UNION-CONNECTIONS AND STRAIGHTFORWARD WINNING STRATEGIES IN HEX

Kohei Noshita[1]

Tokyo, Japan

ABSTRACT

A notion of *union-connection* in the game of Hex is introduced as a generalization of the well-known virtual connection. This is applied to obtain two easy-to-verify winning strategies in 7 × 7 Hex. The extensive use of union-connection along with some related techniques leads to a new proof for the solution of 8 × 8 Hex. It is a win for the first player (Black).

1. INTRODUCTION

In the game of Hex the first player Black always wins if Black follows a *winning* strategy. This is proven by the well-known strategy-stealing argument due to Nash (cf. Gardner, 1957; van den Herik, Uiterwijk, and van Rijswijck, 2002). For a comprehensive Hex bibliography I refer to van Rijswijck (2003). On the 7 × 7 board Yang, Liao, and Pawlak (2001) first published an explicit winning strategy. They had found the strategy by hand, but their proof of correctness was not exhaustive. It used precluding analysis. In his homepage, Enderton (2000) reports about his computer program for obtaining such an exhaustive proof of correctness, and Yang's (2003) homepage reports about results of the 8 × 8 and 9 × 9 cases. Hayward *et al.* (2003) have obtained a winning strategy for each of forty-nine opening positions on the 7 × 7 board by their computer program.

This paper introduces a notion of *union-connection* which is a generalization of the well-known virtual connection (e.g., Yang *et al.* (2001), Anshelevich (2002), Hayward *et al.* (2003)), and demonstrates the power of union-connections by presenting new winning strategies for 7 × 7 and 8 × 8 along with correctness proofs. By combining it with the usual precluding analysis, we can construct complete proof trees representing winning strategies, which consist of 22 nontrivial nodes of Black's turn for the 7 × 7 Hex game and 52 nontrivial moves for the 8 × 8 Hex game. As another application of our union-connection, we show that, for 7 × 7 with the initial position which is different from the above one, the only one node (the root of the game tree) suffices to construct a winning strategy. For 8 × 8 Hex a proof technique named AB-*property* is also introduced as an extension of union-connection for handling interactions between two overlapping areas. Finally, we note that no complete winning strategies for 8 × 8 Hex have been formally published so far. Yet, the author encourages the reader to compare his results with those of Yang's (2003) Java applet that plays a winning strategy according to their findings.

2. DEFINITIONS AND LEMMAS

Let $S[ab]$ be a virtual connection (VC for short) in S between two cells (hexagons) a and b, where S is a set of empty cells (see Anshelevich (2002)). Unless otherwise stated, connections are assumed to be for Black, and thus a player's name will be omitted in our notation.

By definition, $S[ab]$ holds if and only if, for any white move-sequence (even when White moves first), Black connects a and b where both players make moves only at cells in S. S will be called a supporting set (carrier) for this connection.

[1] Department of Computer Science, The University of Electro-Communications, Chofu, Tokyo 182, Japan. Email: noshita@cs.uec.ac.jp.

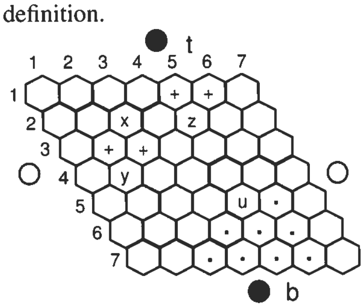
We define a *union-connection* $S[\ a_1b_1 \mid a_2b_2 \mid \ldots \mid a_kb_k\ ]$ if, for any white move-sequence, there exists i $(1 \leq i \leq k)$ such that Black connects a_i and b_i in S. If $k = 1$, union-connection is identical to VC. In this definition, the order of pairs as well as the order of two cells in a pair is arbitrary (interchangeable) as an ordinary set.

We denote $a - b$ for $S[ab]$ when S is understood in the context. Similar notations like $a - b|c - d$ will be used for union-connections.

For any black cell x (i.e., cell x occupied by Black), we simply write x to mean the set of black cells which are directly connected to x (by transitive adjacency) including itself.

Let $\mathbf{t}$ and $\mathbf{b}$ represent the top-border and the bottom-border, respectively, as shown in Figure 1. Thus $\mathbf{t} - \mathbf{b}$ implies a black winning connection.

The following properties will be used for deducing various connections. The proof is straightforward from the definition.



S-lemma: $x - y$. S'-lemma: $\mathbf{t} - z$. T-lemma: $u - \mathbf{b}$.

Figure 1: S- and T-lemmas.

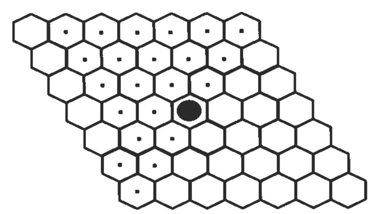
Let σ denote a form of $|a_1b_1\ |a_2b_2|\ \ldots\ |a_kb_k$ $(k \geq 0)$.
(1) $S[ab|ab\ \sigma]$ implies $S[ab\ \sigma]$.
(2) If $S_1[ab\ \sigma]$ and $S_2[bc]$ and $S_1 \cap S_2 = \phi$, we have $S[ac\ \sigma]$ where $S = S_1 \cup S_2$.

In a similar way other types of deduction rules can be introduced when necessary. Two basic connections are shown in Figure 1. They are also well-known properties of VC. We call a type of the upper-left connection Skip Lemma (S-lemma for short), in which $x - y$ (and $\mathbf{t} - z$) holds. We call a type of the lower-right connection Trapezoid Lemma (T-lemma for short), in which $u - \mathbf{b}$ holds. These lemmas will be freely used without mentioning it explicitly.

The following Union Lemma (U_1-lemma) is the key property of our winning strategy.

Lemma 1 (U_1-lemma) In Figure 2 we have $S[\ \mathbf{t}x \mid \mathbf{tb}\ ]$, where S is the set of empty cells as shown there.

