3] [G4] and to the paper by Erdős and Guy [EG], brief updates of which appear periodically in the research problems section of the American Math. Monthly [G5] [G6]. Many recent books on graph theory mention crossings; see [BW] and [BCLF], for example. Finally, the crossing number problem is classified as category #361 in Brown's Reviews of Graph Theory [Br].

1.3.3 On x-numbers and cfhc-numbers

The following tables give the cfhc-optimal and x-optimal numbers for planar drawings of K_n, for n up to 9 and 10 respectively. The cfhc-optimal table, Table I, is taken from [NM], while the x-optimal table, Table II, is from [G4], and also appears in [BW]. Note that in some cases only lower bounds are given.

1.3.4 New Results

This paper contains several new results.

In Chapter 2 we determine lower bounds for $\phi(n)$ and $\bar{\phi}(n)$ for n = 3 to 15, consequently improving Newborn and Moser's lower bound for $\bar{\phi}(9)$ from 1228 to 1252. Compare Table I (from Newborn and Moser) with Table VIII (from our catalogue) . The lower bounds for n = 9 to 15 are new as of this paper.

In Chapter 3 we determine an asymptotic lower bound for $\bar{\phi}(n)$ by developing a counting argument due to Selim Akl [A1] [A2]. Our lower bound is asymptotic to $k * 3.191847754^n$, where k is a constant. The previous best lower bound for $\bar{\phi}(n)$ (due to Akl) was asymptotic to $c * 2.270719168^n$, where c is a constant. See section 1.3.2 and Chapter 3.

All of the results of our preprint [H] are contained in Chapter 3.

Table I. Results of Newborn and Moser

n	3	4	5	6	7	8	9
$\bar{\phi}(n)$	1	3	8	29	≥ 92	≥ 339	≥ 1228
$\phi(n)$	1	3	8	29	≥ 96	≥ 399	——

Table II. Crossing Numbers [G4] [BW]

n	3	4	5	6	7	8	9	10
$\bar{\nu}(n)$	0	0	1	3	9	19	36	61 or 62*
$\nu(n)$	0	0	1	3	9	18	36	60

* $60 < \bar{\nu}(10) < 63$.

2. A CATALOGUE OF DRAWINGS

2.1 Introduction

In this chapter we present a catalogue of drawings of K_n, along with their respective x-numbers and cfhc-numbers. The catalogue consists of Tables III to VIII. Included are all known cfhc-optimal drawings of K_n in the Euclidean and rectilinear planes, as well as a few general constructions of K_n that have appeared in the literature on crossing numbers. A back-tracking algorithm, written by Henk Meijer for Selim Akl in Pascal and altered slightly by the author, was used to count cfhc's. The program is listed in the last chapter. We are indebted to Henk and Selim in that this paper never would have been possible without the use of their program. However, responsibility for any possible errors in the following data is ours alone.

For a given drawing, input to the program was a list of the crossing arc pairs. Validity checks included

i) counting cfhc's by hand,

ii) entering data several different ways,

iii) confirming cfhc equality of isomorphic drawings,

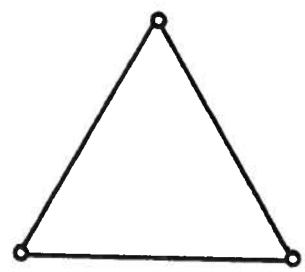
iv) confirming cfhc numbers of all drawings shown in the paper by Newborn and Moser **[NM]**, and

v) confirming the number of cfhc's of a (slightly) non-trivial drawing for which the exact number of cfhc's was calculated.

2.2 Drawings of K_3 to K_6

All of the drawings mentioned in this section are catalogued in Table III, (see the end of this section). Note that all of these drawings, as well as some drawings catalogued in Table VI, have alpha-numeric names. The digit in the name of such a drawing indicates the number of nodes; e.g. drawing 3A is a drawing of K_3, drawing 6Q is a drawing of K_6, etc.

All non-isomorphic good drawings of K_3, K_4, and K_5 are included in Figures 2, 3, and 4 [E]. As there are so few good drawings, it is trivial to determine the optimal cfhc-numbers for n = 3, 4, and 5.



3A

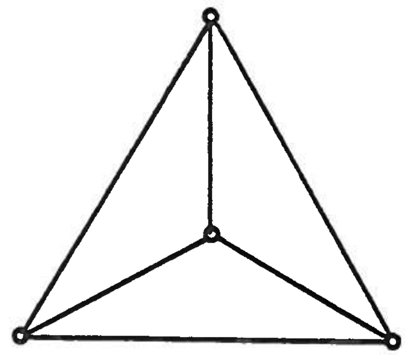
Figure 2. Good Drawing of K_3

It is not much more work to do the same for K_6 in the rectilinear plane; we now show that all non-isomorphic good rectilinear drawings of K_6 appear in Figures 5 and 6 (with some drawings repeated due to isomorphism).

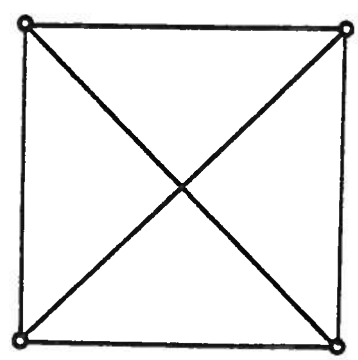
Let D be a good rectilinear drawing of K_6. If D has six nodes on its convex hull, then D must be isomorphic to the drawing 6Z (see Figure 6). Suppose now that D has fewer than six convex hull nodes; then removal of some node not on the convex hull of D leaves a good rectilinear drawing of K_5. As there are only three non-isomorphic good rectlinear drawings of K_5, namely drawings 5A, 5B and 5E, it is feasible to construct all possibly non-isomorphic drawings of D (having fewer than six convex hull nodes) by placing a node in each of the different interior regions of each of the three drawings. Thus, the set of 25 drawings 6A to 6Z (there is no drawing 6X) of Figures 5 and 6 includes the set of all non-isomorphic good rectilinear drawings of K_6. Note that the letter in each of the labelled regions of Figures 5 and 6 corresponds to the name given to that drawing of K_6 obtained by placing a node in that region and joining the node to all five other nodes with straight line segments. For example, drawing 6H is obtained by placing a node in region H.

Note that of all good rectilinear drawings of K_6, 6A has the most cfhc's, namely twenty-nine (see Table III at the end of this section). Note that there is no bad rectilinear drawing with a greater number of cfhc's; thus by exhaustive analysis we have proved that $\bar{\Phi}(6) = 29$.

Drawing 6A is the unique planar cfhc-optimal (and unique x-optimal [G3]) drawing of K_6 , in both the rectilinear and non-rectilinear planes [NM] . As Drawing 6A is also the drawing TS_6 (see section 2.3.4), drawing 6A appears in Figure 9 as well as in Figure 5.



4A



4B

Figure 3. All Good Drawings of K_4 [E]

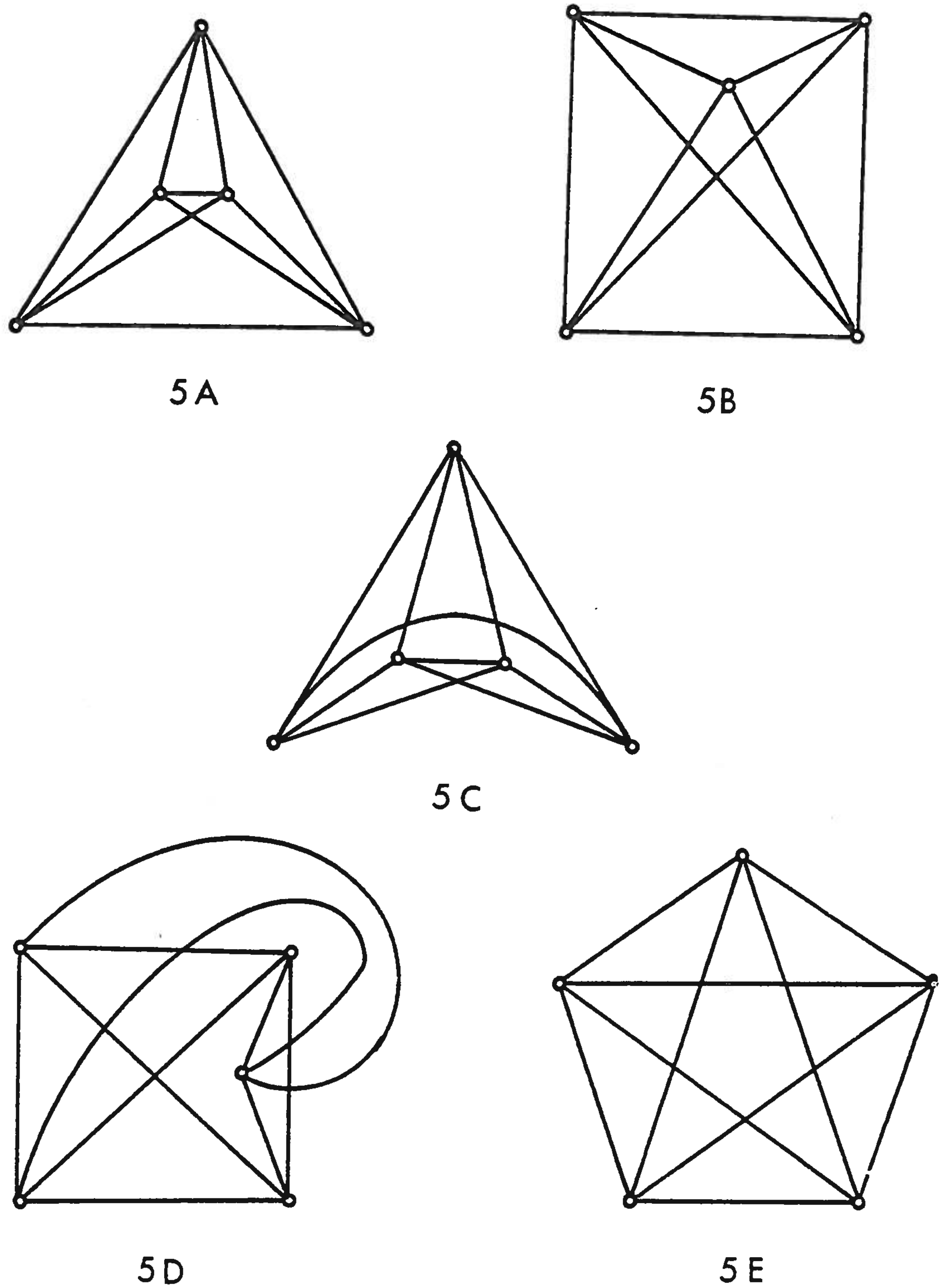


Figure 4. All Good Drawings of K_5 [E]

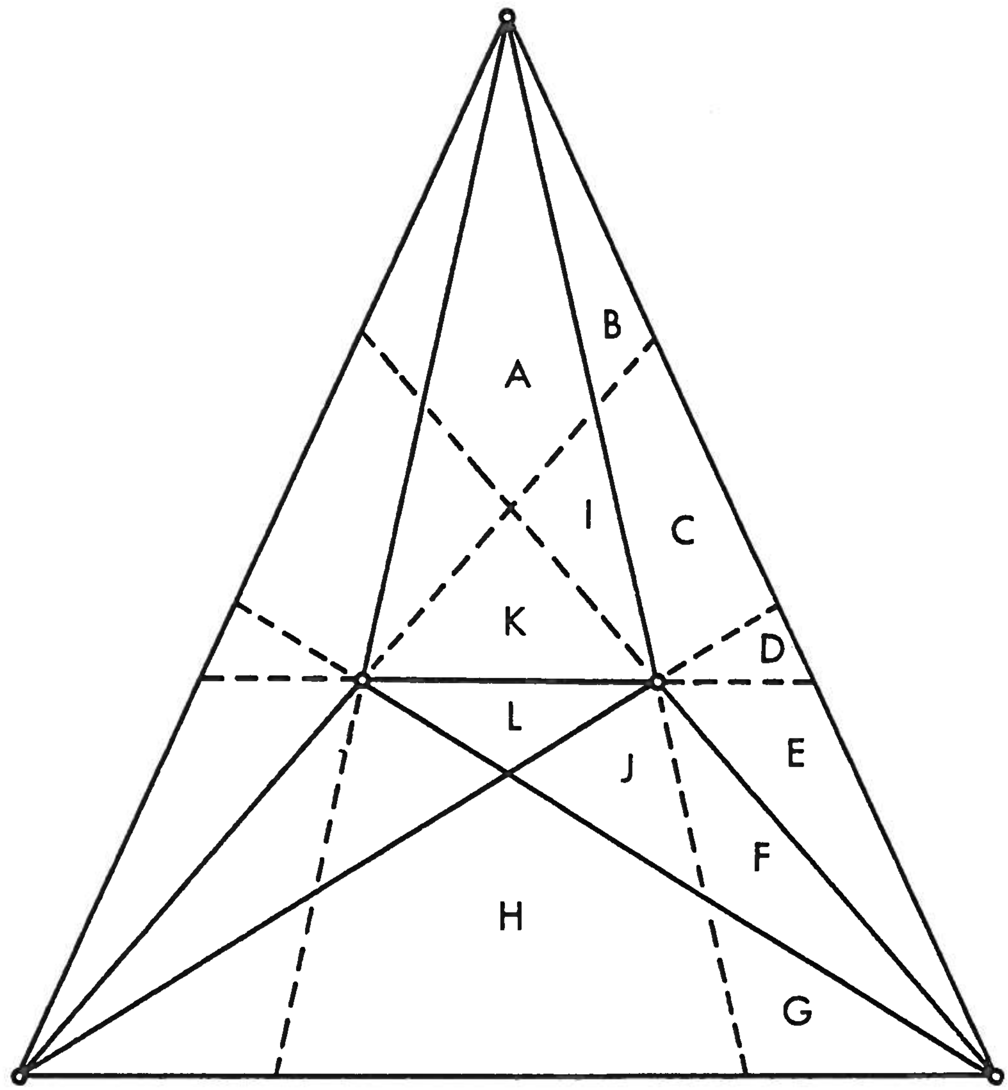


Figure 5. Drawings of K_6: Convex Hull 3

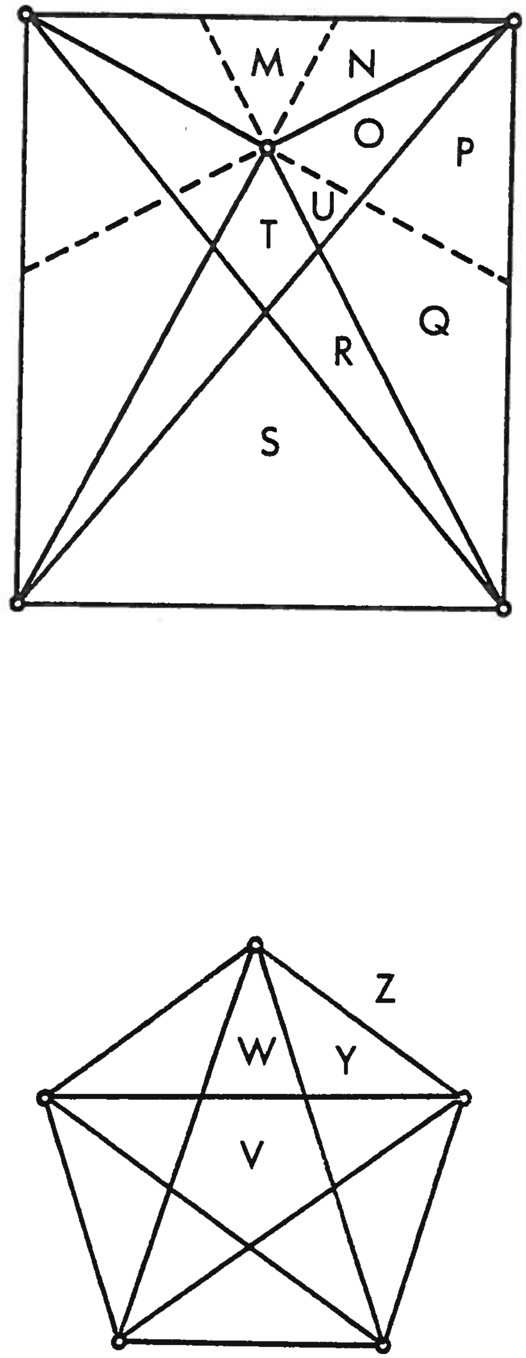


Figure 6. Drawings of K_6: Convex Hull 4, 5 and 6

Table III. Drawings of K_3 to K_6

Drawing	Responsibilities: Arc	Node	x	cfhc	con. hull
K_3					
3A	0 ...	0 0 0	0	1	3
K_4					
4A	0 ...	0 0 0 0	0	3	3
4B	1 1	1 1 1 1	1	1	4
K_5					
5A	1 1	1 1 1 1 0	1	8	3
5B	2 2 1 1	3 3 2 2 2	3	4	4
5C	2 1 1 1 1	3 3 2 2 2	3	3	-
5D	3 2 2 1 1 1	4 4 4 4 4	5	2	-
5E	2 2 2 2 2	4 4 4 4 4	5	1	5
K_6					
6A	1 1 1 1 1 1	2 2 2 2 2 2	3	29	3
6B,F,D,L,I	2 2 1 1 1 1	3 3 3 3 3 1	4	24	3
6C,G,J	3 2 2 1 1 1	4 4 4 4 2 2	5	23	3
6E,K	2 2 2 2 2	4 4 4 4 4 0	5	20	3
6H	3 3 3 1 1 1	5 5 5 3 3 3	6	22	3
6M,T	3 3 2 2 1 1 1 1	6 6 4 4 4 4	7	13	4
6O	3 3 2 2 1 1 1 1	6 6 4 4 4 4	7	13	4
6P,R	4 3 2 2 2 1 1 1	7 5 5 5 5 5	8	13	4
6N,U	3 3 3 2 2 2 1 1 1	7 7 5 5 5 3	8	12	4
6S	4 4 2 2 2 2 2	6 6 6 6 6 6	9	13	4
6Q	4 3 3 3 2 2 1	8 6 6 6 6 4	9	12	4
6V	3 3 3 3 3 1 1 1 1 1	7 7 7 7 7 5	10	6	5
6W	4 3 3 3 3 2 2 1 1	8 8 8 8 6 6	11	5	5
6Y	4 4 3 3 3 3 2 2	9 9 9 7 7 7	12	5	5
6Z	4 4 4 3 3 3 3 3 3	10 10 10...	15	1	6

Note: All drawings listed on any line are all isomorphic. Also, two drawings listed on different lines are non-isomorphic. Note in particular that 6O is non-isomorphic to 6M (and 6T). Thus, there are 15 non-isomorphic good rectilinear drawings of K_6.