Proof: Let ij denote the cell at column i and row j. This position in the figure *forces* White to make the next move only at the five cells, i.e., 51,61,52,43, and 53. Otherwise Black will make a move at 52, leading to $\mathbf{t} - x$. In case of White W61, W52 or W53, Black B33 immediately leads to $\mathbf{t} - x$ by T-lemma and S-lemma. In case of W51, move-sequences beginning with B62 (e.g., B62 W53 B43 W52 B32) lead to $\mathbf{t} - x$. In case of W43, the hardest move-sequence leading to $\mathbf{t} - \mathbf{b}$ is W43 B53 W61 B52 W51 B32 W42 B24 W34 B25 W35 B17.
The idea for these black moves is that, for any (defending) white move, Black always makes a threat move for attaining $\mathbf{t} - x$ which forces the next defending move on White. The second move B53 forces W61. After this move, White has no other choices (except futile moves to try cutting already-secured connections), because otherwise Black can immediately attain $\mathbf{t} - x$. The final move B17 attains the other connection $\mathbf{t} - \mathbf{b}$. By combining this with other forcing sequences, we have $\mathbf{t} - x$ or $\mathbf{t} - \mathbf{b}$. □



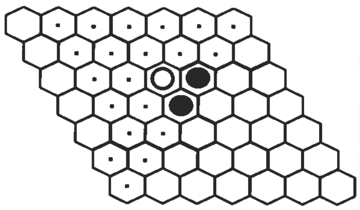
Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$ in the dotted set S. See the proof in the text.

Figure 2: U_1-lemma.

The technique used in the above proof for reaching $\mathbf{t} - \mathbf{b}$ by black threat moves for $\mathbf{t} - x$ in the hardest move-sequence may be called the *sidestepping* technique. As a corollary of this proof, we have U_{1+}-lemma in Figure 3. In a similar way, we can prove several useful variants of the U-lemma as shown in the Figures 4 to 7.

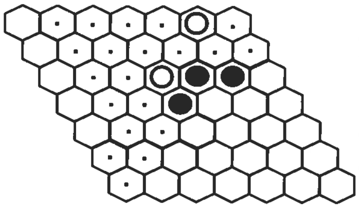
3. OUR GAME TREE FOR THE 7×7 BOARD

The complete game-tree is shown in Figure 8. A node (written in the form of a move) represents a position obtained by the corresponding move. For example, B44 at the root represents the position obtained by the first black move at 44 in the initial position. For each node of White's turn (after the corresponding previous black move), all the white moves that are precluded in the analysis step at that node have been deleted from the game tree.



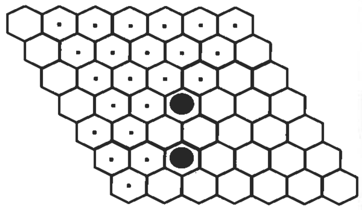
Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$. This position appears in the proof of U_1-lemma.

Figure 3: U_{1+}-lemma.



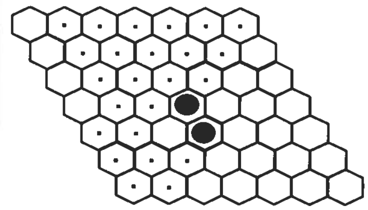
Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$. White blocking of B71 leads to a similar position in the proof of U_{1+}-lemma.

Figure 4: U_{1*}-lemma.



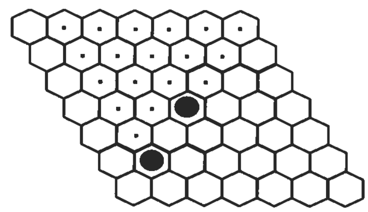
Let x be B44. Let y be B36. We have $\mathbf{t} - x \mid \mathbf{t} - y \mid \mathbf{t} - \mathbf{b}$. The proof is similar to U_1-lemma. If W43, then B53. For this position, we also have $\mathbf{t} - x \mid \mathbf{t} - y \mid \mathbf{t} - \mathbf{b}$, which we call U_{2+}-lemma.

Figure 5: U_2-lemma.



Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$. The proof is similar to U_2-lemma. If W43, then B53. For this position, we also have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$, which we call U_{3+}-lemma.

Figure 6: U_3-lemma.



Let x be B44. Let y be B26. We have $\mathbf{t} - x \mid \mathbf{t} - y$.

Figure 7: U_4-lemma.

As seen in the tree, there are twenty-two nodes for Black's turn, i.e., a to g, $f.p$ to $f.w$ and $g.p$ to $g.v$, among which twenty nodes are terminal leaf positions. Those twenty positions are classified into the final eleven patterns, for each of which Black's winning move is shown in the tree.

In Figures 9 to 29, those winning moves along with the correctness proof and the precluding analyses are described in the order of a depth-first traversal of the tree. The nodes in the tree with their corresponding Figure numbers are listed in Table 1.

We use the precluding analysis for determining the set of white nontrivial moves throughout this paper. This precluding analysis is well-known (see Hayward *et al.* (2003) for a more detailed explanation of *mustplay*), which is summarized as follows in our notation.

In the *analysis* step at a given position of White's turn, we can preclude apparently ineffective white moves by the following naive reasoning. We choose an empty cell z to make a *fictitious* black move (instead of a white move) at z which attains $\mathbf{t} - \mathbf{b}$. Let S_z be a minimal supporting set of empty cells such that $S_z[\mathbf{tb}]$. We take the intersection of all of these minimal sets $S_z \cup \{z\}$. Note that there may be more than one choice for such z and that, for any fixed z, this minimal set is not always unique. Then all empty cells that are not in the intersection can be precluded from the set of candidate cells for White's next move. (The proof is obvious.) If the intersection is empty, this position can be regarded as a terminal leaf node. Note that, if we use an appropriate connection in place of $\mathbf{t} - \mathbf{b}$, we can apply this precluding analysis to a position defined on a part of the board.

Here we remark that Yang's (2003) winning strategy also starts with B44 and is similar to our strategy. Of the 22

node	Figure	node	Figure	node	Figure
B44	9, 10	$f.p$	17	$g.p, g.q$	26
a, b	11	$f.q, f.r$	18	$g.r, g.s, g.t$	27
c, d	12	$f.s$	19	$g.u$	28
e	13	$f.t, f.u, f.v$	20	$g.v$	29
f (B45)	14, 15, 16	$f.w$	21		
g (B35)	22, 23, 24, 25				

Table 1: Nodes and figure numbers.

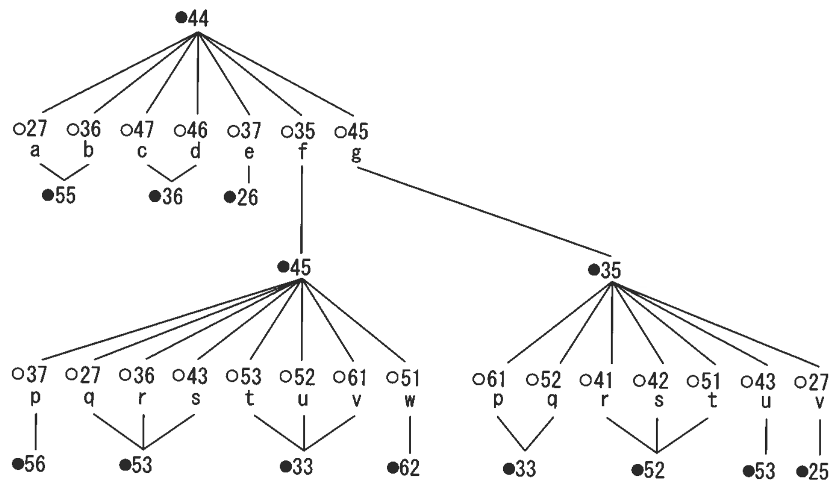
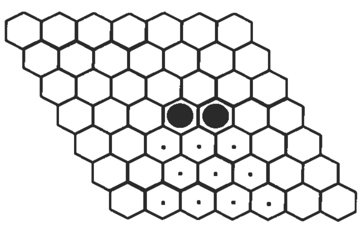


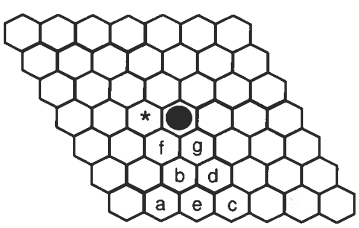
Figure 8: Our game tree of 7×7 Hex.

black responses to the white moves, shown in Figure 8, 17 are the same as in Yang's strategy. Again the reader is encouraged to compare the strategies. For the 7×7 opening move 35, Yang *et al.* (2003) also found and published a winning strategy together with a proof using precluding analysis.



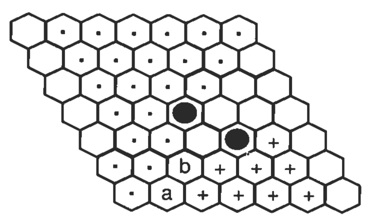
Let x be B44. We have $x - \mathbf{b}$. The proof is easy. Note that W45 B36 W35 B55 leads to $x - \mathbf{b}$.

Figure 9: Supertrapezoid (ST) lemma.



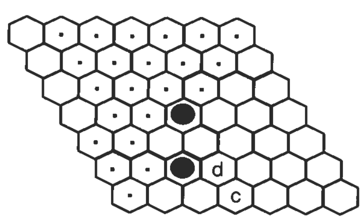
For White's second move, the seven moves (a to g) are to be considered. All other moves are precluded by ST-lemma along with its left-shift (to *) and top-bottom symmetry. For example, if W33, B54 makes $\mathbf{t} - \mathbf{b}$ by two applications of ST-lemma.

Figure 10: White's second moves.



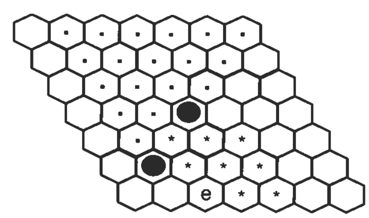
If a (W27) or b (W36), then B55. Let x be B44. Let y be B55. We have $\mathbf{t}-x \mid \mathbf{t}-\mathbf{b}$ by U_1-lemma. We have $x-y$ by S-lemma. We have $y - \mathbf{b}$ by T-lemma. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 11: Case a and b.



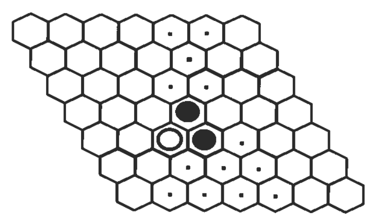
If c (W47) or d (W46), then B36. Let x be B44. Let y be B36. We have $\mathbf{t} - x \mid \mathbf{t} - y \mid \mathbf{t} - \mathbf{b}$ by U_3-lemma. We have $x - y$ by S-lemma. We have $y - \mathbf{b}$ by S'-lemma. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 12: Cases c and d.



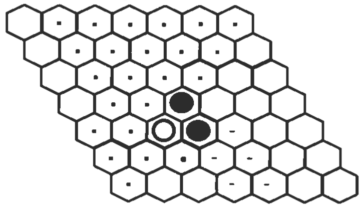
If e (W37), then B26. Let x be B44. Let y be B26. We have $\mathbf{t}-x \mid \mathbf{t}-y$ by U_4-lemma. We have $y - \mathbf{b}$ by S'-lemma. We have $x - y \mid x - \mathbf{b}$ in the asterisk set. (The proof is immediate, since W35 B45 W36 B56.) Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 13: Case e.



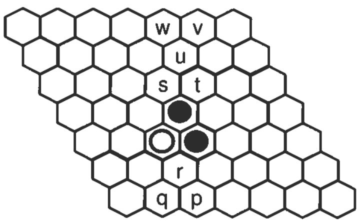
If f (W35), then B45. Let x be B44. If White's next move is not in the dotted set, Black moves at B52. This leads to $\mathbf{t} - \mathbf{b}$, since $\mathbf{t} - x$ and $x - \mathbf{b}$ by T-lemma. Hence all the cells without dots are precluded.

Figure 14: Case f (analysis 1).



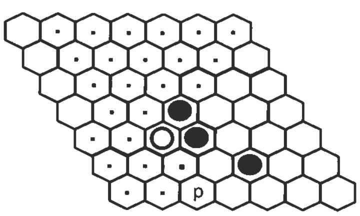
If White's next move is in the minus (-) set, Black moves at B36. This leads to $x - \mathbf{b}$. We also have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$ by U_2-lemma. Hence we have $\mathbf{t} - \mathbf{b}$. All the cells *with* minuses are also precluded.