2.3 Constructions of K_n

2.3.1 Introduction

The reader will probably appreciate that the number of non-isomorphic good drawings of K_n increases rather quickly as n increases. Sooner or later some technique other than brute force (i.e. simply drawing all good drawings and counting their cfhc's) will have to be used in the search for cfhc-optimal drawings. One such technique is to limit the search to drawings that seem likely to have many cfhc's, namely drawings with few crossings. We define a **construction** of K_n to be a rule for drawing K_n for any given n. We have selected five rectilinear and two non-rectilinear constructions for K_n in the plane. They are described below.

One drawing of each construction is included as a figure at the end of section 2.3. However, as the catalogue includes all drawings of each construction of K_n for n up to 13, not all drawings of the catalogue appear as figures.

The two tables at the very end of the section record the number of cfhc's of the constructions of K_n for n up to 13.

2.3.2 The construction WH_n

Suppose that it is a cold and miserable February, and that there are n-1 professors living on or near the 49th parallel who decide to take a holiday at the south pole (after all, it will be summer down south . . .). The **Winter Holiday** drawing WH_n of K_n is obtained by joining the n locations mentioned above with geodesics on the globe, and then transferring the drawing to the plane so that no new crossings are created. Note that WH_n can be drawn in the plane with straight line segments (see Figure 7) ; hence, WH_n is a rectilinear construction. Of all constructions we will give,

WH_n has by far the most crossings, namely $\binom{n-1}{4}$, and the fewest cfhc's. However, it is the only construction for which we have been able to explicitly determine a closed form solution for the number of cfhc's.

Claim: For $n > 2$, $\text{cfhc}(WH_n) = (n-1) * 2^{n-4}$

Proof: A **path** is formed when an edge is deleted from a cycle. A **Hamilton path** is a path that visits all the vertices of a graph. A **crossing-free Hamilton path** (cfhp) is the image in a drawing of a Hamilton path of a graph, such that no two of the resulting arcs cross. The **end-nodes** of a cfhp are the nodes of the cfhp incident to only one arc of the cfhp. Let **cfhc(D)** be the number of cfhp's of drawing D. Let the rectilinear planar **Circle drawing C_n** of K_n be the drawing constructed by placing n nodes on the circumference of an ellipse. Note that the number of cfhc's of WH_n is equal to the number of cfhp's of C_{n-1} (remove the south pole from WH_n).

Label the nodes of C_m from 1 to m. Let $Q(m : s,t)$ be the number of cfhp's of C_m with end-nodes s and t; consider such a cfhp $Z = (1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, t)$, where $s=1$. Note that for $1 < i < j < t$, node i occurs before node j in Z (otherwise Z will have a crossing). [Similarly, for $t < i < j \leq m$, node j occurs before node i in Z.] In fact, for $1 < t \leq m$, there is exactly one cfhp $(1, \ldots, t)$ of C_m for every pair of integers (i,j) such that $1 < i < j < t$. Thus, $Q(m : 1,t) = \binom{m-2}{t-2}$. A similar argument yields $Q(m : s,t) = \binom{m-2}{t-s-1}$.

Finally, note that $\text{cfhp}(C_m)$ is obtained by summing $Q(m : s,t)$ over all pairs (s,t), such that $1 \leq s < t \leq m$. Thus we have

$$\text{cfhc}(WH_n) = \text{cfhp}(C_{n-1}) = \sum_{s=1}^{n-2} \sum_{t=s+1}^{n-1} Q(n-1 : s,t) =$$

$$\sum_{s=1}^{n-2} \sum_{t=s+1}^{n-1} \binom{n-3}{t-s-1} = (n-1) * 2^{n-4}$$

This completes our proof.

Note that the above was used as a check of our computer program.

2.3.3 The construction ALM_n

This construction is due to Saaty, who calls this the Alternating Linear Model of K_n [Sa]. Draw the n nodes 1 to n in order on a straight line L. Join node i to node i+1 with a straight line segment, for i = 1 to n. Join node 1 to nodes 3 to n with semicircles drawn above L. Join node 2 to nodes 4 to n with semicircles drawn below L. Node 3 is connected to remaining nodes with semicirles drawn above L; node 4 with semicircles below L; and so forth. Jensen notes that ALM_n is actually rectilinear [J] : starting with node 3, nodes are drawn alternately above and below L, each additional node being placed increasingly further from L. The nodes can be thought of as the peaks and troughs of an exponentially amplified sinusoid. See Figure 8. Note that as arcs in a rectilinear drawing are straight line segments between pairs of nodes, a rectilinear drawing is completely described once its node locations have been described. The number of crossings of ALM_n is $[(n - 1)(n - 3)(n^2 - 4n +1) / 48]$.

2.3.4 The Construction TS_n

TS_n (**Triangular Spiral**) is a rectilinear construction. The nodes are drawn as three concentric rays of $[\frac{n+2}{3}]$, $[\frac{n+1}{3}]$, and $[\frac{n}{3}]$ nodes each. The rays all spiral slightly in the same direction. The number of crossings is $(11n^4 - 90n^3 + an^2 + bn + c) / 648$, where a, b, and c depend on the residue of n (mod 3).

TS_n is included essentially because its trilateral symmetry renders it useful in establishing an improved lower bound for $\overline{\Phi}(n)$. See the following chapter, which also gives a more complete description of how to draw TS_n. See also Figures 9 to 11.

2.3.5 The Construction FTS_n

This is also a rectilinear construction. To draw FTS_n (**Flip Triangular Spiral**), first draw TS_n. Then take the ray containing the fewest (i.e. $[\frac{n}{3}]$) nodes, and flip it in place so that it curls in the direction opposite to the other two rays. See Figure 12. This construction has been included because of all constructions consisting of three concentric spiralled rays of nodes, this construction has the fewest number of crossings, namely $(11n^4 - 94n^3 + an^2 + bn + c) / 648$, where a, b, and c depend on the residue of n (mod 3).

2.3.6 The Construction JE_n

This construction is due to **Jensen**, and is the last of our rectilinear constructions. As with TS_n and FTS_n, JE_n consists of three rays of nodes, but in this case the rays are not spiralled. Instead, each ray is a copy of (a flattened) ALM_n. Figure 13 shows how JE_{10} is constructed from ALM_3, ALM_3, and ALM_4. See **[J]** for a more complete description of how to draw JE_n. JE_n is included because of all rectilinear constructions that have appeared in the open literature it has the least number of crossings, namely $(7n^4 - 56n^3 + an^2 + bn + c) / 432$, where a, b, and c depend on the residue of n (mod 6). Jensen and Singer have independently found constructions with fewer crossings, but these have not been published **[G5]** **[Si]** .

2.3.7 The Construction BKA_n

This non-rectilinear construction is due to **Blazek and Koman [BK]**, and independently Guy **[G2]**. The number of crossings is the least of all known planar constructions, namely $[\frac{n}{2}][\frac{n-1}{2}][\frac{n-2}{2}][\frac{n-3}{2}] / 4$. This number is also conjectured to be $\nu(n)$ **[G4]** . It has been proved that the above expression is equal to $\nu(n)$ for n up to 10 [G4] . See Table II.

To draw BKA_n, place n nodes at the corners of a regular n-gon. Pick any line of reference. If the slope of the line on a pair of nodes is positive, or if the line is perpendicular to the reference line, then the nodes are joined with a straight line segment. Otherwise, the nodes are joined by an arc outside the n-gon. Another way to picture this is to draw the n nodes on the equator of the sphere. Then in the former case, the nodes are joined by an arc in the upper hemisphere, and in the latter by an arc in the lower hemisphere. [Any drawing that can be drawn on the sphere can be drawn in the plane.] See Figure 14.

2.3.8 The Construction BKB_n

This second construction of **Blazek and Koman [BK]** was discovered independently by Guy [G1], Saaty [Sa], and others [G3]. BKB_n has the same number of crossings as does BKA_n.

Let n be even. To draw BKB_n, place half of the nodes evenly on the upper rim of a cylinder and the other half evenly on the bottom. Any pair of nodes on the top (or bottom) disc is joined with a straight line segment. A bottom node and a top node are joined by the shortest helix between them. If necessary, the bottom disc may be rotated slightly so that there is a unique shortest helix between any pair of upper and lower nodes. This construction can be mapped from the cylinder onto the sphere, and hence onto the plane. In fact, it is possible to realize this construction using geodesics on the sphere. See Figure 15.

If n is odd, draw BKB_{n+1} and remove any node.

Note that for n = 3 to 7, BKA_n and BKB_n are identical.

2.3.9 The Singer Drawing

We thank David Singer for having described (over the telephone) his drawing of K_{10} with 62 crossings, and for having sent us his manuscript **[Si]**. This is actually a very notorious drawing. Its existence refutes a conjecture of Guy's **[G4]**, and yet the drawing has never appeared in the literature. For this reason, we are extremely grateful that Dr. Singer has allowed us to include this drawing in our manuscript. "S-9" is an abbreviation for the drawing obtained by removing node 9 from the Singer drawing; S-7,8 is the Singer drawing minus nodes 7 and 8, etc. The Singer drawing of K_{10} is Figure 16.