Figure 15: Case f (analysis 2).



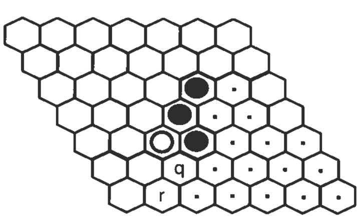
For White's fourth move, the eight moves (p to w) are to be considered. All other moves are precluded by the above analyses in Figures 14 and 15. (We shall consider only the cells in the intersection of sets which are *not* precluded in Analyses 1 and 2.)

Figure 16: Case f (White's fourth moves).



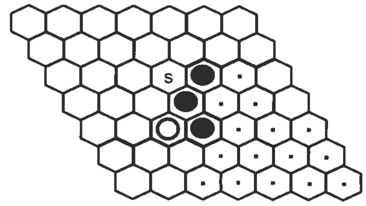
If $f.p$ (W37), then B56. Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$ by U_3-lemma. Let y be B56. We have $x - y$ and $y - \mathbf{b}$. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 17: Case $f.p$.



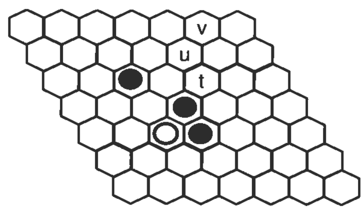
If $f.q$ (W27) or $f.r$ (W36), then B53. Let x be B44. We have $\mathbf{t} - x$ (by T-lemma). We also have $x - \mathbf{b}$ in the dotted set. (The proof is easy. If W46 or W37, B65 suffices. If W47, B46 W37 B66 W56 B65 suffices.) Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 18: Cases $f.q$ and $f.r$.



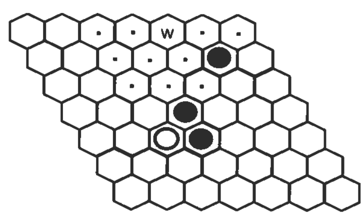
If $f.s$ (W43), then B53. Let x be B44. We have $\mathbf{t} - x \mid \mathbf{t} - \mathbf{b}$ by U_{3+}-lemma in the upper-left area. We also have $x - \mathbf{b}$ as seen in Figure 18. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 19: Case $f.s$.



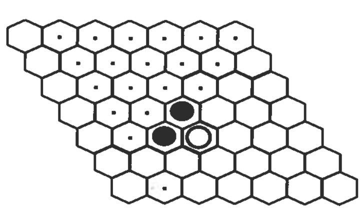
If $f.t$ (W53), $f.u$ (W52) or $f.v$ (W61), then B33. Let x be B44. Let y be B33. We have $\mathbf{t} - y$ and $y - x$ and $x - \mathbf{b}$ by T-, S- and T-lemmas. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 20: Cases $f.t$, $f.u$ and $f.v$.



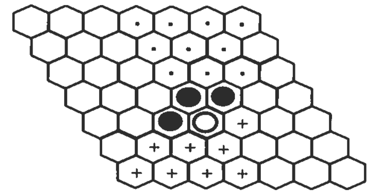
If $f.w$ (W51), then B62. Let x be B44. We have $\mathbf{t} - x$, since W53 B43 W52 B32 makes it. We also have $x - \mathbf{b}$ by T-lemma. Hence we have $\mathbf{t} - \mathbf{b}$. Note that B62 can be replaced by B43.

Figure 21: Case $f.w$.



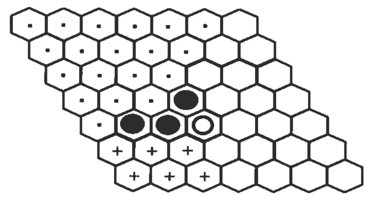
If g (W45), then B35. Let x be B44. We have $\mathbf{t} - x$. The proof is similar to U_4-lemma. If White next move is not at the dotted cells, Black makes $x - \mathbf{b}$. This implies $\mathbf{t} - \mathbf{b}$. Hence all the cells *without* dots are precluded. Note that 54 is precluded.

Figure 22: Case g (analysis 1).



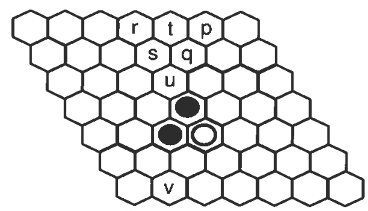
If White next move is not at the cells with dots or pluses (+), B54 makes $\mathbf{t} - x$ (in dots by ST-lemma) and $x - \mathbf{b}$ (in pluses by B26 or B46). Hence 21, 22, 23, 24, 25, 31, 32 33 and 34 are also precluded.

Figure 23: Case g (analysis 2).



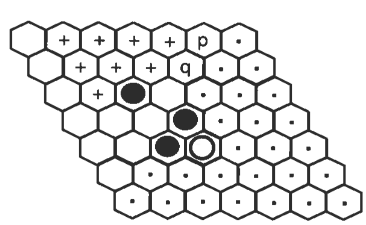
If White next move is not at the cells with dots, B25 makes $\mathbf{t} - x$ (in dots). The proof is similar to U_1-lemma. Note that (a) W43 (W52, W61 or W41) B33, (b) W42 (or W51) B23, (c) W33 B43, and (d) W32 (or W31) B42. B25 also makes $x - \mathbf{b}$ (in pluses). Hence 71, 62 and 53 are also precluded.

Figure 24: Case g (analysis 3).



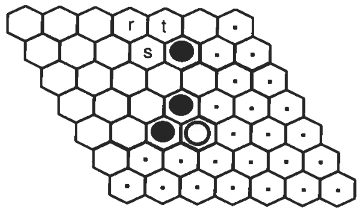
For White fourth move, the seven cells (p to v) are to be considered. All other cells are precluded through Analyses 1 to 3.

Figure 25: Case g (White's fourth moves).