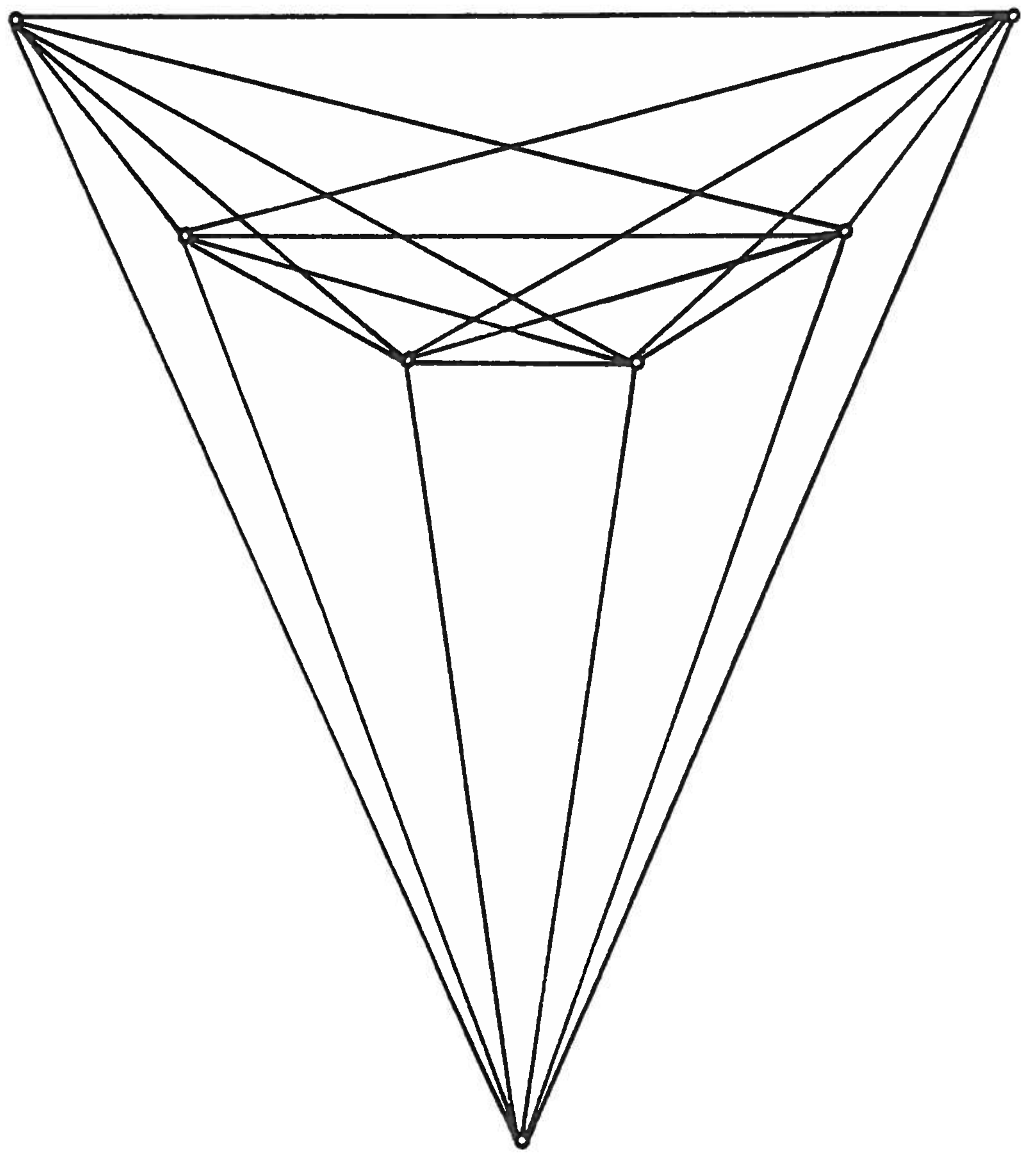


Figure 7. The Drawing WH_7

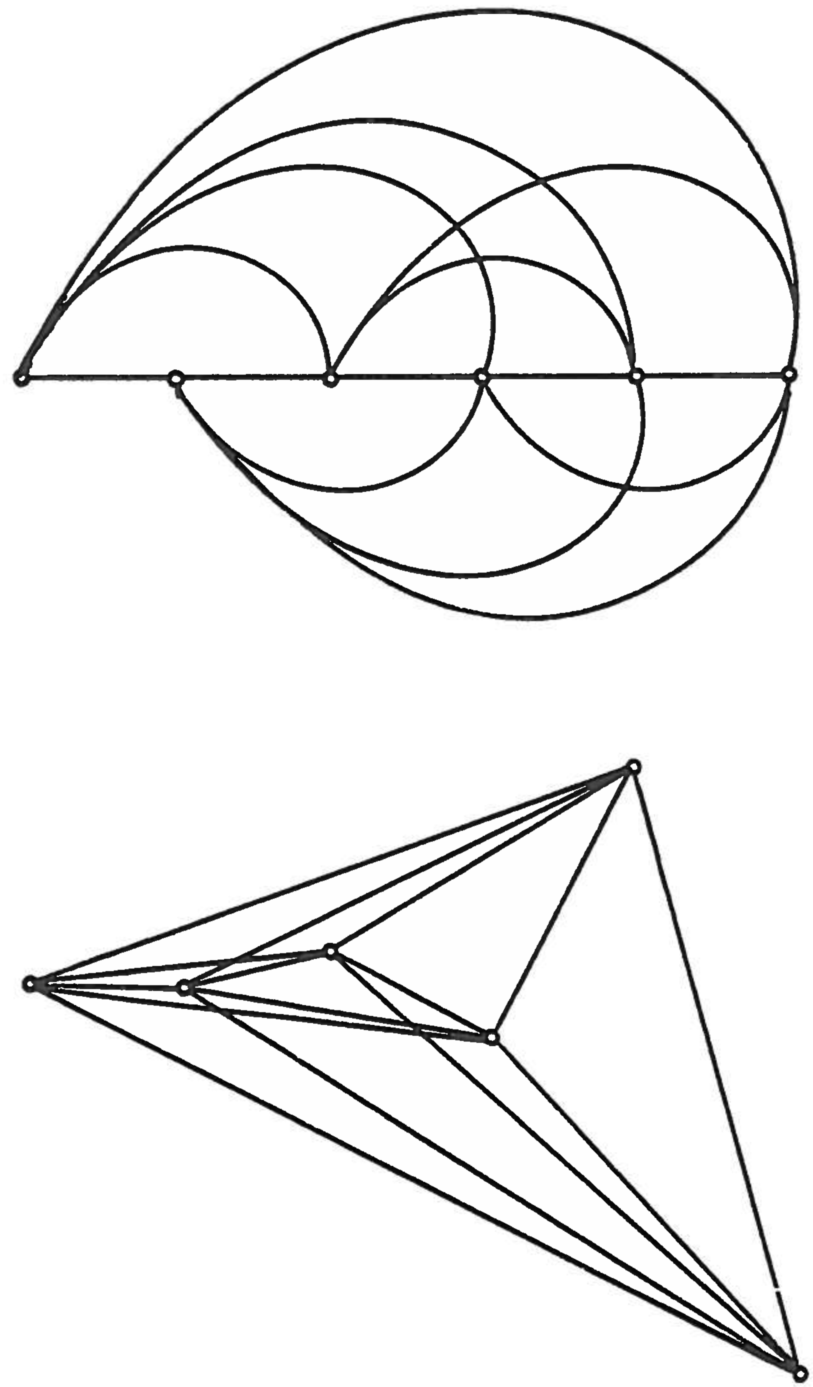


Figure 8. Drawings of ALM_6

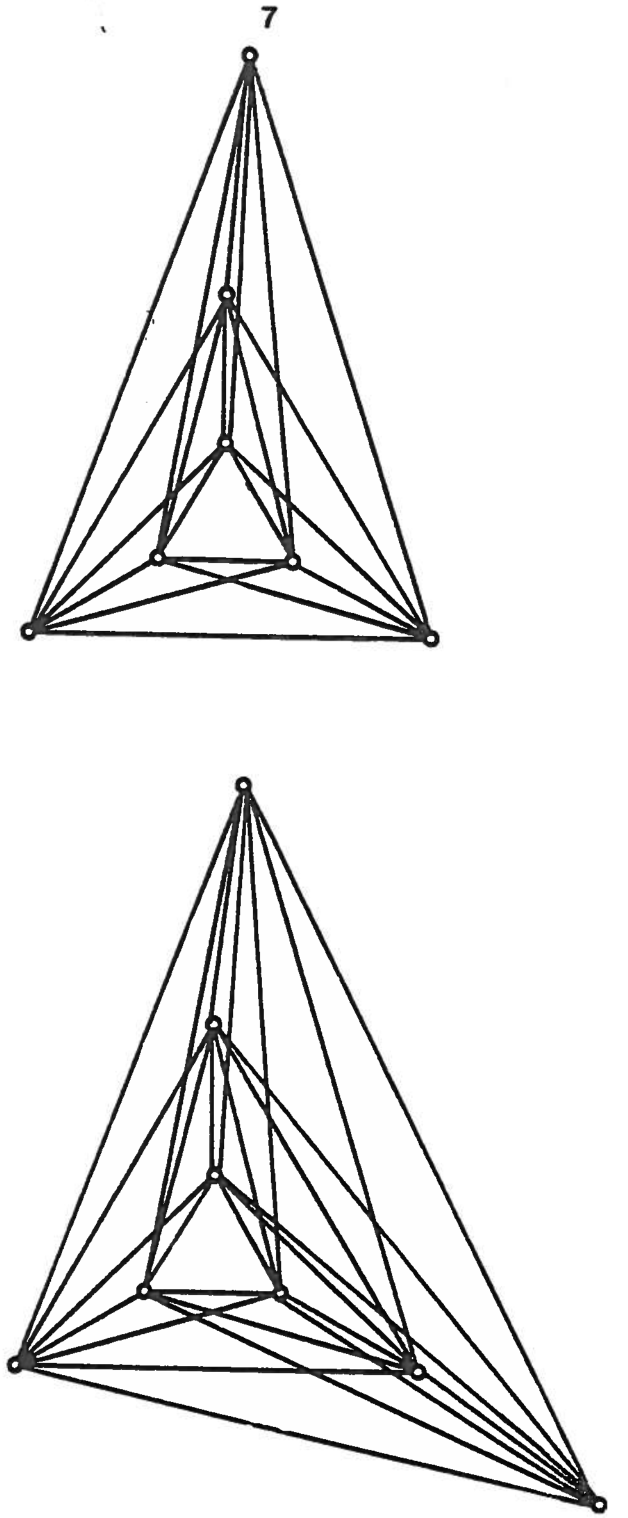


Figure 10. The Drawings TS_7 and TS_8

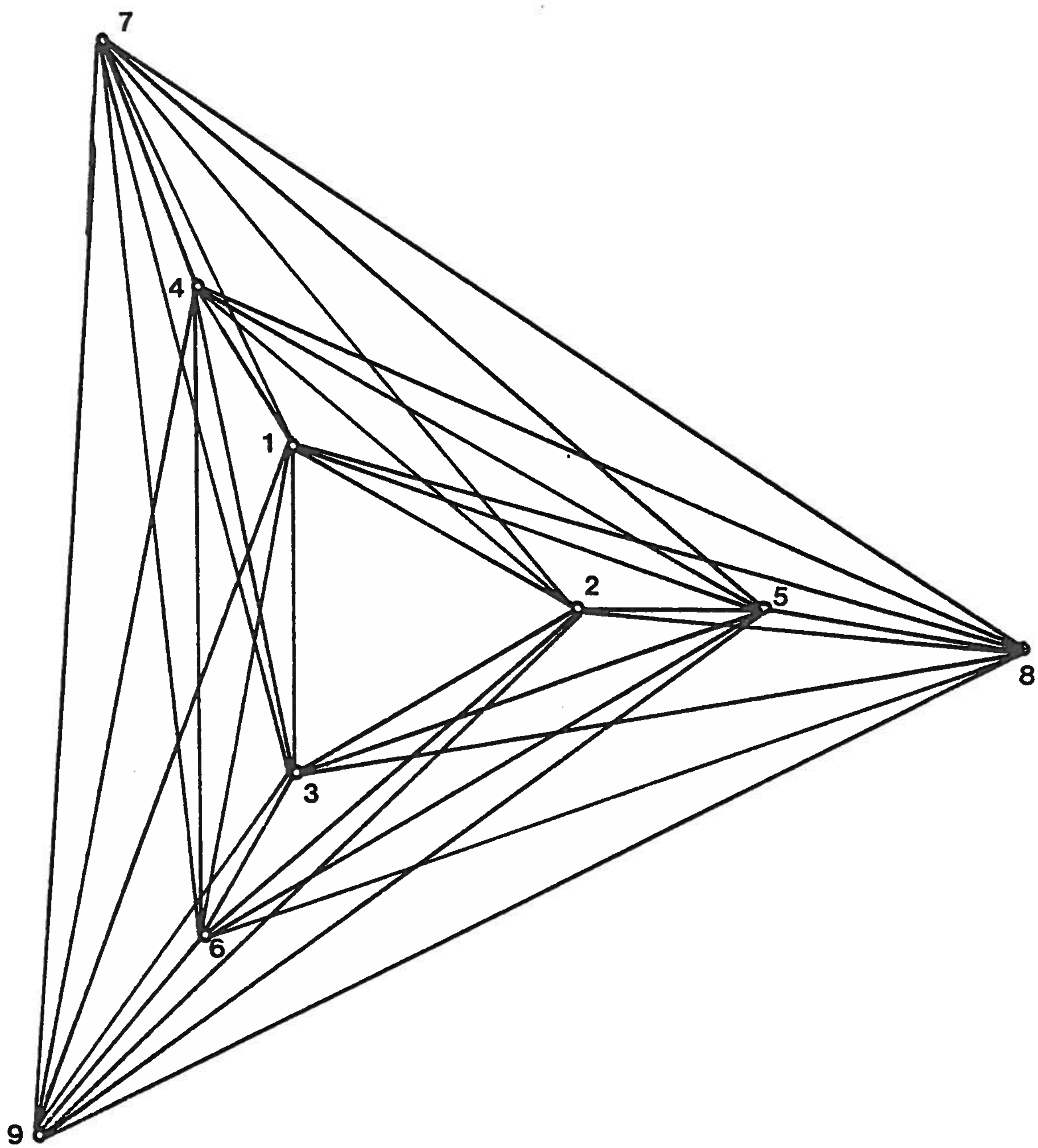


Figure 11. The Drawing TS_9

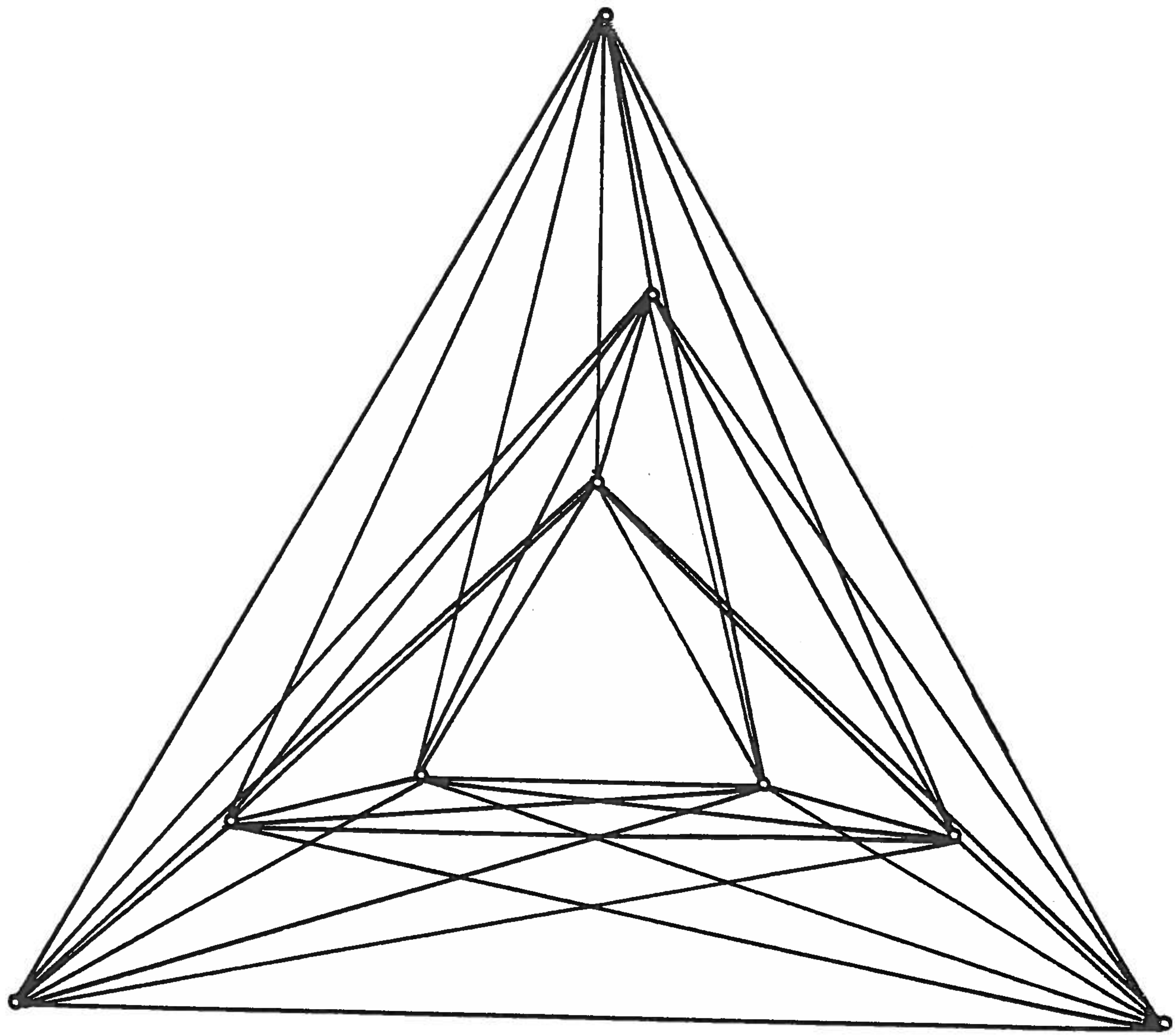


Figure 12. The Drawing FTS_9

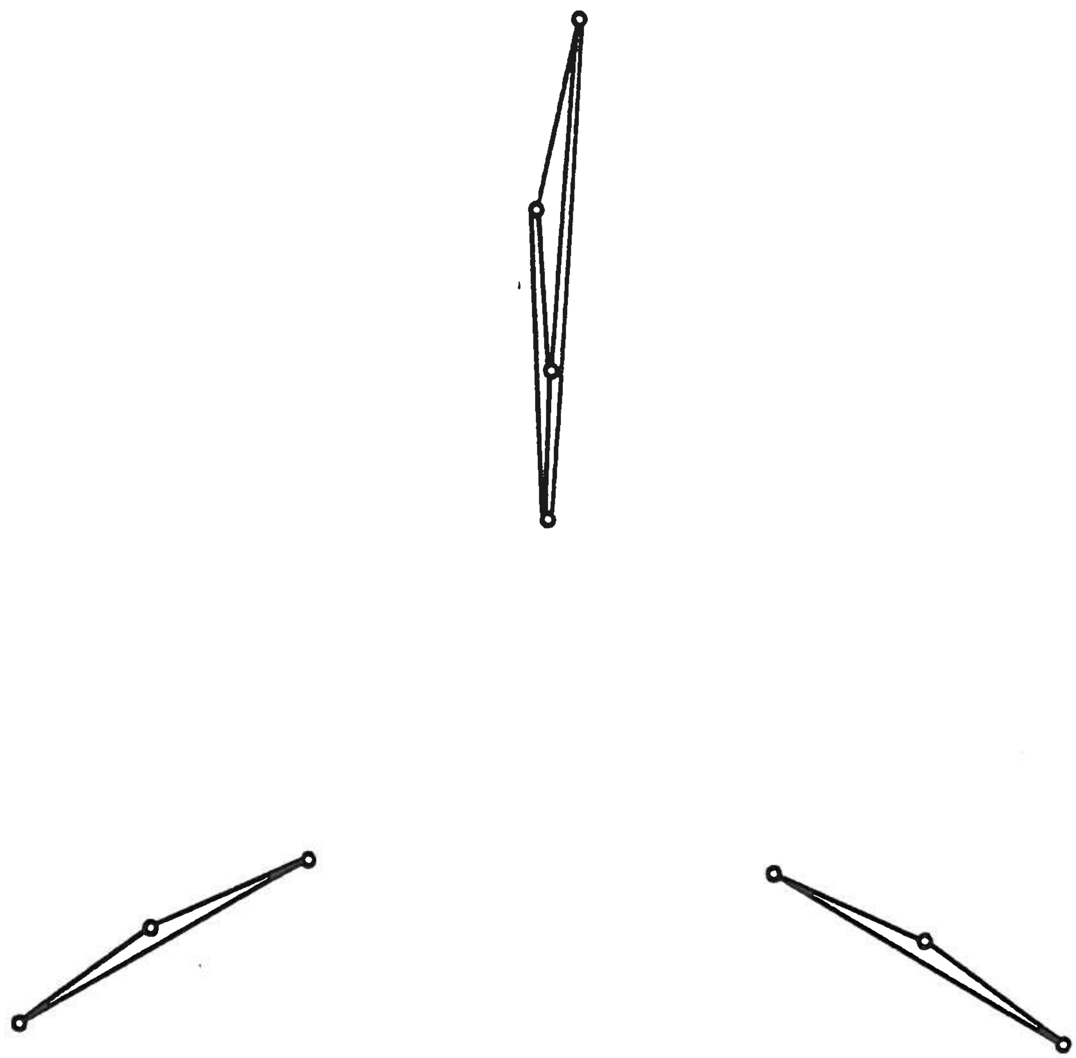


Figure 13. Outline of JE_{10}

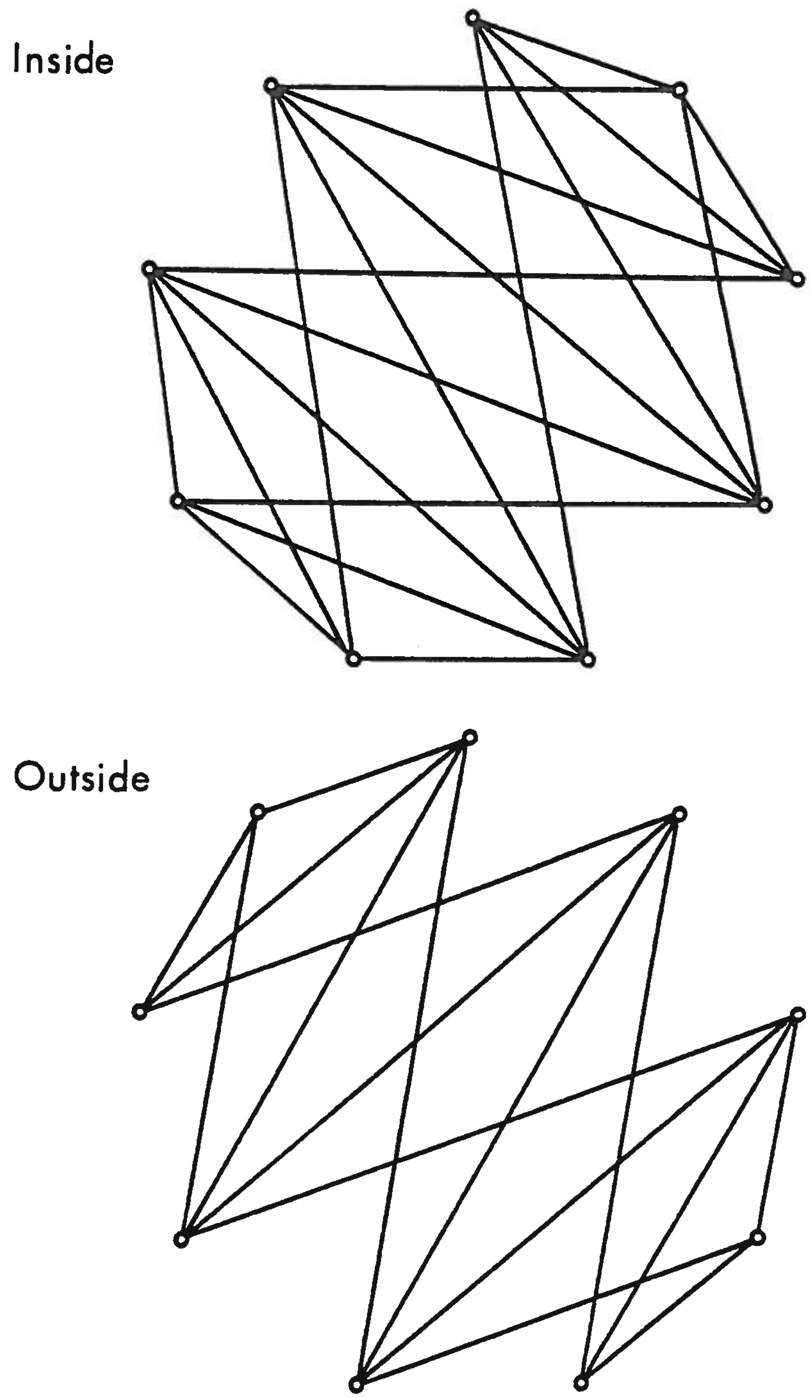


Figure 14. The Drawing BKA_9

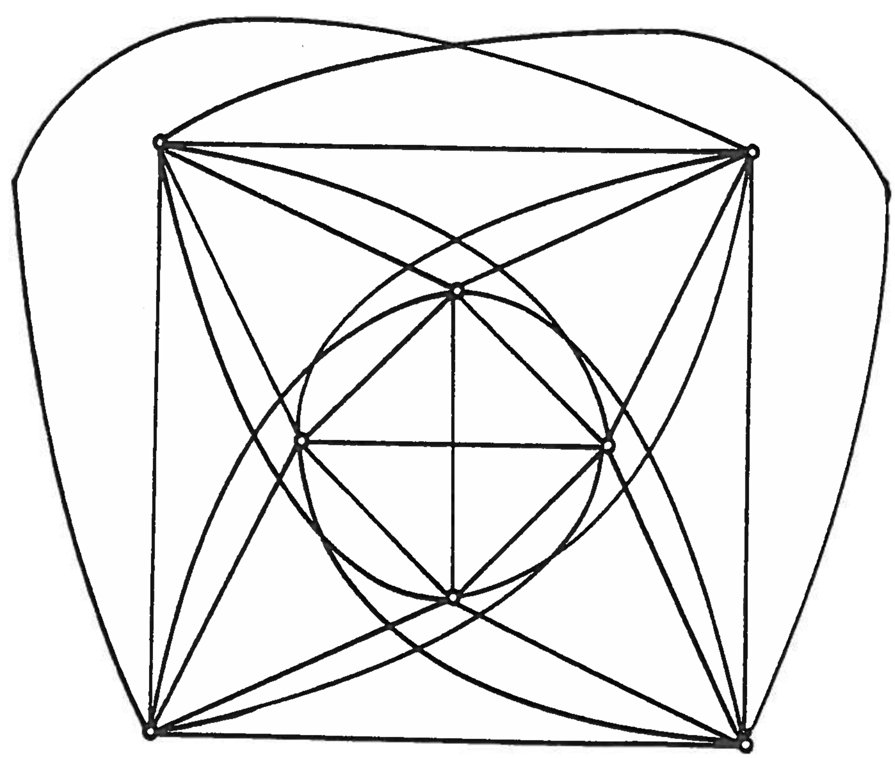


Figure 15. The Drawing BKB_8

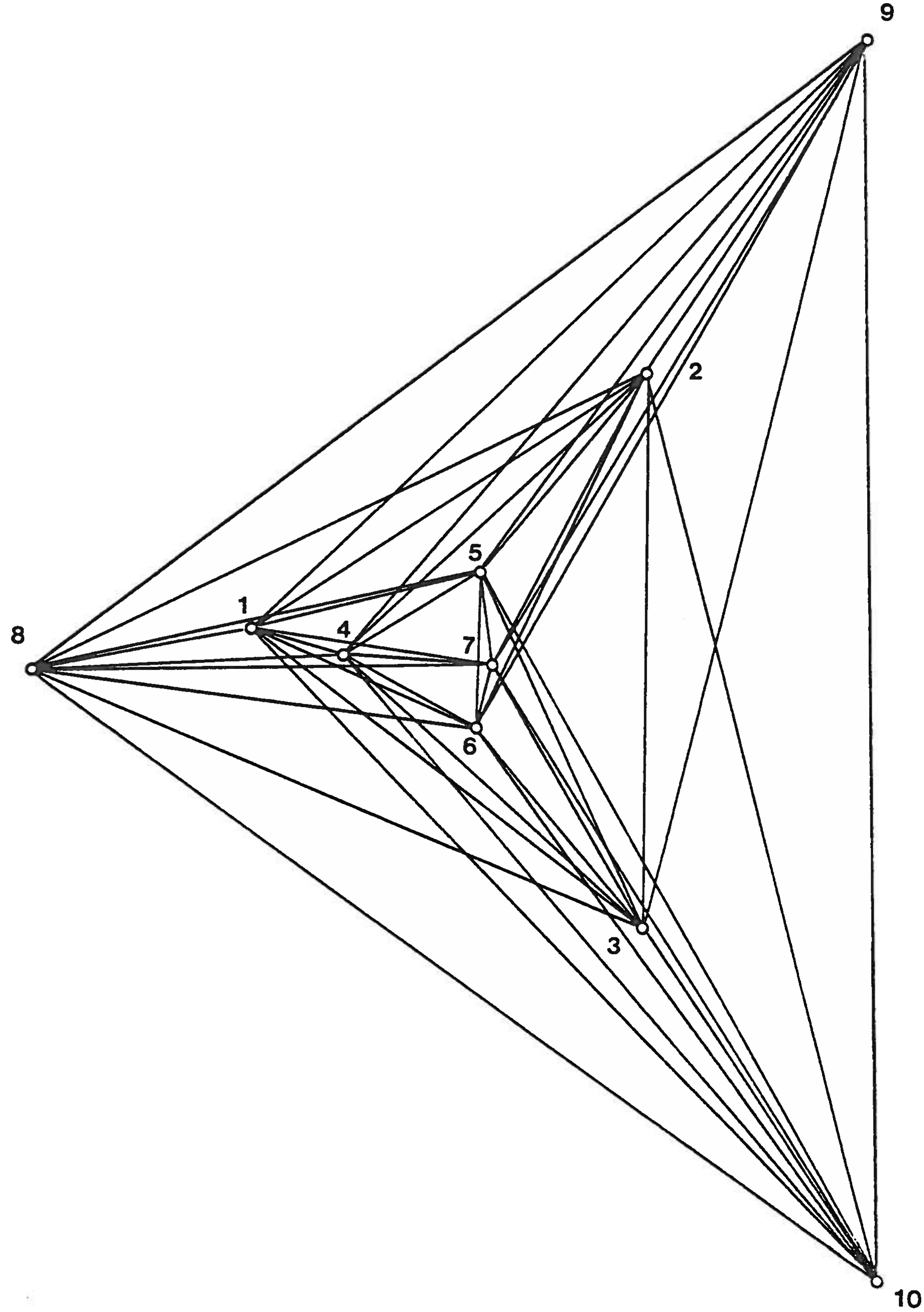


Figure 16. The Singer Drawing of K_{10}

Table IV. Rectilinear Constructions

n	TS_n		FTS_n		JE_n	
	cfhc	x	cfhc	x	cfhc	x
3	1	0	1	0	1	0
4	3	0	3	0	3	0
5	8	1	8	1	8	1
6	29	3	29	3	29	3
7	91	9	91	9	91	9
8	313	20	339	19	339	19
9	1188	36	1228	36	1228	36
10	4154	65	4490	64	4720	63
11	15527	106	18239	102	18383	102
12	62097	159	68387	159	75231	156
13	233042	239	275745	235	306446	231
14	918595	341	-------	331	-------	328
15	3803482	465	-------	465	-------	453

n	WH_n		ALM_n	
	cfhc	x	cfhc	x
3	1	0	1	0
4	3	0	3	0
5	8	1	8	1
6	20	5	24	4
7	48	15	73	11
8	112	35	235	24
9	256	70	778	46
10	576	126	2665	80
11	1280	210	9347	130
12	2816	330	33534	200
13	6144	495	122574	295
14	13312	715	-------	420
15	28672	1001	-------	581

Table V. Spherical (Non-Rectilinear) Constructions

n	x	cfhc(BKA_n)	chfc(BKB_n)
3	0	1	1
4	0	3	3
5	1	8	8
6	3	29	29
7	9	96	96
8	18	369	399
9	36	1403	1461
10	60	5756	6354
11	100	23204	24687
12	150	99649	110162
13	225	434689	446798

2.4 Drawings of K_7 to K_{13}

The five drawings 7A to 7E of Table VI are all of the non-isomorphic x-optimal drawings of K_7 [EG]. Note that three of these can be drawn with straight lines (i.e. are rectilinear). The three figures at the end of this section are the only catalogued drawings not given by any of our constructions. They are included in the catalogue because they are x-optimal. Erdös and Guy state that there are three non-isomorphic Euclidean x-optimal drawings of K_8, of which none are rectilinear [EG]. These are drawings BKA_8, BKB_8, and 8C, all of which have 18 crossings. As there exist rectilinear drawings of K_8 with 19 crossings, the rectilinear crossing number of K_8 is 19. Three such rectilinear-optimal drawings are catalogued in Table VI; there may exist others.

For K_9, the crossing number is equal to the rectilinear crossing number for the last time; Guy has shown that $\bar{\nu}(n) > \nu(n)$ for $n > 9$. Unfortunately, there are about 400 non-isomorphic x-optimal drawings of K_9 [G5] . In Table VII we have included only the 6 given by our 7 constructions (FTS_9 and JE_9 are isomorphic), as well as two others that can be obtained by deleting a pair of nodes from Singer's drawing of K_{10}. Two further drawings, which are also sub-drawings of Singer's drawing but not x-optimal, are included because they both contain more cfhc's than any of the other catalogued rectilinear drawings, and also improve Newborn and Moser's lower bound for $\bar{\phi}(9)$ from 1228 to 1252.

Table VII includes only the 7 drawings of K_{10} given by our constructions, plus the Singer drawing. Of these, the two BK drawings are x-optimal in E. It is not known how many x-optimal drawings of K_{10} there are. The catalogue also includes the Singer drawing, which is the only known rectilinear drawing of K_{10} with 62 or fewer crossings. It is known that the rectilinear crossing number of K_{10} is 61 or 62 [Si] [G3] [BW] . For

K_{11} to K_{13}, our catalogue includes only those drawings given by our 7 constructions. See Tables IV and V of the previous section. The catalogue ends at about K_{13} due to computer time constraints: about 20 hours of c.p.u. time is necessary to count cfhc's of a drawing of K_{14}. For K_{14} and K_{15}, the only drawings for which the number of cfhc's was counted were TS_{14} and TS_{15}.

All drawings of this section are catalogued in Tables IV to VII. The last table in the chapter, Table VIII, summarizes the lower bounds for $\Phi(n)$ and $\overline{\Phi}(n)$ that are established by drawings of our catalogue. The tables appear after the following three figures, which are the only x-optimal drawings of K_7 or K_8 not given by any of our constructions.

Note that the lower bounds for $\Phi(n)$ and $\overline{\Phi}(n)$ for $n \geq 9$ are new as of this paper. See Table I and Table VIII.

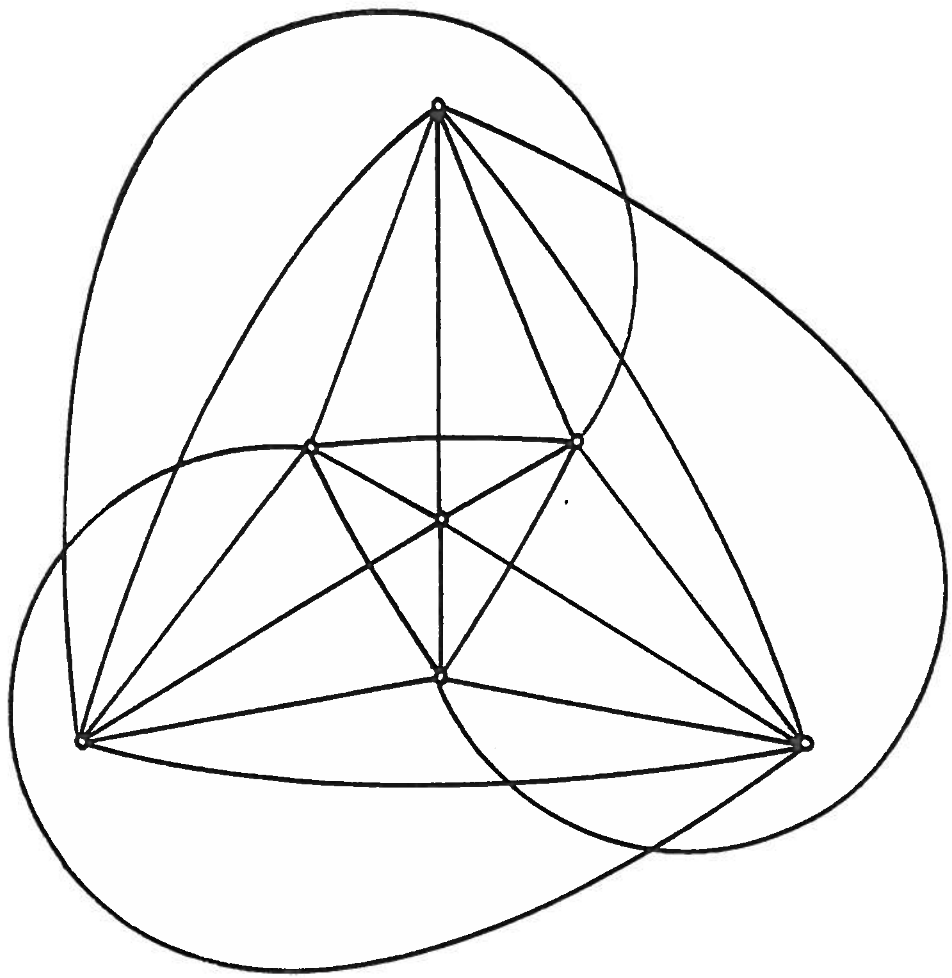


Figure 17. The Drawing 7B

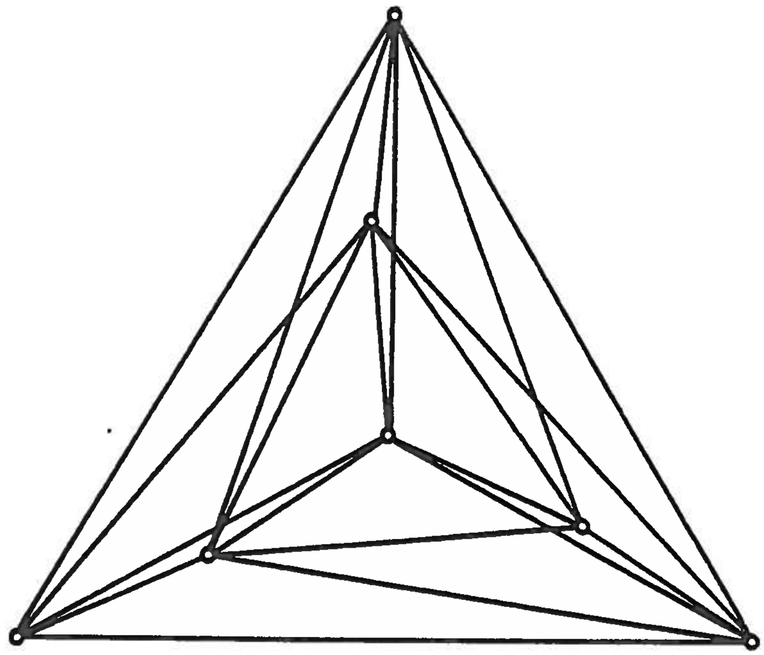


Figure 18. The Drawing 7E

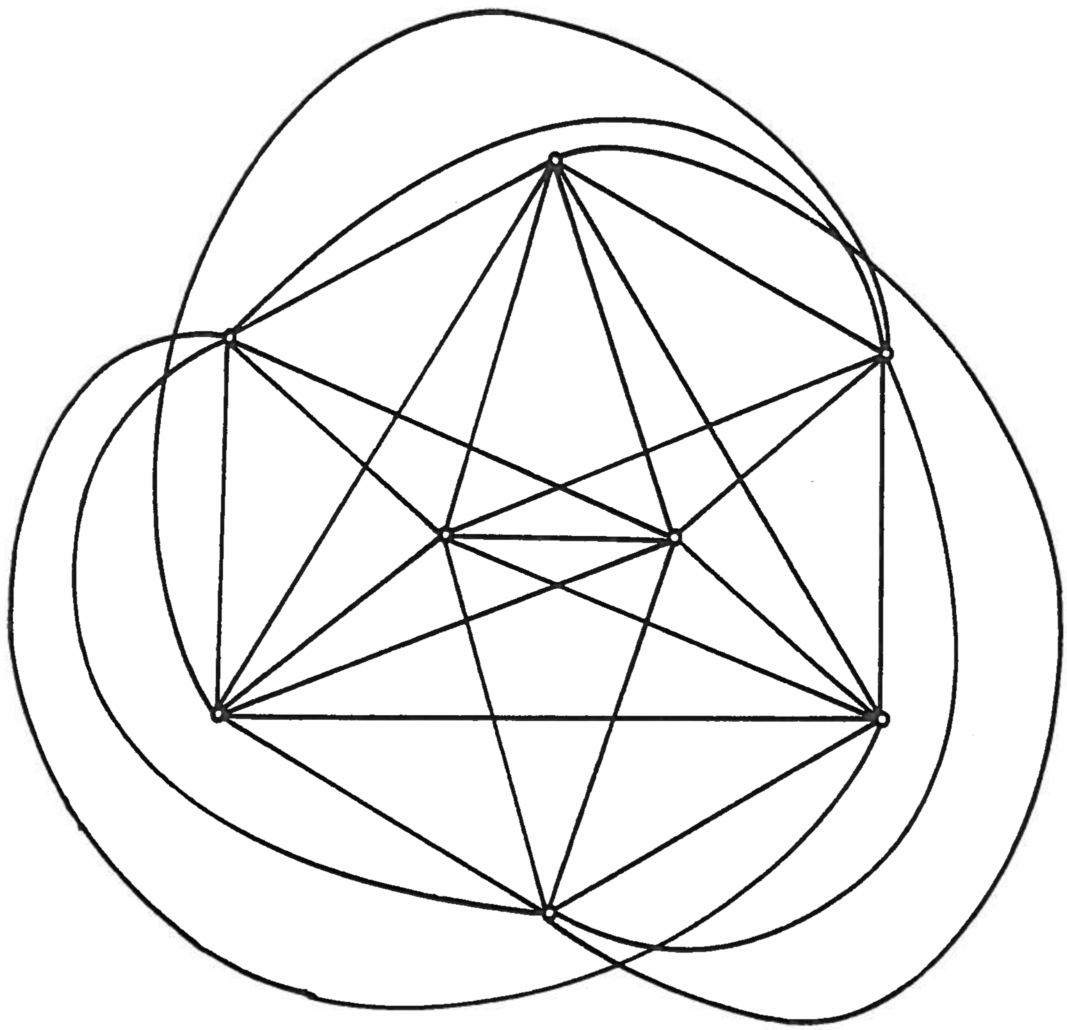


Figure 19. The Drawing 8C

Table VI. Drawings of K_7 and K_8

Drawing	Arc Resp.			Node Resp.	x	cfhc
K_7	# with					
	3	2	1			
7A *	2	2	8	6 6 6 5 5 4 4	9	96
7B *	3		9	6 6 6 5 5 5 3	9	96
7C	1	4	7	6 6 5 5 5 5 4	9	92
7D		7	4	6 6 6 5 5 4 4	9	91
7E		6	6	6 5 5 5 5 5 5	9	87
7F					11	73
7G					15	48

K_8							
	# with						
	4	3	2	1			
BKB *		8		12	9 9 9 9 9 9 9 9	18	399
BKA *		6	4	10	9 9 9 9 9 9 9 9	18	369
8C		5	7	7	9 9 9 9 9 9 9 9	18	354
FTS,JE	2		15		10 10 10 10 10 10 8 8	19	339
S-8,10	1	4	9	4	10 10 10 10 10 10 8 8	19	335
S-8,9	2	5	6	5	11 11 11 11 9 9 9 9	20	331
TS	2	4	9	2	11 11 11 11 11 11 7 7	20	313
ALM						24	235
WH						35	112

The arc responsibilities' column gives the number of arcs with responsibilities 1 to 4; none of the above drawings have any arcs with greater responsibility. All above rectilinear drawings have three convex hull arcs. Non-rectilinear drawings are followed by *.

Drawing 7A is both BKA_7, BKB_7; drawing 7C is S-8,9,10, i.e. the Singer drawing with nodes 8, 9, and 10 removed. Drawing 7D is TS_7, FTS_7, and JE_7. Drawing 7F is ALM_7; drawing 7G is WH_7. FTS_8 is Singer-8,7 ; TS_8 is Singer-8,6.

Table VII. Drawings of K_9 and K_{10}

Drawing	x	cfhc
K_9		
BKB *	36	1461
BKA *	36	1403
Singer-8	38	1252
Singer-9	38	1233
Singer-7, FTS, JE	36	1228
Singer-10	36	1218
TS	36	1188
Singer-5	36	1184
ALM	46	778
WH	70	256
K_{10}		
BKB *	60	6354
BKA *	60	5756
Singer	62	4956
JE	63	4720
FTS	64	4490
TS	65	4154
ALM	80	2665
WH	126	576

All above rectilinear drawings have three convex hull arcs. Non-rectilinear drawings are followed by *.

Table VIII. Lower bounds for $\overline{\phi}(n)$ and $\phi(n)$ from Catalogue

n	lower bound for $\overline{\phi}(n)$	drawing	lower bound for $\phi(n)$	drawing
3	*1	3A	*1	3A
4	*3	4A	*3	4A
5	*8	5A	*8	5A
6	*29	6A	*29	6A
7	**92	Singer-8,9,10	**96	BKB
8	**339	Singer-8,7	**399	BKB
9	1252	Singer-8	1461	BKB
10	4956	Singer	6354	BKB
11	18383	JE	24687	BKB
12	75231	JE	110162	BKB
13	306446	JE	446798	BKB
14	918595	TS	-------	----
15	3803482	TS	-------	----

For K_{14} and K_{15}, TS_n was the only construction for which the number of cfhc's was computed.

* These numbers are cfhc-optimal [NM] ; see Table I.

** These are the same lower bounds obtained by Newborn and Moser; see Table I.

All other lower bounds are new as of this paper.