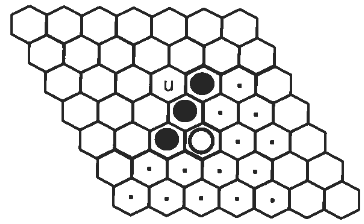
If $g.p$ (W61) or $g.q$ (W52), then B33. Let x be B44. Let y be B33. We have $\mathbf{t} - y$ by T-lemma and $x - y$ by S-lemma. We also have $x - \mathbf{b} \mid \mathbf{t} - \mathbf{b}$ by U_{1+}-lemma. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 26: Cases $g.p$ and $g.q$.



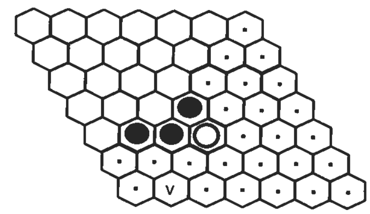
If $g.r$ (W41), $g.s$ (W42) or $g.t$ (W51), then B52. Let x be B44. Let y be B52. We have $x - y$. We also have $\mathbf{t} - y$. (Cases $g.r$ and $g.s$ are obvious. For $g.t$, the hardest sequence B52 W61 B32 W42 B24 suffices to see this.) We have $x-\mathbf{b} \mid y-\mathbf{b} \mid \mathbf{t}-\mathbf{b}$ by U_{2+}-lemma in the lower-right area. Hence we have $\mathbf{t} - \mathbf{b}$. Note that $g.t$ can also be done by B62.

Figure 27: Cases $g.r$, $g.s$ and $g.t$.



If $g.u$ (W43), then B53. Let x be B44. We have $x - \mathbf{b}$. (The proof is similar to U_{1+}-lemma.) By the top-bottom symmetry, We also have $\mathbf{t} - x$. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 28: Case $g.u$.



If $g.v$ (W27), then B25. Let x be B44. We have $\mathbf{t} - x$. (See Figure 24.) We also have $x - \mathbf{b} \mid \mathbf{t} - \mathbf{b}$ by U_{1*}-lemma. Hence we have $\mathbf{t} - \mathbf{b}$.

Figure 29: Case $g.v$.

By examining our 7×7 tree more carefully along this reasoning, we can further reduce the size of the tree in Figure 8 (e.g., from twenty-five nodes to five), although the analysis at each node becomes more complicated.

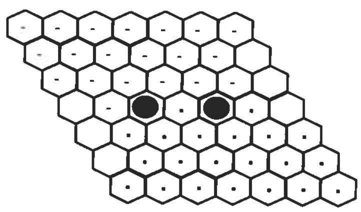


Figure 30: A precluding analysis for the 7×7 game tree with only one node.

For 7×7 Hex, if we pursue the smallest possible number of nodes in the game tree, we can prove that only *one* node suffices. We choose Black's first move at 34, rather than 44. For the initial precluding analysis at the root, B54 is used as a fictitious move as shown in Figure 30. Let x and y be B34 and B54, respectively. In the lower-half area of the board including 44 and 64, we can prove that both $x - \mathbf{b}$ and $y - \mathbf{b}$ hold, which may be called AND-lemma. Given this position, if White's next move is at 44, Black moves at 26 for $x - \mathbf{b}$. Then Black also secures $y - \mathbf{b}$. For any other white move, Black moves at 44, which directly connects x and y. The desired connection $x - \mathbf{b}$ can be proved by straightforward case analyses.

In the upper-half area including 24 (disjoint from the above area), we can prove that, if we restrict the analysis within this area, all the empty cells are precluded for attaining $\mathbf{t}-x|\mathbf{t}-y$, i.e., two consecutive white moves in this area cannot cut this union-connection. The proof is almost the same as the above.

By the top-bottom symmetry, we can deduce $\mathbf{t} - \mathbf{b}$, unless White's second move is at 44 or 54. These two surviving moves can be precluded by B26. The proof is easy.

4. A WINNING STRATEGY IN 8×8 HEX

Our method has been applied to the 8×8 board to make a complete winning strategy. The game tree consists of 108 nodes of positions as shown in Figure 31. Actually it is drawn in the form of a DAG including confluent paths to (almost) identical positions. In the figure, $P+$ means that P is a representative of White blocking moves in that there are some other blocking moves which are regarded as trivial variations of P. As usual, we traverse the tree in the depth-first order for the correctness proof. In each of the 52 positions of White's turn (● nodes), we make a precluding analysis. The remaining 56 nodes (plus the initial position of the game) represent Black's positions to make winning moves. Each of those black moves is depicted at its succeeding ● node. The tree has 12 terminal leaf nodes. Among them, 8 leaf nodes (denoted by ϕ) represent positions in which no white next moves survive the precluding analyses. The other 4 leaf nodes (denoted by ω) represent final-stage positions in which Black has the straightforward forcing winning move-sequences, i.e., in which all White's moves (possibly having some trivial variations) are forced.

The description of the whole proof is reasonably simple (for the complexity of 8×8), though it is still too long to include here, as requiring about 200 figures for the precluding analyses in the 52 cases. Here we shall sketch the outline of our proof and explain *all* the basic ideas used there.

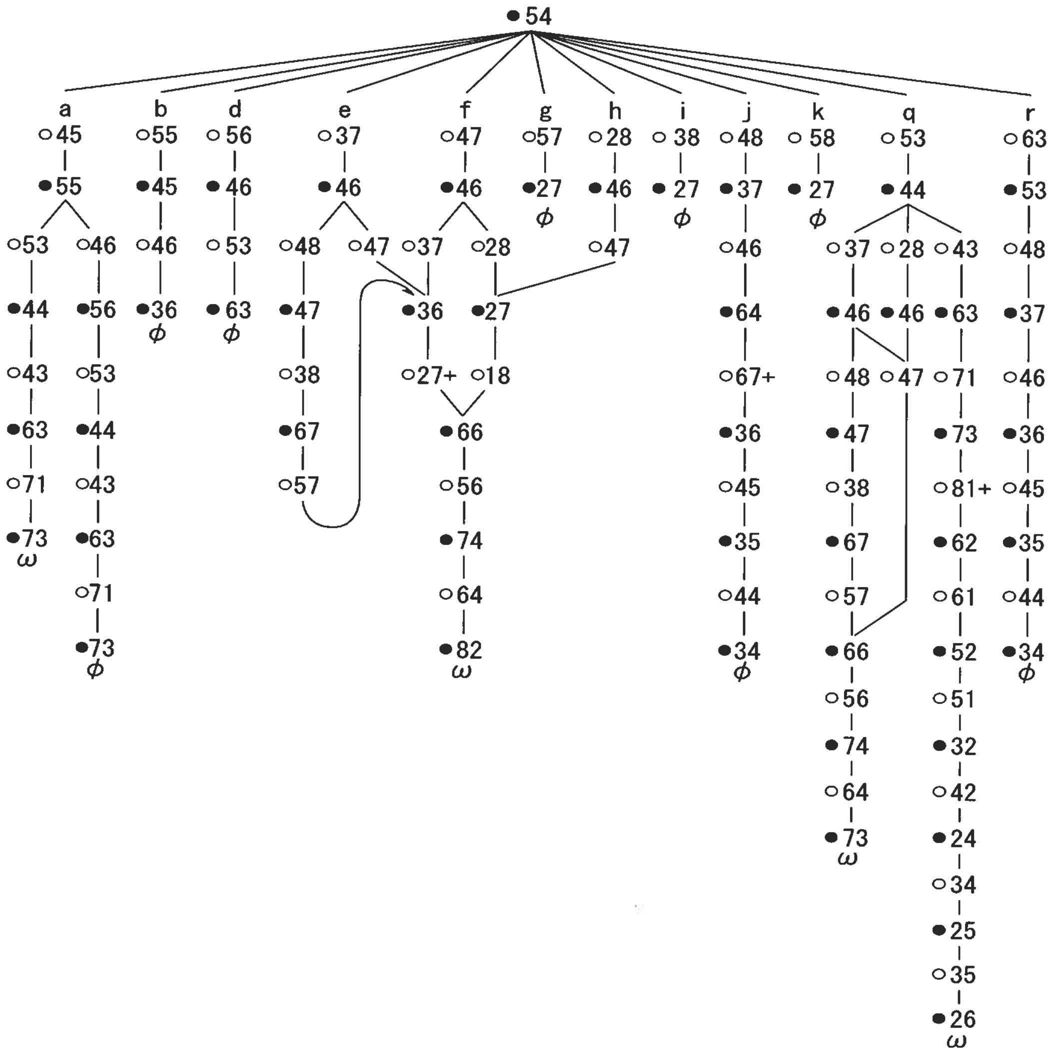
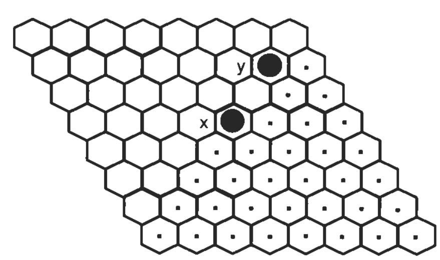


Figure 31: Our proof tree of 8×8 Hex.

The basis of our proof for 8×8 is the following union-connection.

Lemma 2 (U_8-lemma)
In Figure 32 we have $x - \mathbf{b} \mid y - \mathbf{b}$, where the supporting set of empty cells for this connection is shown there.



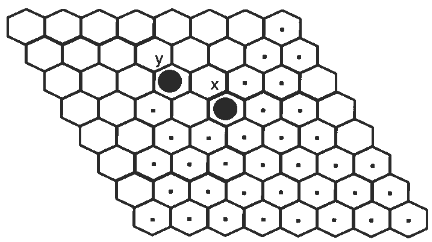
Let x and y be B54 and B72, respectively. We have $x - \mathbf{b} \mid y - \mathbf{b}$.

Figure 32: U_8-lemma.

Proof: The proof is similar to U_1-lemma for 7×7. Eight cases W45, W55, W46, W37, W47, W28, W38 and W48 need to be checked, since in other cases B46 suffices.

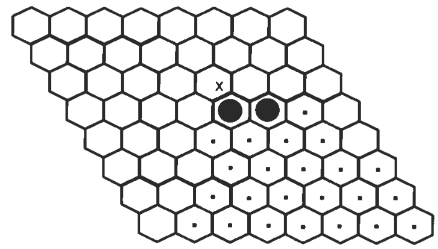
For W45, Black's move is B55. Each of five blocking moves (W46, W56, W47, W38 and W48) can be easily shown to lead to $x - \mathbf{b}$.

For W55, Black's move is B45. This leads to a position similar to U_1-lemma, which can be shown to attain $x - \mathbf{b} \mid y - \mathbf{b}$. More specifically, after W55 and B45, five white moves W36, W46, W37, W28 and W38 need to be checked. For W36 or W38, B46 easily attains $x - \mathbf{b}$. For W46, B36 leads to a similar position in U_1-lemma, which can be dealt with by the sidestepping technique (W55 B45 W46



Let x and y be B54 and B43, respectively.
We have $x - \mathbf{b} \mid y - \mathbf{b} \mid \mathbf{t} - \mathbf{b}$.

Figure 33: A huge lemma.



Let x be B54. We have $x - \mathbf{b}$.

Figure 34: Hypertrapezoid (HT) lemma.

B36 W28 B37 W38 B47 W48 B67 W57 B66 W56 B74). For W28 or W37, B46 manages White's succeeding blocking moves by the sidestepping technique in a similar way as above.

For W46, Black's move is B65. Each of five blocking moves (W56, W66, W57, W48 and W58) except W66 can be easily shown to lead to $x - \mathbf{b}$. The remaining W66 can be dealt with by the sidestepping technique starting with B56. The hardest move-sequence leading to $y - \mathbf{b}$ is W46 B65 W66 B56 W48 B57 W58 B77 W67 B85 W75 B84 W74 B83.

For W38, Black's move is B46. The proof is easy by left-right symmetry. For the remaining four cases, Black move is also B46, since White's blocking moves can be dealt with by the sidestepping technique (which is similar to the subcases of W55 above). □

Many variants of this lemma as well as the sidestepping technique are extensively used throughout the analyses. As one of such variants we can prove a huge lemma in Figure 33. Proving $x - \mathbf{b}|y - \mathbf{b}|\mathbf{t} - \mathbf{b}$ may be fun. (Find a superb move for W46.) The hypertrapezoid (HT) lemma in Figure 34 is another example, which can be proved by using the sidestepping technique. Note that, for W57, Black's move is B47.