3. AN IMPROVED LOWER BOUND FOR $\overline{\phi}(n)$

3.1 Introduction

In this chapter we develop and generalize slightly the recursive counting argument used by Selim Akl in [A1] and [A2]. In the first section, we present a more detailed description of TS_n than that given in the catalogue, as this drawing of K_n plays an integral role in our argument. In the second section we present a summary of Akl's techniques. The following section shows how his arguments can be generalized. Finally, in the last section is a refinement of our generalization.

Selim Akl counted the number of cfhc's of TS_n that have arcs only from the outermost three nodes to the next outermost three nodes, and from the latter set of nodes to the next outermost three nodes, etc. We improve his result by counting the number of cfhc's of TS_n that have arcs from the outermost three nodes to any set of three nodes up to the fifth outermost set, and from that set of nodes anywhere up to the ninth outermost set of three nodes, etc.

Our exposition is such that we are able to use computer generated data in our counting argument. In fact, with access to greater computer resources, our lower bound for $\overline{\phi}(n)$ could be easily improved.

3.2 A Description of TS_n

3.2.1 Definition of TS_n

TS_n is a mnemonic for "triangular spiral on n nodes". To draw TS_n, begin with $[\frac{n+2}{3}]$ concentric, similar, symmetrically parallel triangles (hereafter referred to as template triangles, or tt's). [We call two similar concentric triangles **symmetrically parallel** if and only if the outer triangle can be mapped onto the inner triangle without rotation.] Note that the nodes of the tt's lie on three rays with the same origin. In order to avoid having any three nodes collinear (as this is a rectilinear construction, the above would violate the definition of a drawing), gently spiral the three rays clockwise. That is, rotate the outermost tt clockwise by some very small but positive angle θ, rotate the next outermost tt clockwise an even smaller angle θ', etc. Label the nodes of the innermost tt in clockwise order as nodes 1, 2, 3 of TS_n. Complete the labelling so that corresponding nodes of adjacent tt's have labels that differ by three. Note that only n of the $3\,[\frac{n+2}{3}]$ nodes of the tt's are nodes of TS_n. To complete the drawing of TS_n, draw straight line segments between all $\binom{n}{2}$ pairs of nodes.

The drawings TS_3 to TS_9 are included in Figures 9 to 11 of the previous chapter.

3.2.2 Crossings of TS_n

We relabel the nodes in order to facilitate listing the crossings (i.e. crossing arc pairs) of TS_n. Let $a = [\frac{n+2}{3}]$, $b = [\frac{n+1}{3}]$, and $c = [\frac{n}{3}]$. The three rays of TS_n will be called α, β and γ, in clockwise order with the first containing the most nodes, and the last the fewest. More precisely, we relabel

nodes	1, 4, . . . , 3a - 2	as	$\alpha_1, \alpha_2, \ldots, \alpha_a$
nodes	2, 5, . . . , 3b - 1	as	$\beta_1, \beta_2, \ldots, \beta_b$
nodes	3, 6, . . . , 3c	as	$\gamma_1, \gamma_2, \ldots, \gamma_c$.

The following is the list of all crossings of TS_n. We point out that the reason for gently spiralling the tt's was to effect exactly the following set of crossings. The set of crossings will obviously be different if the spiralling is exaggerated. See Figures 9 to 11 of the previous chapter.

1) $(\alpha_i, \alpha_k)(\alpha_j, \alpha_m)$ $\quad 1 \leq i < j < k < m \leq a$

2) $(\beta_i, \beta_k)(\beta_j, \beta_m)$ $\quad 1 \leq i < j < k < m \leq b$

3) $(\gamma_i, \gamma_k)(\gamma_j, \gamma_m)$ $\quad 1 \leq i < j < k < m \leq c$

4) $(\alpha_i, \beta_k)(\alpha_j, \beta_m)$ $\quad 1 \leq i < j \leq a \quad 1 \leq k < m \leq b$

5) $(\beta_i, \gamma_k)(\beta_j, \gamma_m)$ $\quad 1 \leq i < j \leq b \quad 1 \leq k < m \leq c$

6) $(\gamma_i, \alpha_k)(\gamma_j, \alpha_m)$ $\quad 1 \leq i < j \leq c \quad 1 \leq k < m \leq a$

7) $(\alpha_i, \alpha_k)(\alpha_j, \beta_m)$ $\quad 1 \leq i < j < k \leq a \quad 1 \leq m \leq b$

8) $(\beta_i, \beta_k)(\beta_j, \gamma_m)$ $\quad 1 \leq i < j < k \leq b \quad 1 \leq m \leq c$

9) $(\gamma_i, \gamma_k)(\gamma_j, \alpha_m)$ $\quad 1 \leq i < j < k \leq c \quad 1 \leq m \leq a$

Case 1): The number of crossings of this type is the number of distinct four-tuples of nodes α_i, namely $\binom{a}{4}$ if $a \geq 4$, else 0.

Case 4): There is a crossing of this type for every distinct pair of nodes α_i, α_j coupled with any distinct pair of nodes β_k, β_m. The number of such crossings is $\binom{a}{2}\binom{b}{2}$ if a and b are both ≥ 2 , else 0.

Case 7): Every triple of nodes $\alpha_i\ \alpha_j\ \alpha_k$, together with a node β_m gives such a crossing, namely $b\binom{a}{3}$ if $a \geq 3$, else 0.

The total number of crossings of TS_n is therefore

$$\begin{aligned} &\binom{a}{4} + \binom{b}{4} + \binom{c}{4} \\ + &\binom{a}{2}\binom{b}{2} + \binom{b}{2}\binom{c}{2} + \binom{c}{2}\binom{a}{2} \\ + &\binom{a}{3} b + \binom{b}{3} c + \binom{c}{3} a \end{aligned}$$

where we define $\binom{x}{y}$ to be 0 if $y > x$. This gives

$$x(TS_n) = \frac{11n^4 - 90n^3 + 225n^2 - 162n}{648},$$

$$\frac{11n^4 - 90n^3 + 249n^2 - 290n + 120}{648}, \quad \text{and}$$

$$\frac{11n^4 - 90n^3 + 249n^2 - 250n + 48}{648}$$

for n congruent to 0, 1 and 2 (mod 3) respectively.

3.3 Akl's Counting Argument

In **[A1]** Akl computes a lower bound for $\overline{\phi}(n)$. We now present the essential details of his work, following his notation as closely as possible.

A lower bound for $\overline{\phi}(n)$ is obtained by counting all cfhc's of the rectilinear drawing D_n. We paraphrase Akl's description of D_n : n nodes are placed in the plane to form [n/3] concentric triangles, and the nodes of each triangle are then connected to those of the next smaller one by all possible arcs **[A1]** .

Define a **sub-drawing** of a drawing D to be a drawing obtained by removing some arcs and possibly nodes from D. We will alter Akl's definition slightly by defining D_n as a sub-drawing of TS_n.

3.3.1 Recursive definition of D_n

Nodes of D_n are placed and numbered as for TS_n.

0) Define D_4, D_5, D_6 to be TS_4, TS_5, TS_6.

We now define D_n in terms of D_{n-3}, for $n \geq 7$:

1) D_n contains all arcs of D_{n-3}. Also, all pairs of nodes of $\{ n, n-1, \ldots, n-5 \}$ are joined with arcs.

Thus, the following is all arcs of D_n:

all node pairs of $\{ n, n-1, \ldots, n-5 \}$

all node pairs of $\{ n-3, n-4, \ldots, n-8 \}$

. . .

all node pairs of $\{ n-3t, n-3t-1, \ldots, n-3t-5 \}$, where $t = [\frac{n-6}{3}]$

plus all node pairs of $\{ 1, 2, \ldots, q \}$, where q = 4, 5, and 6 for n congruent to 4, 5, and 6 respectively.

3.3.2 The number of cfhc's of D_n

For non-negative i, define an **i-cfhc** of a rectilinear drawing to be a cfhc that includes exactly i of the convex hull arcs of the drawing. Figure 20 shows one way in which the convex hull arc of a 1-cfhc of D_{n-3} can be removed to construct a 2-cfhc of D_n.

Note that every 1-cfhc of D_{n-3} yields 5 2-cfhc's of D_n, and that every 2-cfhc yields 4 1-cfhc's and 10 2-cfhc's. See Figures 21 to 23.

Let d_i^n be the number of i-cfhc's of D_n. Note that for n > 6, D_n has no 0-cfhc's or 3-cfhc's. Thus for n > 6, $\text{cfhc}(D_n) = d_n^1 + d_n^2$. The above gives the following system of equations for n > 6:

$$\begin{aligned} d_n^1 &= \qquad\qquad 4\, d_{n-3}^2 \\ d_n^2 &= 5\, d_{n-3}^1 + 10\, d_{n-3}^2 \end{aligned} \qquad (1)$$

Once d_6^1 and d_6^2 are calculated (this is easily done by simply drawing all 29 cfhc's of D_6), a closed form expression for $\text{cfhc}(D_n)$ can be determined from (1) for n congruent to 0 (mod 3). Similarly, counting d_4^1 and d_4^2 from the three cfhc's of D_4 leads to a closed form expression for $\text{cfhc}(D_n)$ for n congruent to 1 (mod 3), and finding d_5^1 and d_5^2 from the 8 cfhc's of D_5 gives $\text{cfhc}(D_n)$ for n congruent to 2 (mod 3). In all three cases, for large n $\text{cfhc}(D_n) \doteq c * 2.270719168^n$, where c is a constant.

3.3.3 Akl's improvement

In **[A2]**, Akl used the exact same techniques described in section **3.3.2**, except that 0) of the recursive definition of D_n was changed to defining D_7, D_8, and D_9 to be drawings FTS_7, FTS_8, and FTS_9. The result is that one again arrives at (1) , but with improved initial conditions. Although the coefficients in the solution of the new system will be larger than in (1), $\text{cfhc}(D_n)$ remains asymptotically $c * 2.270719168^n$.