The initial analysis at the root for White's second moves (depth 1) leaves 18 nontrivial cases as shown in Figure 35. In this analysis the fictitious black moves are B44 and B64 (both using ST- and HT-lemmas) and B46 (using an 8×8 version of U_2-lemma in Figure 5 plus S- and T-lemmas). Among the 18 cases, six cases (c, l, m, n, o, p) are precluded by more detailed analyses. The fictitious black moves are B35 for c (using a variant of HT-lemma), B72 for l, m (each using $U8$-lemma), B44 for o (using HT-lemma), and B43 for n, p. In the last case n and p, we have $x - \mathbf{b}|y - \mathbf{b}|\mathbf{t} - \mathbf{b}$ (the huge lemma in Figure 33), $x - y$ (S-lemma) and $y - \mathbf{t}$ (T-lemma) without including n and p in the supporting set. Hence the 12 cases survive as shown in Figure 31.

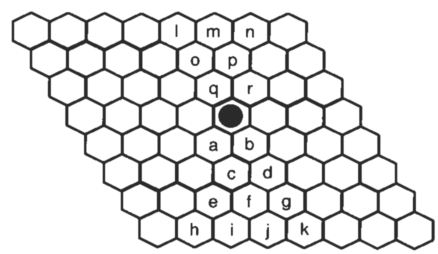


Figure 35: The initial analysis for White's second moves in 8×8 Hex.

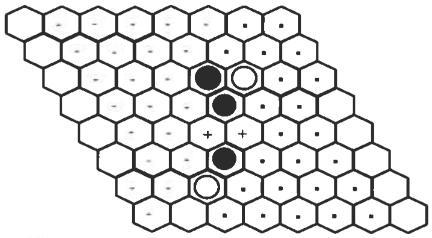
In a similar way, the remaining 51 nodes can be analyzed by appropriately choosing the set of fictitious black moves at each node. As another example, Figure 36 shows an interesting position, among many others, which demonstrates the power of union-connection. In this position we can prove that Black wins by applying two variants of union-connection in the left and right disjoint areas as seen in the figure. Thus in the descendent of node e (after B54 W37 B46), the fictitious move B53 precludes W63. This position also implies that, in the descendent of node r (after B54 W63 B53), the fictitious move B46 precludes W37.

In order to make our case-analyses simpler, the notion of union-connection is extended. This is best explained by means of an example. The upper-left T-lemma in Figure 37 can be given an additional property named AB-*property*.

Lemma 3 (AB-property for T-lemma)

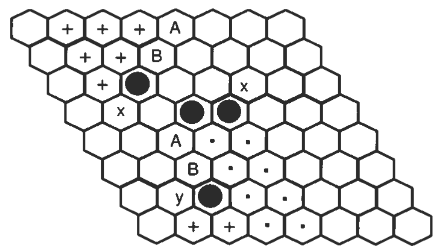
Let x be B33 in the upper-left of Figure 37. We have not only $x - \mathbf{t}$ but also the following property: starting with the empty T area (8 empty cells including A and B), at any time of White's turn, if White occupies A, then Black occupies B.

Proof: The proof is done by the case analysis. For White A, Black moves at B. This forces White to move at 41



Let x and y be B54 and B46, respectively. We have $\mathbf{t}-x \mid \mathbf{t}-y \mid \mathbf{t}-\mathbf{b}$ in the left area, and $x-\mathbf{b} \mid y-\mathbf{b} \mid \mathbf{t}-\mathbf{b}$ in the right area. Hence we have $\mathbf{t}-\mathbf{b}$, since $x-y$.

Figure 36: Applying two variants of U-lemma.



In the lower-right, let x and y be B54 and B37, respectively. We have $x-y \mid x-\mathbf{b}$ and AB-property.

Figure 37: Two examples of AB-property.

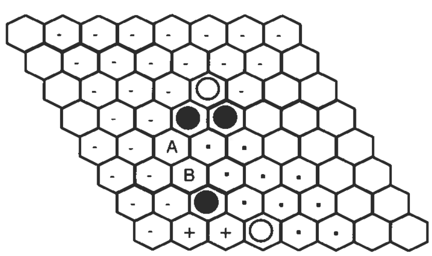
and then Black moves at 22. For White playing at B, Black moves at 32, leaving A empty. For W41 as well as other white moves, Black moves at B. In all these cases, we have $x-\mathbf{t}$ and the desired property. □

Note that, at any time, when White occupies both A and B (necessarily A after B), Black has already secured $x-\mathbf{t}$ and is able to make a move outside the T area.

The AB-property serves as a connecting interface between two adjacent areas whose supporting sets of empty cells are mutually disjoint. Actually the above lemma 3 need not be used in our proof. Two types of AB-property explained below are used at three nodes (among the descendents of a, j, and q).

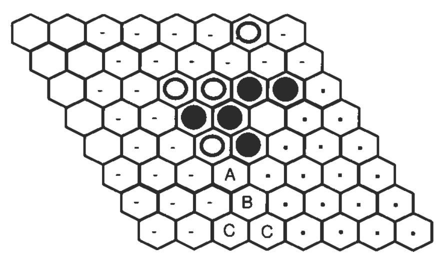
The lower-right in Figure 37 shows a useful lemma with $x-y|x-\mathbf{b}$ as well as AB-property. The proof is similar to the above.

Consider the position in Figure 38, which appears in the descendents of node j and q. In this position we can prove $\mathbf{t}-\mathbf{b}$, without any deeper searching. For any white move-sequence, starting with this position, Black attains $\mathbf{t}-x|\mathbf{t}-y|\mathbf{t}-\mathbf{b}$ in the upper-left area. Among White's candidate moves, W43 is the hardest case, since in other cases Black easily attains $\mathbf{t}-x$. W43 can be dealt with by using the sidestepping technique plus AB-property. Consider the sidestepping move-sequence leading to B24 and W34 (i.e., W43 B63 W71 B62 W61 B52 W51 B32 W42 B24 W34). In this position, Black can choose either B16 or B25 depending on the configuration at A and B, for attaining the desired connections $\mathbf{t}-x|\mathbf{t}-y|\mathbf{t}-\mathbf{b}$. If A is White, Black chooses B16, because B is Black by AB-property. If A is empty (or Black), Black chooses B25. (If White has occupied both A and B, Black can make an additional move in the upper-left area.) In the lower-right area we have $x-y|x-\mathbf{b}$ and $y-\mathbf{b}$. Hence we have $\mathbf{t}-\mathbf{b}$. This implies that W48 or W53 can be precluded in the analysis for the fourth move in q or j, respectively.