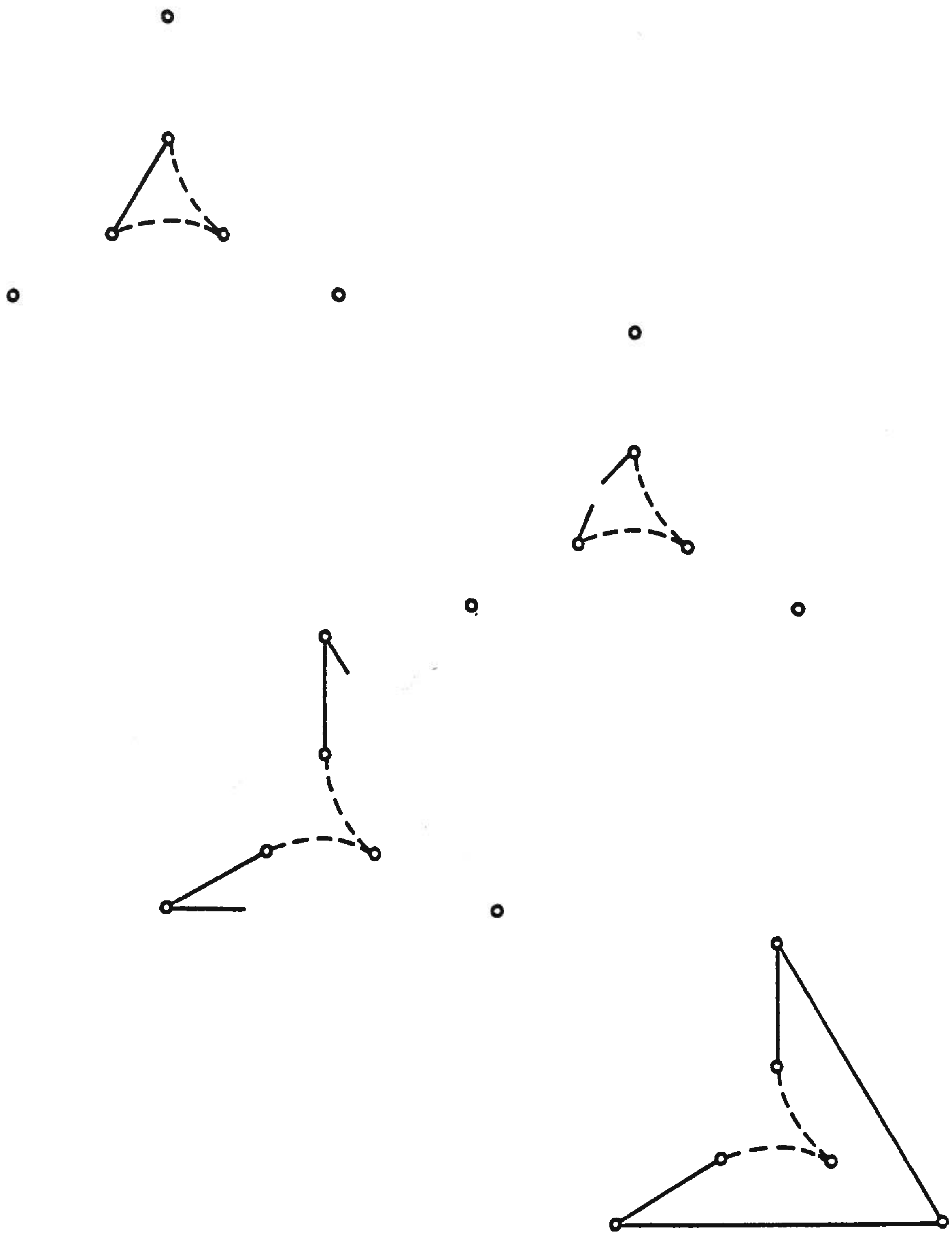


Figure 20. A 2-cfhc of D_n from a 1-cfhc of D_{n-3}

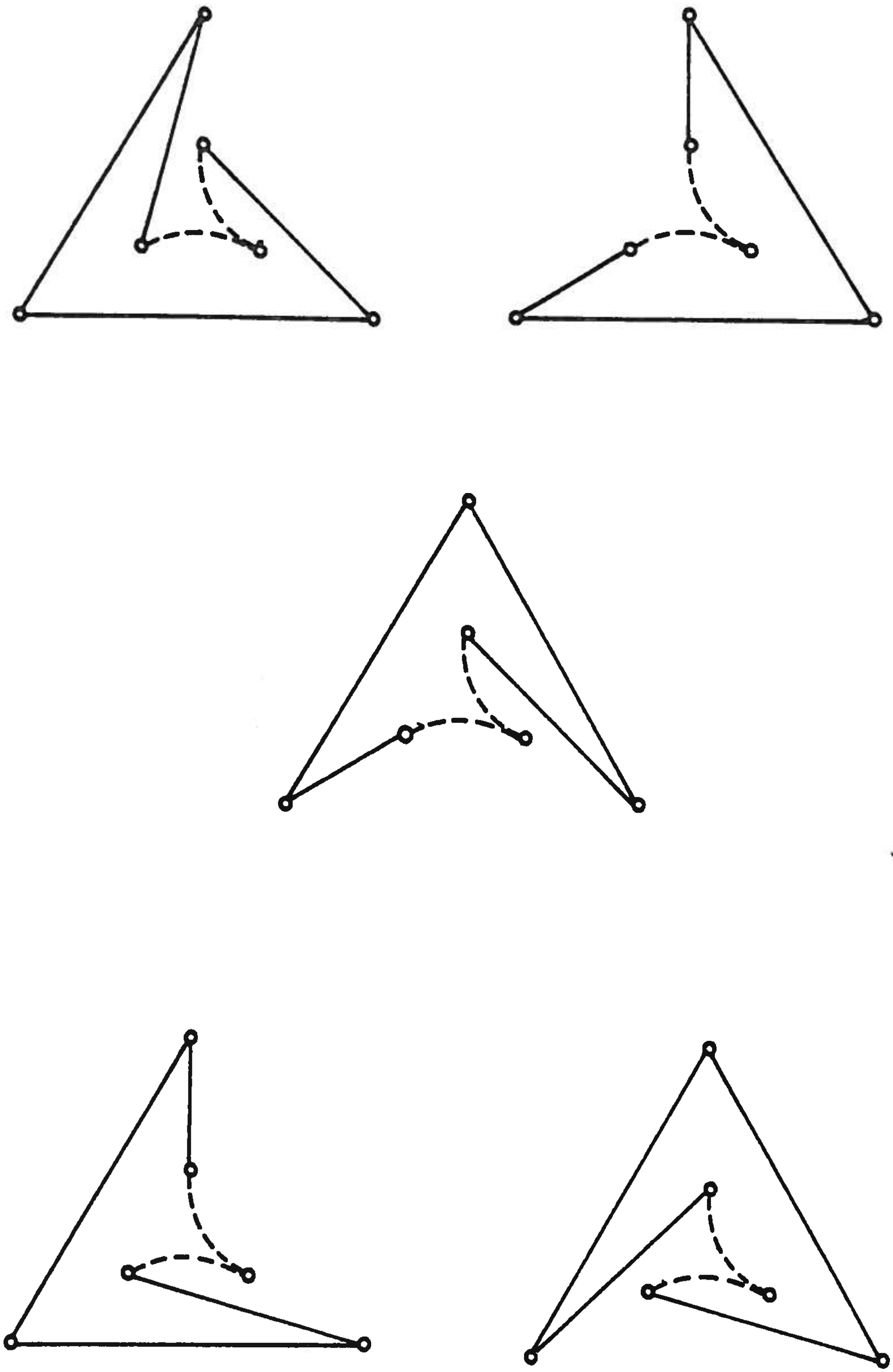


Figure 21. 2-cfhc's of D_n from 1-cfhc's of D_{n-3}

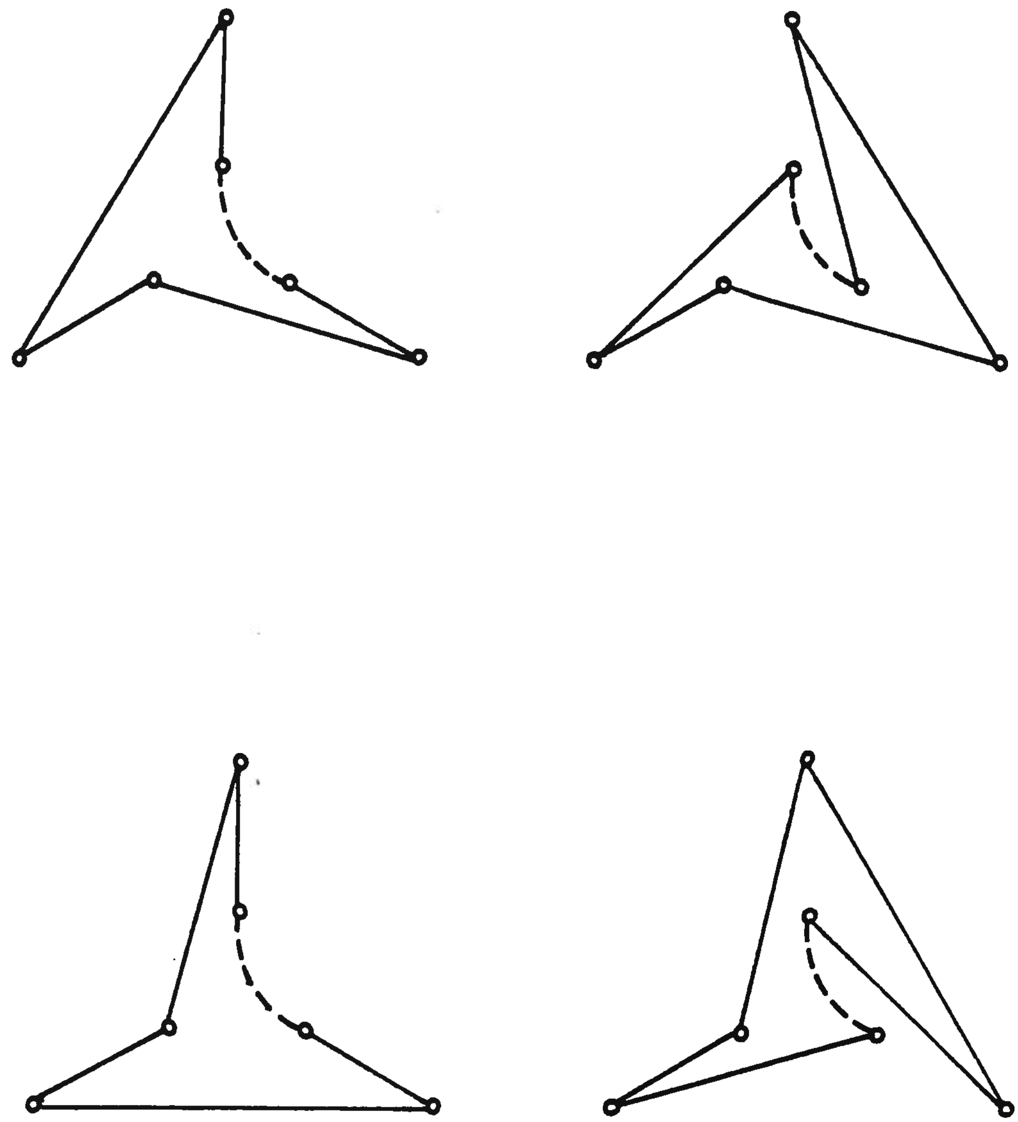


Figure 22. 1-cfhc's of D_n from 2-cfhc's of D_{n-3}

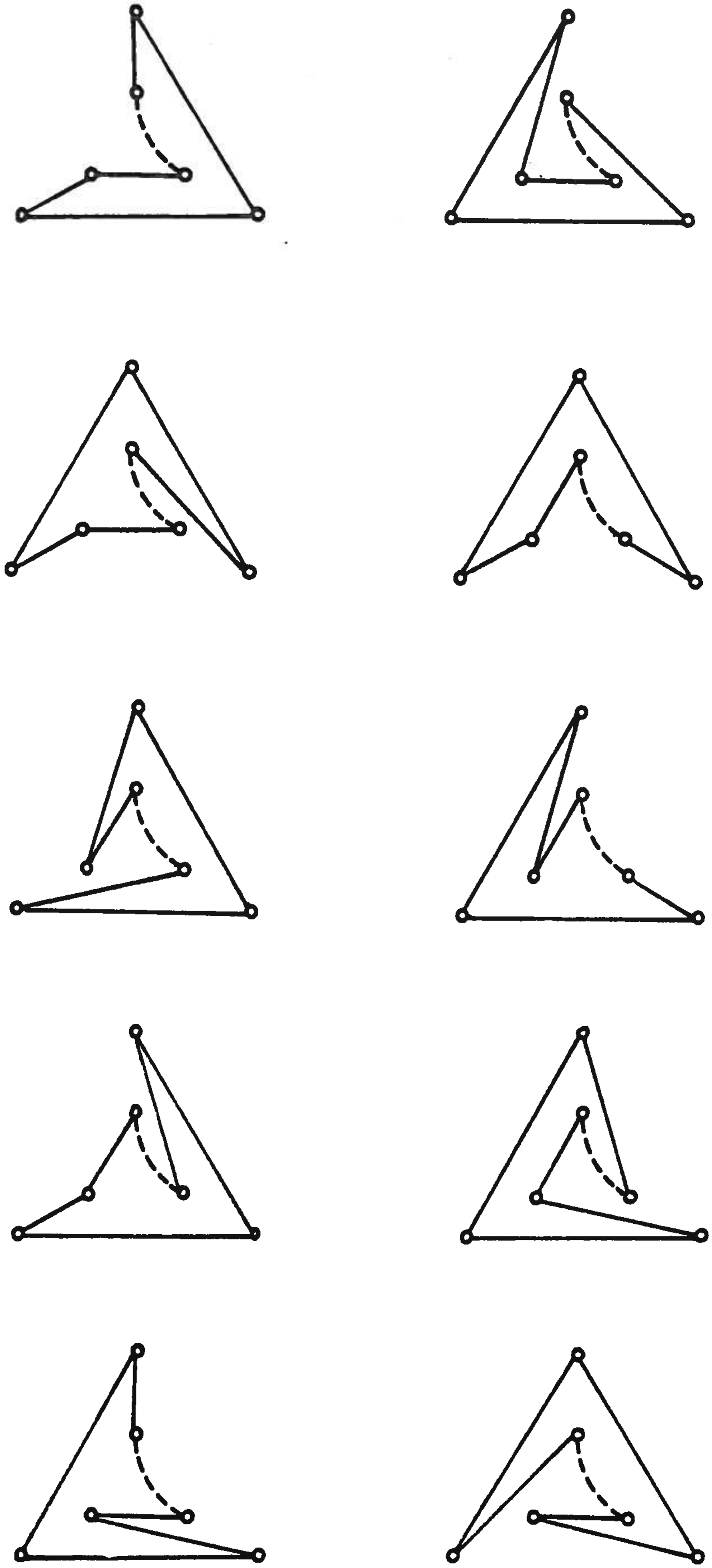


Figure 23. 2-cfhc's of D_n from 2-cfhc's of D_{n-3}

3.4 Generalization of Akl's Method

3.4.1 The basic idea

Recall that D_n is constructed from D_{n-3} by placing three nodes outside D_{n-3}, and joining each of these three nodes to each of the three nodes on the convex hull of D_{n-3}. We expand on this idea by starting with a "kernel" drawing, and adding multiples of three nodes outside the kernel, and again joining all outer nodes to the kernel's convex hull. E_n, F_n, and G_n are the drawings thus created when respectively six, nine, and twelve nodes are added outside the kernel. As the techniques used to count cfhc's of these three sub-drawings of TS_n are very similar, we will provide a meticulous description of how to compute cfhc(E_n), and only a brief description of how to compute cfhc(F_n) and cfhc(G_n).

3.4.2 Recursive definition of E_n

Nodes of E_n are placed and numbered as for TS_n.

0) Define E_{10} to E_{15} to be TS_{10} to TS_{15} respectively.

We now define E_n in terms of E_{n-6}, for $n \geq 16$:

1) E_n contains all arcs of E_{n-6}. Also, all pairs of nodes of $\{$ n, n-1, . . . , n-8 $\}$ are joined with arcs.

The following is a list of all the arcs of E_n:

all node pairs of $\{$ n, n-1, . . . , n-8 $\}$

all node pairs of $\{$ n-6, n-7, . . . , n-14 $\}$

. . .

all node pairs of $\{$ n-6t, n-6t-1, . . . , n-6t-8 $\}$, where $t = [\frac{n-15}{6}]$

plus all node pairs of $\{$ 1, 2, 3, . . . , q $\}$, where q = 10 to 15 for n congruent to 10 to 15 (mod 6) respectively.

3.4.3 The number of cfhc's of E_n

We will count cfhc's of E_n as we counted cfhc's of D_n. For example, we ask: how many 0-cfhc's, 1-cfhc's and 2-cfhc's of E_n can be constructed from a 1-cfhc of E_{n-6} ? (*?*)

Let us consider this question more carefully. Figure 24 shows how a cfhc of E_n is constructed from a 1-cfhc of E_{n-6}. The lone convex hull arc of E_{n-6} is removed and the resulting path is joined with the outermost 6 nodes of E_n. The question is : in how many different ways can the above procedure be performed ? As we shall see, this question is easily answered. But first, we need some more definitions.

Define a **shape** of TS_9 to be a set of cfhc's of TS_9, such that any of the cfhc's of the set can be rotated to give any of the other cfhc's. Note that the set of all cfhc's of TS_9 partitions into shapes. Define the set of **inner arcs** of TS_n as $\{ (1, 2), (1, 3), (2, 3) \}$. The set of **convex hull arcs** of TS_n is $\{ (n, n-1), (n, n-2), (n-1, n-2) \}$.

Figure 25 is a visual description of all shapes of TS_9 whose cfhc's all have exactly two inner arcs. Note the correspondence between Figure 24 and Figure 25 ! In fact, the number of different ways a cfhc of E_n can be constructed from a 1-cfhc of E_n is exactly the number of different shapes of TS_9 that have exactly two inner arcs. Ergo, in order to answer (*?*) we need to know how many shapes of TS_9 with 2 inner arcs have 0, 1, and 2 convex hull arcs.

There exist cfhc's of TS_9 which are unchanged by rotation. However, all cfhc's that have either 1 or 2 inner arcs can be changed by rotation. Thus, all shapes of TS_9 with either 1 or 2 inner arcs contain exactly three cfhc's. Consequently, we are now in good shape to answer (*?*) :

the number of 0, 1, and 2-cfhc's of E_n that can be constructed from a 1-cfhc of E_{n-6} is equal to

the number of shapes of TS_9 having 0, 1, and 2 convex hull arcs and 2 inner arcs, which is equal to

one third the number of cfhc's of TS_9 with 0, 1, and 2 convex hull arcs and 2 inner arcs .

Define an **i,j-cfhc** of TS_n to be a cfhc that includes exactly i convex hull arcs and j inner arcs. Let **tsn(i,j)** be the number of i,j-cfhc's of TS_n. Let e_n^i be the number of i-cfhc's of E_n. Above we showed that the number of i-cfhc's of E_n constructed from a 1-cfhc of E_{n-6} is $\frac{ts9(i,2)}{3}$. Figure 26 shows all ways in which cfhc's of E_n are constructed, and gives the corresponding number of cfhc's that arise from each cfhc of E_{n-6}. Summing the terms gives the following system of equations: for i = 0 to 2

$$e_n^i = 2\,\frac{ts9(i,2)}{3}\,e_{n-6}^2 + \frac{ts9(i,1)}{3}\,e_{n-6}^2 + \frac{ts9(i,2)}{3}\,e_{n-6}^1 \qquad (2)$$

Thus we have established a system of equations that corresponds exactly to the system of equations (1) determined by Akl. Note that with E_n (unlike D_n) it is possible to construct 0-cfhc's. Thus (2) holds for i = 0 to 2, and not just for i = 1 and 2. However, (as was the case with D_n) cfhc's of E_n cannot be constructed from 0-cfhc's of E_{n-6}. This corresponds to the absence of ts9(i,0) from (2).

Note that in order to use (2) to find cfhc(E_n), we must first find values for ts9(i,j). From the catalogue we see that TS_9 has 1188 cfhc's. The author had classified over 900 cfhc's of TS_9 by hand before he became aware of the existence of Henk Meijer's cfhc-counting computer program ! The reader is invited to laugh heartily. Needless to say, Henk's program was rapidly modified, and values of ts9(i,j) were computed. A table of values of tsn(i,j) for n = 3 to 15 appears at the end of this section (see Table IX).

Summing tsn(i,j) over j gives the total number of i-cfhc's of TS_n. Thus, as E_{10} to E_{15} are by definition TS_{10} to TS_{15}, the initial conditions of (2), namely e_n^i for n = 10 to 15 and for i = 0 to 2, can also be determined from the aforementioned table. One then finds that for large n, $\text{cfhc}(E_n) \doteq c * 2.551263446^n$, where c is a constant.

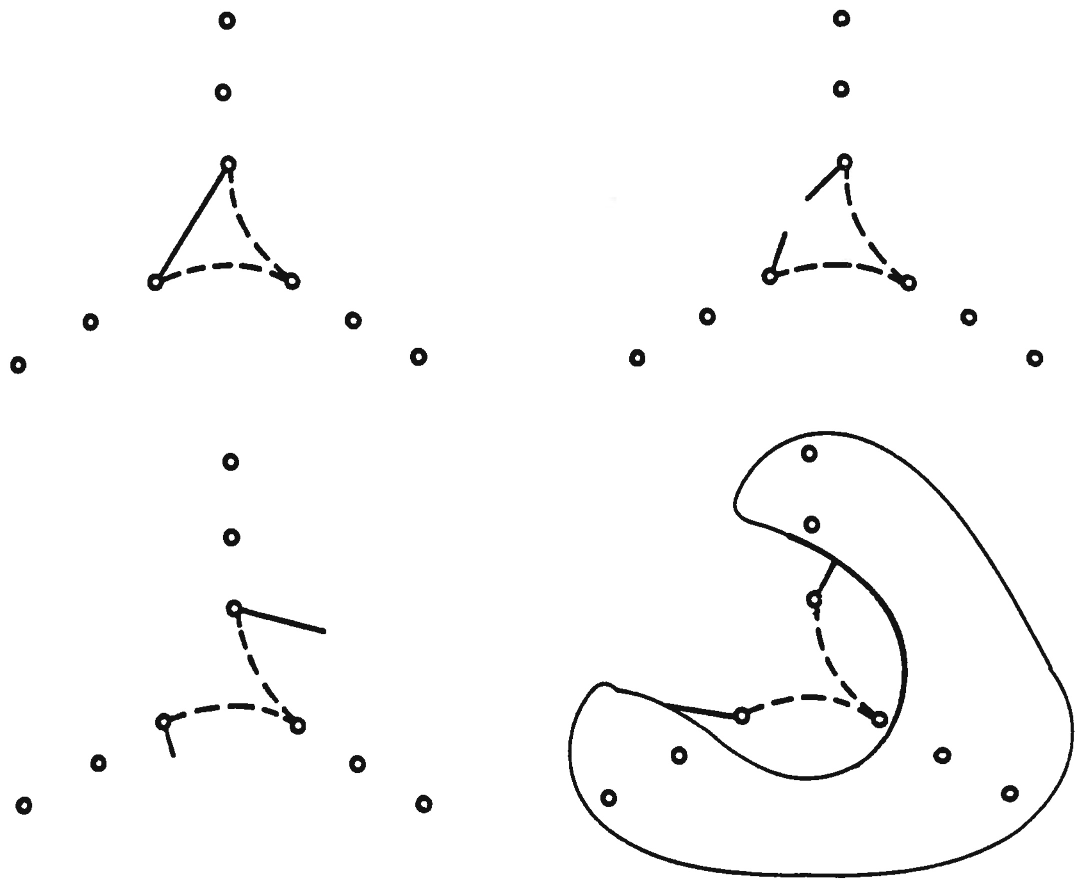


Figure 24. A cfhc of E_n from a 1-cfhc of E_{n-6}

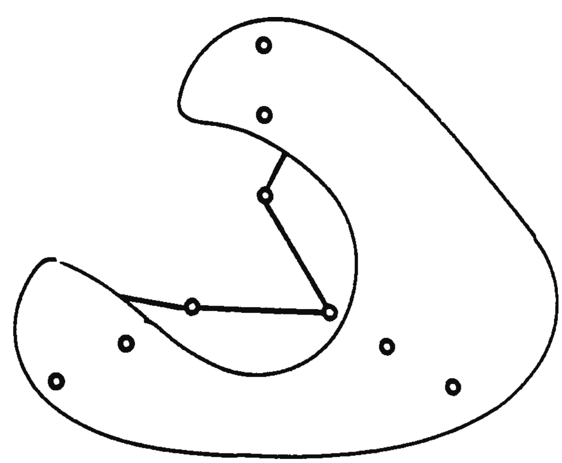


Figure 25. Shapes of TS_9 with 2 inner arcs

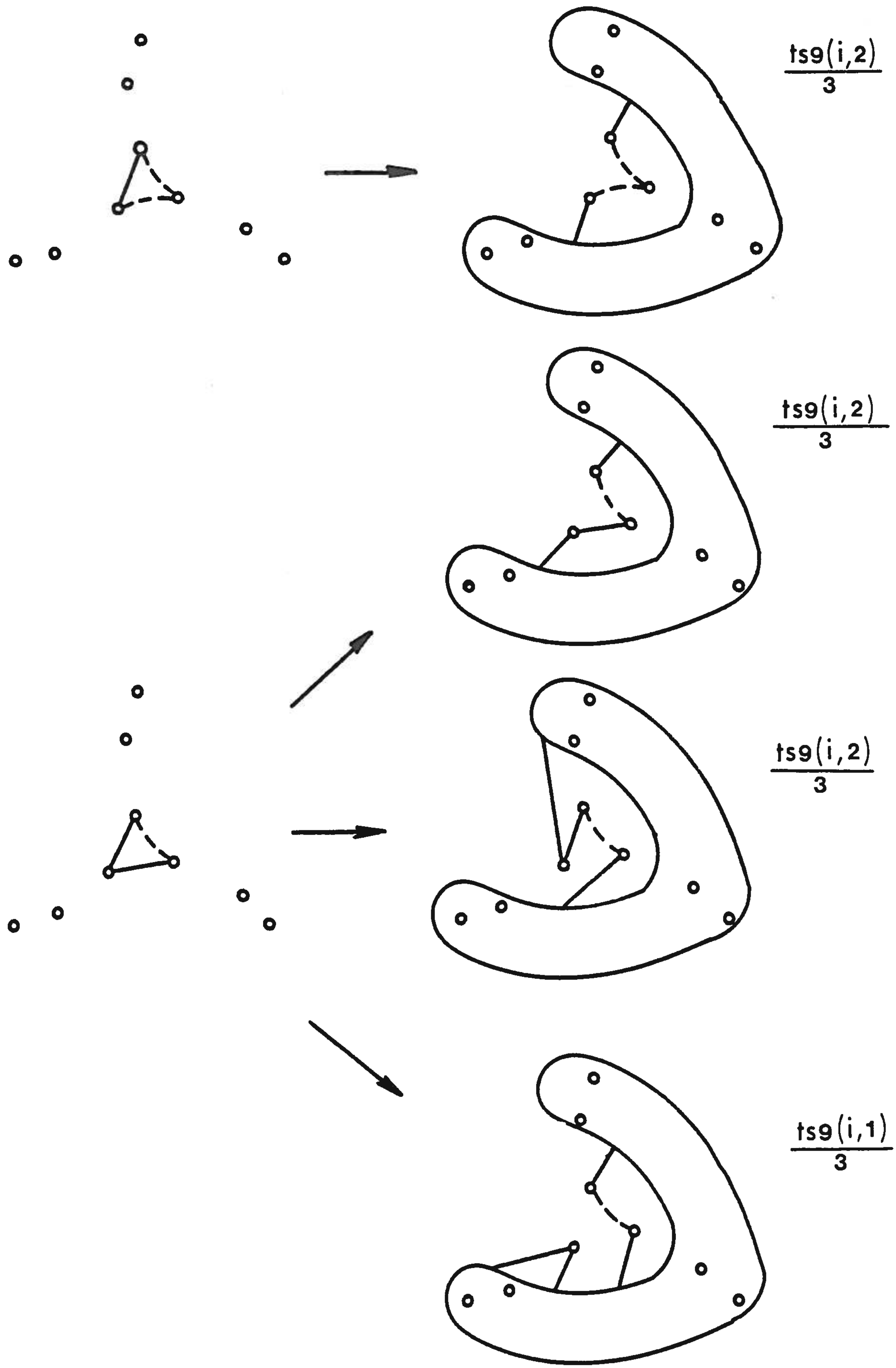


Figure 26. All cfhc's of E_n , from cfhc's of E_{n-6}

Table IX. Values of tsn(i,j)

j=	0	1	2	rowsum
ts4	0	0	0	0
	0	0	0	0
	0	0	3	3
ts5	0	0	0	0
	0	2	1	3
	0	1	4	5
ts6	2	0	0	2
	0	12	0	12
	0	0	15	15
ts7	4	4	0	8
	4	26	14	44
	0	14	25	39
ts8	12	18	4	34
	18	90	53	161
	4	53	61	128
ts9	48	72	27	147
	72	330	210	612
	27	210	192	429
ts10	160	289	125	574
	289	1188	718	2195
	125	718	542	1385
ts11	580	1180	535	2295
	1180	4538	2610	8328
	535	2610	1759	4904
ts12	2232	4950	2262	9444
	4950	18432	10215	33597
	2262	10215	6579	19056
ts13	8412	20056	9266	37734
	20056	69668	37141	126865
	9266	37141	22036	68443
ts14	33265	83239	38373	154877
	83239	276074	142845	502158
	38373	142845	80342	261560
ts15	137236	355365	162636	655237
	355365	1147386	582327	2085078
	162636	582327	318204	1063167

3.4.4 Further Generalisation

We define F_n and G_n as we defined E_n and D_n, only we will add respectively nine and twelve nodes in our recursive construction, instead of six nodes as in E_n or three nodes as in D_n. More precisely. . .

3.4.4.1 Recursive definition of F_n

Nodes of F_n are placed and numbered as for TS_n.

0) Define F_7 to F_{15} to be TS_7 to TS_{15} respectively.

We now define F_n in terms of F_{n-9}, for $n \geq 16$:

1) F_n contains all arcs of F_{n-9}. Also, all pairs of nodes of $\{ n, n-1, \ldots, n-11 \}$ are joined with arcs.

In exactly the same manner as before, we find for i = 0, 1, and 2

$$f_n^i = 2\ \frac{ts12(i,2)}{3}\ f_{n-9}^2 + \frac{ts12(i,1)}{3}\ f_{n-9}^2 + \frac{ts12(i,2)}{3}\ f_{n-9}^1 \qquad (3)$$

where f_n^i is the number of i-cfhc's of F_n. Values for ts12(i,j) and initial conditions f_7^i to f_{15}^i can be determined from the tsn(i,j) table. From (3) one then finds that cfhc(F_n) is asymptotically $c * 2.822303776^n$.

3.4.4.2 Recursive definition of G_n

Nodes of G_n are placed and numbered as for TS_n.

0) Define G_4 to G_{15} to be TS_4 to TS_{15} respectively.

We now define G_n in terms of G_{n-12}, for $n \geq 16$:

1) G_n contains all arcs of G_{n-12}. Also, all pairs of nodes of $\{ n, n-1, \ldots, n-14 \}$ are joined with arcs.

As before, we have for i = 0, 1, and 2

$$g_n^i = 2\ \frac{ts15(i,2)}{3}\ g_{n-12}^2 + \frac{ts15(i,1)}{3}\ g_{n-12}^2 + \frac{ts15(i,2)}{3}\ g_{n-12}^1 \qquad (4)$$

where g_n^i is the number of i-cfhc's of G_n. Values for ts15(i,j) and initial conditions g_4^i to g_{15}^i can be determined from the tsn(i,j) table. One can then use (4) to show that cfhc(G_n) is asymptotically c * 3.0326381^n.

3.4.4.3 Recursive definition of H_n

Alas, all good things must come to an end. There is no H_n, at least we do not bother to define one, as we have not calculated values of tsn(i,j) for $n > 15$. Note that the time required for our computer program to compute values of tsn(i,j) was almost exactly 6 times the amount of time required to compute values of ts(n-1)(i,j), for each n from 6 to 15. Approximately 120 c.p.u. hours were required to compute ts15(i,j). [That's not too bad when you consider that over 43 billion Hamilton cycles had to be checked !] This extrapolates to approximately 2.95 years of c.p.u. time that would be required to compute ts18(i,j) [i.e., using our program on a DEC-Vax 11-780] . Those readers with access to vast amounts of time on a fast computer should note that computation of ts18(i,j) will almost certainly lead to an improved lower bound for $\bar{\Phi}(n)$.

3.5 A Refinement

We will now introduce sub-drawings E_n', F_n', and G_n' of TS_n, which are similar to but have more cfhc's than E_n, F_n, and G_n respectively. Whereas E_n, F_n, and G_n are defined in terms of E_{n-6}, F_{n-9}, and G_{n-12} respectively, E_n', F_n' and G_n' will be defined in terms of E_{n-3}', F_{n-3}', and G_{n-3}' respectively. At the very end of this section, Table XII gives the asymptotic number of cfhc's of all these sub-drawings of TS_n.

3.5.1 Recursive Definition of E_n'

Nodes of E_n' are placed and numbered as for TS_n.

0) Define E_{13}' to E_{15}' to be E_{13} to E_{15}, i.e. TS_{13} to TS_{15}.

We now define E_n' in terms of E_{n-3}', for $n \geq 16$:

1) E_n' contains all arcs of E_{n-3}'. Also, all pairs of nodes of $\{ n, n-1, \ldots, n-8 \}$ are joined with arcs.

Thus, the following is a list of all arcs of E_n':

all node pairs of $\{ n, n-1, \ldots, n-8 \}$

all node pairs of $\{ n-3, n-4, \ldots, n-11 \}$

. . .

all node pairs of $\{ n-3t, n-3t-1, \ldots, n-3t-8 \}$, where $t = [\frac{n-15}{3}]$

plus all node pairs of $\{ 1, 2, 3, \ldots, q \}$, where $q = 13$ to 15 for n congruent to 13 to 15 (mod 3) respectively.

Compare the definitions of E_n and E_n', and note that every arc of E_n is an arc of E_n', while E_n' contains many arcs not in E_n.

For the sake of completeness, we include the definitions of F_n' and G_n'. For the sake of symmetry, we define D_n' to be the drawing D_n.

3.5.2 Recursive Definitions of F'_n and G'_n

Nodes of F'_n are placed and numbered as for TS_n.

0) Define F'_{13} to F'_{15} to be F_{13} to F_{15}, i.e. TS_{13} to TS_{15}.

We now define F'_n in terms of F'_{n-3}, for $n \geq 16$:

1) F'_n contains all arcs of F'_{n-3}. Also, all pairs of nodes of $\{n, n-1, \ldots, n-11\}$ are joined with arcs.

Nodes of G'_n are placed and numbered as for TS_n.

0) Define G'_{13} to G'_{15} to be G_{13} to G_{15} (i.e. TS_{13} to TS_{15}).

We now define G'_n in terms of G'_{n-3}, for $n \geq 16$:

1) G'_n contains all arcs of G'_{n-3}. Also, all pairs of nodes of $\{n, n-1, \ldots, n-14\}$ are joined with arcs.

3.5.3 Counting cfhc's of E'_n, F'_n, and G'_n.

We will now show how to compute cfhc(E'_n), cfhc(F'_n), and cfhc(G'_n), using data already gathered in the previous sections. We first unify and condense our notation.

Let $\underline{d}_n$ be the 3-vector whose zero-th, first, and second components are d_n^0, d_n^1, and d_n^2 (recall that these are the number of cfhc's of D_n with respectively 0, 1, and 2 convex hull arcs). Similarly, let $\underline{e}_n$, $\underline{f}_n$ and $\underline{g}_n$ give the number of 0, 1, and 2-cfhc's of E_n, F_n and G_n, and $\underline{e}'_n$, $\underline{f}'_n$, and $\underline{g}'_n$ the number of 0, 1, and 2-cfhc's of E'_n, F'_n, and G'_n. We can now write system (1) of the previous section as the following matrix equation.

$$\underline{d}_n = N_1 \ \underline{d}_{n-3} \quad , \text{where} \quad N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 5 & 10 \end{bmatrix} \qquad (1^*)$$

Define N_2 to N_4 as the corresponding matrices from systems (2) to (4).

Thus we have

$$\underline{e}_n = N_2\, \underline{e}_{n-6} \qquad (2^*)\ ;$$

$$\underline{f}_n = N_3\, \underline{f}_{n-9} \qquad (3^*)\ ;$$

$$\underline{g}_n = N_4\, \underline{g}_{n-12} \qquad (4^*)\ .$$

In fact, from systems (1) to (4) and the above definitions of N_1 to N_4, note that for $k = 1$ to 4

$$N_k = \frac{1}{3} T_{3+3k}\, Q \qquad \text{where } Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and where T_n is the 3 by 3 matrix whose component (i,j) is tsn(i,j). Recall that tsn(i,j) is the number of cfhc's of TS_n having i convex hull arcs and j inner arcs, for i and j = 0, 1, and 2.

We now define matrices M_k, for $k = 1$ to 4, such that

$$M_k = \frac{1}{3} U_{3+3k}\, Q \ , \qquad \text{where } Q \text{ is as above}$$

and where U_n is the matrix whose component (i,j) is the number of cfhc's of TS_n having i convex hull arcs and j inner arcs, **and at least one arc from** $\{$ **n, n-1, n-2** $\}$ **to** $\{$ **1, 2, 3** $\}$ **.**

Note that $M_1 = N_1$, and that for $k = 2$ to 4

$$M_k = N_k - \sum_{j=1}^{k-1} M_j * N_{k-j}$$

We have introduced the matrices M_k because we use them to compute cfhc(E'_n), cfhc(F'_n), and cfhc(G'_n). In fact,

$$\underline{e}'_n = M_1\, \underline{e}'_{n-3} + M_2\, \underline{e}'_{n-6} \qquad (5)\ ,$$

$$\underline{f}'_n = M_1\, \underline{f}'_{n-3} + M_2\, \underline{f}'_{n-6} + M_3\, \underline{f}'_{n-9} \qquad (6)\ , \text{ and}$$

$$\underline{g}'_n = M_1\, \underline{g}'_{n-3} + M_2\, \underline{g}'_{n-6} + M_3\, \underline{g}'_{n-9} + M_4\, \underline{g}'_{n-12} \qquad (7)\ .$$

From the above systems of equations (5) to (7) and the matrices M_k [the matrices N_k and M_k are given in Table X at the end of this chapter] it is a straightforward exercise to calculate the asymptotic values of cfhc(E'_n), cfhc(F'_n), and cfhc(G'_n). For example, system (5) can be rewritten as

$$\begin{bmatrix} \underline{e}'_n \\ \underline{e}'_{n-3} \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ I & O \end{bmatrix} \begin{bmatrix} \underline{e}'_{n-3} \\ \underline{e}'_{n-6} \end{bmatrix} \qquad (5*)$$

where I is the 3 by 3 identity matrix, and O is the 3 by 3 zero matrix. For large n, cfhc(E'_n) / cfhc(E'_{n-3}) is approximately equal to the dominant eigenvalue z of the 6 by 6 matrix of system (5*). Thus for large n we have cfhc(E'_n) $\doteq$ c * $(z^{1/3})^n$, which is c * $(19.21...^{1/3})^n$ or c * 2.978084514^n. Table XI gives the characteristic polynomials of the matrices corresponding to the systems of equations that give cfhc(D'_n), cfhc(E'_n), cfhc(F'_n), and cfhc(G'_n), as well as the dominant eigenvalues of the matrices, and the cubed root of the dominant eigenvalues. Table XII gives the asymptotic rate of growth of the number of cfhc's of D_n, E_n, F_n, G_n, and D'_n, E'_n, F'_n and G'_n.

In closing, we point out that $\bar{\Phi}(n) \geq$ cfhc(TS_n) > cfhc(G'_n), which we find in the tables to be asymptotically c * 3.1918447754^n, for some constant c. This disproves the conjectures (folklore) that $\bar{\Phi}(n)$ is asymptotically equal to c * 3^n or c * π^n. The previous best lower bound for $\bar{\Phi}(n)$ was Akl's (asymptotic) c * 2.270719168^n [A1] [A2] .

Table X. Matrices N_i and M_i

	0	0	0		0	0	0
N_1	0	0	4	M_1	0	0	4
	0	5	10		0	5	10
	0	9	42		0	9	42
N_2	0	70	250	M_2	0	50	210
	0	64	198		0	14	78
	0	754	3158		0	544	2702
N_3	0	3405	12954	M_3	0	2099	9862
	0	2193	7791		0	813	3725
	0	54212	226879		0	37384	187117
N_4	0	194109	770680	M_4	0	119087	578420
	0	106068	406245		0	42516	204119

Table XI. Characteristic Polynomials and Eigenvalues

$p_1(z) = z^2 - 10\ z - 20$

$p_2(z) = z^4 - 10\ z^3 - 148\ z^2 - 606\ z + 960$

$p_3(z) = z^6 - 10\ z^5 - 148\ z^4 - 6430\ z^3 - 30612\ z^2 + 41174\ z - 199031$

$p_4(z) = z^8 - 10\ z^7 - 148\ z^6 - 6430\ z^5 - 353813\ z^4$

$- 1830120z^3 - 2269465\ z^2 - 17503396\ z - 284185367$

z^* = largest root of $p_j(z)$

j	z^*	$(z^*)^{1/3}$
1	11.708204	2.270719168
2	19.21068856	2.678228602
3	26.41259396	2.978084514
4	32.51820048	3.191847754

$p_j(z)$ is the characteristic polynomial of matrix P_j, the matrix associated with the system of equations that gives, for j = 1 to 4, cfhc(D_n'), cfhc(E_n'), cfhc(F_n'), and cfhc(G_n'), respectively. Thus, $(z^{*1/3})$ is the asymptotic rate of growth of the number of cfhc's of the respective drawings.

Table XII. Number of cfhc's of sub-drawings of TS_n

Drawing	Asymptotic Number of cfhc's
D_n	$c * 2.270719168^n$
E_n	$c * 2.551263446^n$
F_n	$c * 2.822303776^n$
G_n	$c * 3.0326381^n$
D'_n	$c * 2.270719168^n$
E'_n	$c * 2.678228602^n$
F'_n	$c * 2.978084514^n$
G'_n	$c * 3.191847754^n$

The constant c may be different for different drawings.

4. THE COMPUTER PROGRAM

4.1 Introduction

The following is a listing of the Pascal computer program used to compute values of tsn(i,j) for n up to 15. The original program was written by Henk Meijer for Selim Akl. We are indebted to both Selim and Henk for having allowed us the use of their program.

Given an input list of crossing arc pairs and convex hull arcs of a drawing D, the original program counted the number of cfhc's of D with 0, 1, 2, or 3 convex hull arcs, i.e. the number of i-cfhc's of D for i = 0 to 3. We modified the original program so that given an input list of crossing arc pairs, convex hull arcs, and "inner hull arcs", our program now counts the number of i,j-cfhc's of D. That is, our program (which is listed in the next section) counts the number of crossing-free Hamilton cycles of D with i convex hull arcs and j inner arcs, for i and j = 0 to 3.

The program was run on a DEC-Vax 11-780. Approximately 120 hours of c.p.u. time were needed to compute values of ts15(i,j). The amount of c.p.u. time needed to compute values of ts(n-1)(i,j) was almost exactly one sixth the amount of time needed to compute values of tsn(i,j).

The program assumes that the nodes of the input drawing D are labelled from 1 to n. The program counts all cfhc's from (1 a . . .) to (1 b . . .) , where a is the variable "second node of the cycle", and b is "last node of the cycle". The program is constructed in this way so that the program can be run in segments. If it is desired to compute all cfhc's with one run of the program, then a is set to 2, and b to n.

Whenever n (the number of nodes of D) is changed the constants n and n1 (n1 is the variable n+1) must be changed and the program recompiled.

Input to the program is the list of crossing arc pairs of D, followed by the list of convex hull arcs of D, followed by the list of inner hull arcs of D. The output is the matrix of the numbers of values of tsn(i,j), as well as (optionally) a list of all cfhc's of D.

Note that the version of the program listed in the next section is set to compute cfhc's of a drawing with n=6 nodes. The next section is the listing, the following section is the input file for the drawing TS_6, and the last section is the resulting output file. This version of the program took approximately 2 seconds of c.p.u. time on a DEC-Vax 11-780 to compute the values of ts6(i,j) and list all 29 cfhc's.