Let x and y be B54 and B37, respectively. We have $\mathbf{t}-\mathbf{b}$. See the proof in the text.

Figure 38: Proving $\mathbf{t}-\mathbf{b}$ by AB-property (1).



Let x be B54. We have $\mathbf{t}-x \mid \mathbf{t}-\mathbf{b}$ and $x-\mathbf{b}$, which implies $\mathbf{t}-\mathbf{b}$.

Figure 39: Proving $\mathbf{t}-\mathbf{b}$ by AB-property (2).

A slightly more complicated pattern of AB-property appears in a descendent of node a. In the lower-right in Figure 39, starting with the supporting set of empty cells, not only $x-\mathbf{b}$ but also the following property can be proved. If White occupies A, then Black occupies B and secures $B-\mathbf{b}$ (S-lemma) and $x-\mathbf{b}$. This $x-\mathbf{b}$ can be verified by noting the sequence W46(A) B47(B) W56 B66 W65 B84 W75 B85 W76 B86 W78 B77 W68 B67. If White occupies B, then Black occupies 76 for $x-\mathbf{b}$, which makes A and C useless for White's moves any more. If White occupies C, then Black occupies B and secures $x-B$ (S-lemma) and $x-\mathbf{b}$. For any other White move, Black occupies B.

In the position in Figure 39, it is now easy to see that, in the upper-left area, the sidestepping move-sequence can be dealt with by this AB-property. In this case, after W81 B62 W61 B52 W51 B32 W42, the finishing sequence is B24 W34 B25 W35 B17 W18 B27, leading to $\mathbf{t} - \mathbf{b}$ (or $\mathbf{t} - x$ and $x - \mathbf{b}$).

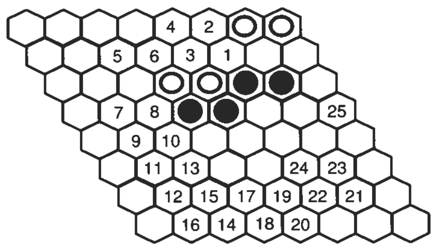


Figure 40: Black winning move-sequence in the descendent of case q.

Finally, a notable move-sequence is shown in Figure 40, which leads to ω in case of q. At White's tenth move (W35), all other white moves can be precluded by the fictitious black moves at B35 and B36. Note that W36 is precluded by B35. We can also prove that, for any white move prior to W34 in the lower area in the figure, Black has an immediate way to attain $\mathbf{t} - \mathbf{b}$. For example, if White moves at 37 instead of 61, then Black moves at 35, implying that $\mathbf{t} - x$ and $x - \mathbf{b}$ in two disjoint supporting sets, where x is B54. In fact, in the position just after B62 (1 in the figure), Black's fictitious moves B61 and B35 preclude all the empty cells except 61. In Figure 40, after W35 (10), the forcing move-sequences beginning with B26 (11) easily attains $\mathbf{t} - x$ and $x - \mathbf{b}$ (or $\mathbf{t} - \mathbf{b}$). Several minor variations of White's forced moves after B26 (11) can be dealt with similarly. Hence we conclude that all white moves from 2 to 24 may be regarded to be *forced* for preventing Black from immediate attaining of $\mathbf{t} - \mathbf{b}$.

5. CONCLUDING REMARKS

We have introduced a notion of union-connection and presented some techniques based on it for proving the correctness of the winning strategies for the 7×7 and 8×8 boards in a simple and constructive way. All the results here were reported at the workshop (Noshita, 2004), though the proof for 8×8 has been refined since then.

Acknowledgements
The author thanks the referees for their useful comments to revise the paper. This work is supported by the Grant-in-Aid from the Ministry of Education, Science, Sports and Culture of Japan, No. 15500021.

6. REFERENCES

Anshelevich, V. (2002). A Hierarchical Approach to Computer Hex. *Artificial Intelligence*, Vol. 134, pp. 101–120.

Enderton, B. (2000). Hexfaq. http://www.cs.cmu.edu/hde/hex/hexfaq.

Gardner, M. (1957). Mathematical Games. *Scientific American*, Vol. 197.

Hayward, R., Bjornsson, Y., Johanson, M., Kan, M., Po, N., and Rijswijck, J. van (2003). Solving 7×7 Hex: Virtual Connections and Game-State Reduction. *Advances in Computer Games: many games many challenges* (eds. H. J. van den Herik, H. Iida, and E. A. Heinz), pp. 261–278, Kluwer Academic Publishers, Norwell, MA.

Herik, H. J. van den, Uiterwijk, J. W. H. M., and Rijswijck, J. van (2002). Games Solved: Now and in the Future. *Artificial Intelligence*, Vol. 134, pp. 277–311.

Noshita, K. (2004). Union-Connections and a Simple Readable Winning Way in 7×7 Hex. *Proc. of 11th Game Programming Workshop*, pp. 72–79.

Rijswijck, J. van (2003). Queenbee. www.cs.ualberta.ca/queenbee.

Yang, J. (2003). Jingyang. www.ee.umanitoba.ca/jingyang.

Yang, J., Liao, S., and Pawlak, M. (2001). On a Decomposition Method for Finding Winning Strategy in Hex Game. *International Conference on Application and Development of Computer Games in the 21st Century*, pp. 96–111.

Yang, J., Liao, S., and Pawlak, M. (2003). New Winning and Losing Posisitions for 7×7 Hex. *Computers and Games, Third International Conference, CG 2002* (eds. J. Schaeffer, M. Muller, and Y. Bjornsson), Vol. 2883 of *Lecture Notes in Computer Science*, pp. 230–248.