4.2 Pascal Program Used to Count tsn(i,j)

```
      program cfhc (input,output,h,hin);

(***********************************************************************)
(*
(* This program finds all crossing-free Hamilton cycles.
(* All edge crossings have to be entered as data, as well as
(* the edges that are lying on the convex hull.
(* The total number of crossing-free Hamilton cycles with
(* i edges on the convex hull is computed for i = 0, 1, 2, or 3.
(* This is an interactive program, using files hin and h.
(*
(* Program is written by Henk Meijer, as part of his Ph. D. research
(* and assistantship for Selim Akl.
(*                                                   October, 1979.
(*
(*
(* A subroutine to compute the inner convex hull has also been added.
(*                  Arrbee (alias Ryan B. Hayward)    November, 1981.
(*
(***********************************************************************)



const n      =  6;
      nl     =  7;  (* nl equals n + 1 *)

var   city                : integer;
      lastsecondcity      : integer;
      totalonconvexhull   : integer;
      inntotalconvexhull: integer;
      tourlength          : integer;
      i,j,sum             : integer;
      crossings           : integer;
      longoutput          : boolean;
      finished            : boolean;

      h,hin               : text;
      visitedcities     : array[1..nl]  of boolean;
      tour                : array[1..n ] of integer;
      ch                  : array[1..4,1..2] of integer;
      ich                 : array[1..4,1..2] of integer;
      intersections     : array[1..n,1..n,1..n,1..n] of boolean;
      data                : array[0..3,0..3]   of integer;
```

```
procedure heading;
begin
      if longoutput then
      begin
            writeln (h);
            writeln (h);
            writeln (h);
            writeln (h);
            writeln (h,'Crossing-free Tours',
                  '                                   Edges ');
            writeln(h,'                                        ',
                  '  Outer Convex Hull      Inner Convex Hull');
            writeln (h);
      end;
end (* heading *);

procedure initialize;
var i,j,a,b,c,d      : integer;
begin
      writeln;
      writeln('*** Program to find Crossing-Free Hamilton Cycles ***');
      writeln;

      crossings := 0;

      for i := 2 to n do
      begin
            visitedcities[i] := false;
            tour          [i] := 0;
      end;
      visitedcities[1]   := true;
      visitedcities[n+1] := false;

      writeln('Enter the first "second node" of the cycle:');
      read (a);
      tour[1] := 1;
      tour[2] := a;
      visitedcities[a] := true;
      tourlength := 2;
      writeln('Enter the last "second node" of the cycle:');
      read (lastsecondcity);

      for a := 1 to n do
      for b := 1 to n do
      for c := 1 to n do
      for d := 1 to n do
```

```
        intersections[a,b,c,d] := false;

  reset (hin);
  rewrite(h);
  writeln;
  writeln;
  writeln ('Edge Crossings (followed by 4 zeroes)',
           ' Convex Hull Edges (f.b. 2 zeroes)');
  writeln (' and Inner Convex Hull Edges (f. b. 2 zeroes)',
           ' should be put in the file "hin".');

  writeln(h);
  writeln(h);
  writeln(h,'Crossing Edge Pairs');
  writeln(h);

  read (hin,a,b,c,d);
  writeln(h,a,b,c,d);
  while (a <> 0) do
  begin
        crossings := crossings + 1;
        intersections[a,b,c,d] := true;
        intersections[a,b,d,c] := true;
        intersections[b,a,c,d] := true;
        intersections[b,a,d,c] := true;
        intersections[c,d,a,b] := true;
        intersections[c,d,b,a] := true;
        intersections[d,c,a,b] := true;
        intersections[d,c,b,a] := true;

        read (hin,a,b,c,d);
        writeln(h,a,b,c,d);
  end (* while *);

   writeln(h);
   writeln(h);
   writeln(h);
   writeln(h,'Edges:  Outer Convex Hull');
   writeln(h);

  read (hin,a,b);
  writeln(h,a,b);
  i := 1;
  while (a <> 0) do
  begin
        ch [i,1] := a;
```

```
        ch [i,2] := b;

        read (hin,a,b);
        writeln(h,a,b);
        i := i+1;
end (* while *);

for j := i to 4 do
begin
        ch [j,1] := 0;
        ch [j,2] := 0;
end;


  writeln(h);
  writeln(h);
  writeln(h);
  writeln(h,'Edges:  Inner Convex Hull');
  writeln(h);


read (hin,a,b);
writeln(h,a,b);
i := 1;
while (a <> 0) do
begin
        ich [i,1] := a;
        ich [i,2] := b;

        read (hin,a,b);
        writeln(h,a,b);
        i := i+1;
end (* while *);

for j := i to 4 do
begin
        ich [j,1] := 0;
        ich [j,2] := 0;
end;

for i := 0 to 3 do
   for j := 0 to 3 do
      data [i,j]:= 0;


writeln;
writeln('Enter 1 for listing of all tours, else 0:');
```

```
        read (i);
        if i = 1 then longoutput := true
                 else longoutput := false;

        heading;

end (* initialize *);

function convexhull (city1,city2 : integer) : boolean;
var i       : integer;
begin
        i := 0;
        repeat
              i := i + 1;
        until ( ((city1 = ch[i,1]) and (city2 = ch[i,2]))
             or ((city1 = ch[i,2]) and (city2 = ch[i,1]))
             or ( i = 4 ) );

        if i = 4 then convexhull := false
                 else convexhull := true;
end (* convexhull *);


function innconvexhull (city1,city2 : integer) : boolean;
var i       : integer;
begin
        i := 0;
        repeat
              i := i + 1;
        until ( ((city1 = ich[i,1]) and (city2 = ich[i,2]))
             or ((city1 = ich[i,2]) and (city2 = ich[i,1]))
             or ( i = 4 ) );

        if i = 4 then innconvexhull := false
                 else innconvexhull := true;
end (* innconvexhull *);


procedure backtrack (var city : integer);
begin
        if tourlength > 1 then
        begin
              visitedcities[tour[tourlength]] := false;
              city := tour[tourlength];
              tourlength := tourlength - 1;
        end
        else
```

```
            finished := true;
end (* backtrack *);

function nointersection (city2 : integer) : boolean;
var int      : boolean;
    i,city1 : integer;
begin
      if (tourlength <= 2) or (city2 = 0) then
            nointersection := true
      else
      begin
            i := 2;
            city1 := tour [tourlength];
            repeat
                  int := intersections[tour[i-1],tour[i],
                                             city1,    city2    ];
                  i := i + 1;
            until ((int) or (i > tourlength));

            nointersection := not int;
      end (* else *);
end (* no intersection *);

procedure selectnextnonvisitedcity (var city: integer);
begin
      repeat
            city := city + 1;
      until ( not visitedcities[city]);

      if ((tourlength = 1) and (city > lastsecondcity)) then
            city := 0;
            (* the next backtrack will cause *)
            (* the execution to stop.        *)

      if city = n+1 then city := 0;
      (* i.e. no non-visited city left *)
end (*select next non-visited city *);

procedure report;
var i : integer;
begin
      if longoutput then
      begin
            for i := 1 to n do
                  write (h,tour[i]:3);
            write (h,tour[1]:3);
```

```
        end;

        totalonconvexhull := 0;
        inntotalconvexhull  := 0;
        for i := 2 to n do
        begin
             if convexhull(tour[i-1],tour[i]) then
                  totalonconvexhull := totalonconvexhull + 1;
             if innconvexhull(tour[i-1],tour[i]) then
                  inntotalconvexhull  := inntotalconvexhull + 1;
        end;

        if convexhull(tour[n],tour[1]) then
             totalonconvexhull := totalonconvexhull + 1;
        if innconvexhull(tour[n],tour[1])  then
             inntotalconvexhull  := inntotalconvexhull + 1;

        data [totalonconvexhull,inntotalconvexhull] :=
             data [totalonconvexhull,inntotalconvexhull] + 1;
        if longoutput then
             writeln (h,'          ',totalonconvexhull,
                                          inntotalconvexhull: 18);

   end (* report *);

   begin (* main program *)

        initialize;
        finished := false;
        city     := 1;

        repeat
             repeat
                  selectnextnonvisitedcity (city);
                  (* returns zero if no city left *)
             until (nointersection(city) or (city = 0));

             if (city = 0) then
                  backtrack (city)
             else
             begin
                  tour[tourlength + 1] := city;
                  tourlength             := tourlength + 1;
                  visitedcities[city]  := true;
                  city                   := 1;
             end (* else *);
```

```
        if (tourlength = n) then
        begin
              (* to prevent creation of symmetric tours: *)
              if tour[n] > tour[2] then
                    if nointersection(1) then report;
              backtrack (city);
        end;

until finished;

writeln (h);
writeln (h);
writeln (h);
writeln (h);
writeln (h,'                    Number of Crossing Free Cycles');
writeln (h);
writeln (h);
writeln (h,'Edges on                    Edges on Inner Convex Hull ');
writeln (h,'Outer');
writeln (h,'Convex');

sum :=0;
write(h,'Hull       ');
for j := 0 to 3 do
      write(h,j:15);
writeln (h);
write (h,'               ------------------------------');
writeln (h,'--------------------------');
for i :=0 to 3 do
      begin
      writeln(h,'               |');
      write(h,i:6,'        |',data[i,0]:11);
      sum := sum + data[i,0];
      for j := 1 to 3 do
            begin
                  write(h,data[i,j]:15);
                  sum := sum + data[i,j];
            end;
            writeln(h);
      end;


writeln (h);
writeln (h);
writeln (h);
writeln (h);
```

```
        writeln (h,'Number of Nodes                         =',n:5);
        writeln (h);
        writeln (h,'Number  of  Crossings                   =',crossings:5);
        writeln (h);
        writeln (h,'Total Number of Crossing Free Tours =',sum:12);
        writeln (h);

        writeln;
        writeln ('Results are written on the file "h".');
        writeln ;
        writeln ('*** End of Crossing Free Hamilton Cycle Algorithm   ***');
        writeln ('***             by Henk Meijer. -- October, 1979 --  ***');
        writeln ('***                                                   ***');
        writeln ('*** Addition of "inner convex hull"                   ***');
        writeln ('***                          r.b.h.   Nov. 1981       ***');
        writeln;

end.
```

4.3 Input File for TS_6

```
1 5 2 4
1 6 3 4
2 6 3 5
0 0 0 0
4 5
5 6
6 4
0 0
1 2
2 3
3 1
0 0
```

4.4 Output File

Crossing Edge Pairs

1	5	2	4
1	6	3	4
2	6	3	5
0	0	0	0

Edges: Outer Convex Hull

4	5
5	6
6	4
0	0

Edges: Inner Convex Hull

1	2
2	3
3	1
0	0

Crossing-free Tours							Edges Outer Convex Hull	Edges Inner Convex Hull
1	2	3	4	6	5	1	2	2
1	2	3	5	4	6	1	2	2
1	2	3	5	6	4	1	2	2
1	2	3	6	4	5	1	2	2
1	2	3	6	5	4	1	2	2
1	2	4	5	3	6	1	1	1
1	2	4	5	6	3	1	2	2
1	2	4	6	5	3	1	2	2
1	2	5	3	6	4	1	1	1
1	2	5	4	6	3	1	2	2
1	2	5	6	3	4	1	1	1
1	2	5	6	4	3	1	2	2
1	2	6	3	4	5	1	1	1
1	2	6	5	4	3	1	2	2
1	3	2	4	5	6	1	2	2
1	3	2	5	4	6	1	2	2
1	3	2	5	6	4	1	2	2
1	3	2	6	4	5	1	2	2

1	3	2	6	5	4	1	2	2
1	3	4	6	2	5	1	1	1
1	3	5	2	4	6	1	1	1
1	3	6	2	5	4	1	1	1
1	3	6	5	2	4	1	1	1
1	4	2	3	5	6	1	1	1
1	4	2	5	3	6	1	0	0
1	4	3	2	6	5	1	1	1
1	4	3	6	2	5	1	0	0
1	4	5	2	3	6	1	1	1
1	4	6	3	2	5	1	1	1

Number of Crossing Free Cycles

Edges on Outer Convex Hull	Edges on Inner Convex Hull			
	0	1	2	3
0	2	0	0	0
1	0	12	0	0
2	0	0	15	0
3	0	0	0	0

Number of Nodes = 6

Number of Crossings = 3

Total Number of Crossing Free Tours = 29

5. CONCLUSIONS, COMMENTS, AND OPEN QUESTIONS

5.1 On crossings and cfhc's

Let S be the set of crossings of a drawing D. Then x(D), the number of crossings of D, is |S|, the cardinality of S. As cfhc(D) is a function solely of S, one asks what information about cfhc(D) can be determined from |S|, i.e from x(D). The answer is that in general, very little information can be determined in this way. For instance, consider the following :

Let A and B be two drawings of the same graph. Then

i) x(A) $<$ x(B) does not imply cfhc(A) $\geq$ cfhc(B) ,

ii) x(A) $=$ x(B) does not imply cfhc(A) $=$ cfhc(B) ,

iii) cfhc(A) $<$ cfhc(B) does not imply x(A) $\geq$ x(B) ,

iv) cfhc(A) $=$ cfhc(B) does not imply x(A) $=$ x(B) .

Proofs:

i) Let A be drawing TS_9 with 36 crossings and 1188 cfhc's, and B drawing Singer-8 with 38 crossings and 1252 cfhc's.

ii) Let A be drawing TS_9 with 36 crossings and 1188 cfhc's, and B drawing FTS_9 with 36 crossings and 1228 cfhc's.

iii) Let A be drawing 6N with 8 crossings and 12 cfhc's, and B drawing 6S with 9 crossings and 13 cfhc's.

iv) Let A be drawing 6P with 8 crossings and 13 cfhc's, and B drawing 6S with 9 crossings and 13 cfhc's.

The reader is invited to find other pairs of drawings listed in the catalogue which confirm the above.

It seems that of all drawings with a given number of crossings, there will be a wide variation in the number of cfhc's. Again, see the catalogue. For instance, drawings of K_9 with 36 crossings may have as few as 1184 or as many as 1461 cfhc's. See Table VII.

As Newborn and Moser have remarked, x-optimal drawings are not necessarily cfhc-optimal: there are five non-isomorphic x-optimal drawings of K_7, and three of these have fewer cfhc's than the other two. Nabil Rafla asks whether or not cfhc-optimal drawings of K_n are necessarily x-optimal **[NM]**. Note that there is some evidence to support the Rafla conjecture : all cfhc-optimal drawings of K_n for n = 3 to 6 are indeed x-optimal. Also, for n = 7 and 8 the drawings that establish the current best lower bounds for $\phi(n)$ and $\bar{\phi}(n)$ are x-optimal. However, note that the rectilinear drawing that establishes the current best lower bound for $\bar{\phi}(9)$, namely the Singer drawing minus node 8, has 38 crossings (and is therefore not x-optimal) and 1252 cfhc's, and that all known rectilinear drawings with fewer crossings have fewer cfhc's. As there are about 400 non-isomorphic x-optimal drawings of K_9 and we have counted cfhc's of only 6, it is quite possible that there exists a rectilinear x-optimal drawing with 1252 or more cfhc's. Even so, because the number of cfhc's of a drawing is so heavily dependent on the "structure" of crossings, we conjecture that there is a drawing of K_n that is cfhc-optimal, but not x-optimal.

5.2 Open Questions

One might say that the optimal cfhc problem itself remains an open question. There obviously remains a great deal to be discovered about $\phi(n)$ and $\bar{\phi}(n)$. Specifically, $\phi(n)$ and $\bar{\phi}(n)$ are both unknown for n > 6. Also, the current asymptotic upper and lower bounds (for constant c)

$$c * 3.1918...^n \quad < \quad \bar{\phi}(n) \leq \phi(n) \quad < \quad 10000000000000^n$$

leave quite a gap of uncertainty.

Thus, open questions are:

i) What are the values for $\phi(n)$ and $\bar{\phi}(n)$ for $n > 6$?

ii) What drawings are cfhc-optimal for $n > 6$?

iii) Find an improved lower bound for $\phi(n)$ or $\bar{\phi}(n)$.

iv) Find an improved upper bound for $\phi(n)$ or $\bar{\phi}(n)$.

A related question is

v) Determine necessary or sufficient conditions for a drawing of K_n to be cfhc-optimal.

Note that the optimal-cfhc problem could also be posed for other surfaces (e.g. the torus) and for other classes of graphs (e.g. the complete bipartite graphs, the k-cube).

It appears that $\phi(n)$ and $\bar{\phi}(n)$ are at least as difficult to determine as $\nu(n)$ and $\bar{\nu}(n)$. As values for the latter two have remained unknown for $n > 10$ for many years, it would be surprising if much progress could be easily made on i) or ii) . However, we feel that much more can be done on iii) . Crude extrapolation of the known cfhc values of the drawings in the catalogue suggests that all six constructions have $c * r^n$ cfhc's for some constant c, where r is at least 3.6 . It also appears as though BKB_n has (asymptotically) at least $c * 4^n$ cfhc's. We feel that the symmetry of some of these drawings could be exploited in determining an exact expression for the number of cfhc's they contain. We conjecture that $\phi(n)$ is asymptotically $c * q^n$, and that $\bar{\phi}(n)$ is asymptotically $k * p^n$, where c and k are constants, and where p and q are both ≥ 4 .

The proof establishing the current upper bound is essentially an elegant counting argument; it is purely combinatorial [ACNM]. Success with iv) might come with combining some geometric or topological result with the work that has been done. It would be interesting to establish upper bounds for the number of cfhc's of the constructions given in the catalogue. Although this would not solve iv), it would perhaps shed some light on whether either of the upper or lower bounds is actually close to $\phi(n)$ or $\overline{\phi}(n)$.

We also feel that much can be done with v). Some success along these lines has been made for the crossing-number problem. In particular, Singer has shown that rectilinear x-optimal drawings of K_n for $n > 9$ have three nodes on the convex hull, and that if these three nodes are removed, the resulting sub-drawing also has three nodes on its convex hull. We have been unable to prove anything along these lines, but we believe that the following are true:

Conjecture: Let D" be a rectilinear drawing with exactly $k > 3$ nodes on its convex hull. Then D" can be redrawn as D′, where D′ has exactly k-1 nodes on its convex hull. (Thus D" can be redrawn as D, where D has exactly three nodes on its convex hull).

Conjecture: All cfhc-optimal rectilinear drawings have exactly three nodes on their convex hull.

Note that the first conjecture may not imply the second, even in the case where D" is cfhc-optimal. Let D" and D′ be as above, and let S" and S′ be their respective sets of crossings. Let T be the (non-empty) set of crossings in S" but not S′. Then it may be that every Hamilton cycle of D" that contains a crossing of T also contains a crossing not in T. Thus, there will be no Hamilton cycles that have a crossing in D" but not in D′. This, together with the definition of a redrawing, would imply that the set of cfhc′s of D" is exactly the set of cfhc′s of D′.

Topological results might also prove useful in v). For instance, the observation that there can be at most one crossing on any set of four nodes in a good drawing might come in handy.

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7. VITA

Name :		Ryan Bruce Hayward
Birth :		June 15, 1958 Vancouver, B.C., Canada
Education :		Alpha Secondary School, Burnaby, B.C. 1976 University of British Columbia, September 1976 - May 1977 Queen's University at Kingston, September 1977 - Appril 1978, September 1979 - September 1982, B. Sc. (Honours) 1981, M. Sc. (Mathematics) 1982
Experience :		Part-time instructor, Frontenac County Board of Education, summer 1980 Part-time teaching assistant, Queen's University, 1980-1982
Awards :	1976	Government of B.C. High School Scholarship U. B. C. Chris Spencer Entrance Scholarship U. B. C. Norman Mackenzie Alumni Scholarship
	1980	N. S. E. R. C. Summer Undergraduate Scholarship
	1981	N. S. E. R. C. Summer Undergraduate Scholarship Queen's Graduate Scholarship N. S. E. R. C. Postgraduate Scholarship
	1982	N. S. E. R. C. Postgraduate Scholarship