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NeST graphs ☆

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Abstract

We establish results on NeST graphs (intersection tolerance graphs of neighborhood subtrees of a tree) and several subclasses. In particular, we show the equivalence of proper NeST graphs and unit NeST graphs, the equivalence of fixed distance NeST graphs and threshold tolerance graphs, and the proper containment of NeST graphs in weakly chordal graphs. The latter two results answer questions of Monma, Reed and Trotter, and Bibelnieks and Dearing.

Keywords: NeST graph; Tolerance graph; Interval graph; Threshold tolerance graph; Weakly chordal graph; Unit NeST representation; Proper NeST graph; Fixed distance NeST graph; Fixed tolerance NeST graph

1. Introduction

A graph $G=(V,E)$ is a *tolerance graph* if there is a set $\{I_v \mid v \in V\}$ of intervals of the real line and a set $\{\tau_v \mid v \in V\}$ of positive tolerances such that

$$xy \in E \quad \Leftrightarrow \quad |I_x \cap I_y| \geqslant \min\{\tau_x, \tau_y\}.$$

Introduced by Golumbic and Monma [11], tolerance graphs generalize interval graphs [8] by incorporating tolerance for overlap. The class of tolerance graphs contains not only the class of interval graphs but also the classes of permutation graphs [12], threshold graphs and threshold tolerance graphs [17].

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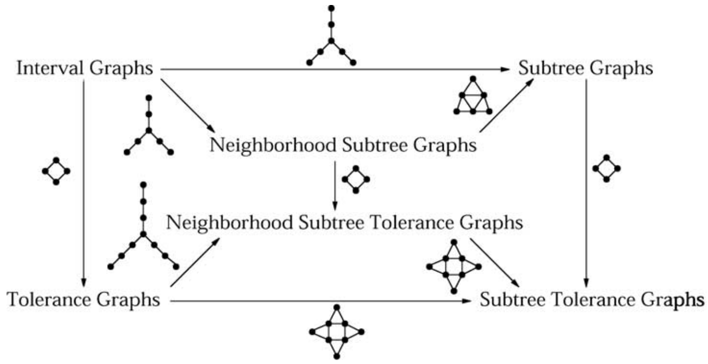

Fig. 1. NeST graphs and related graph classes. The bottom three classes are superclasses of the top three classes obtained by incorporating tolerances. For each superclass relation shown (indicated by an arc), the graph beside the arc is in the superclass but not the class.

In 1993, Bibelnieks and Dearing generalized tolerance graphs to the class of *neighborhood subtree tolerance* (NeST) *graphs* [2]. NeST graphs generalize tolerance graphs by replacing neighborhoods of the line (namely intervals) with neighborhoods of an embedded tree. Generalizations of interval graphs obtained by replacing intervals of the line with subtrees of a tree have been studied before: in [6] *subtree graphs* (intersection graphs of subtrees of a tree) are studied and shown to be exactly the chordal graphs; other models are studied in [7,9].

Generalizing subtree graphs by incorporating tolerance yields *subtree tolerance graphs*. Thus interval graphs are to tolerance graphs as subtree graphs are to subtree tolerance graphs. Subtree tolerance graphs are uninteresting, since every graph is a subtree tolerance graph [2]. NeST graphs are obtained from subtree tolerance graphs by restricting subtrees to neighborhood subtrees (defined shortly), a kind of subtree studied by Tamir in relation to network location problems [19]. Fig. 1 shows the containment relations among NeST graphs and other graph classes discussed here.

Bibelnieks and Dearing showed that NeST graphs are *weakly chordal* [2], namely contain no induced cycle on five or more vertices in the graph or in its complement [14]. It follows that NeST graphs are *perfect*, namely have chromatic number equal to clique number for all induced subgraphs [1,8], and that the optimization problems maximum clique, maximum stable set, minimum coloring and minimum clique cover, can be solved in polynomial time on NeST graphs [10].

In this paper we establish results on NeST graphs and four subclasses, namely

- *unit NeST graphs*, in which all neighborhood subtrees have unit diameter,
- *proper NeST graphs*, in which no neighborhood subtree is properly contained in another,
- *fixed distance NeST graphs*, in which neighborhood subtree centers are equidistant, and
- *fixed tolerance NeST graphs*, in which all tolerances are the same.

The first and third of these subclasses are new while the second and fourth were introduced in [2]. Our main results can be summarized as follows:

- *Unit NeST graphs are exactly proper NeST graphs*. This extends an analogous property of interval graphs [18] not shared by tolerance graphs [3], and yields a succinct description of proper NeST graphs.
- *Fixed distance NeST graphs are exactly threshold tolerance graphs*. This answers a question posed by Monma et al. [17].
- *Fixed tolerance NeST graphs.* These are easily seen [2] to be equivalent to the intersection graphs of neighborhood subtrees of a tree, and their characterization remains an open problem.
- *NeST graphs form a proper subclass of weakly triangulated graphs*. This answers a question posed by Bibelnieks and Dearing [2].

2. Background and definitions

We use standard graph theory terminology [4]. Graphs are simple and undirected; $G=(V,E)$ denotes the graph with vertex set V and edge set E. We abbreviate edge $\{x,y\}$ as xy. Distinct vertices x,y in a graph are *neighbors* if xy is an edge and *nonneighbors* otherwise. For any vertex x in the graph $G=(V,E)$ the *neighborhood* of x is $N(x)=\{y\in V\colon xy\in E\}$ and the *nonneighborhood* of x is $M(x)=\{y\in V-\{x\}\colon xy\notin E\}$.

Let $\mathscr{T}$ be a tree (a connected, acyclic graph) and let T be an embedding of $\mathscr{T}$ in the plane. The notion of tree embedding used here is consistent with that found in [19]. $P(x,y)$ denotes the unique path in T between the points x and y and $d(x,y)$ denotes the length of $P(x,y)$. We call $x\in T$ an *endpoint* of T if x corresponds to a leaf of $\mathscr{T}$.

The *neighborhood subtree* of T with center $c\in T$ and radius $r\geqslant 0$, denoted $T(c,r)$, is the set of points $\{x\in T\colon d(x,c)\leqslant r\}$. Note that this is a set of points in the embedding T and not vertices of the tree $\mathscr{T}$.

If T' is a connected subset of points (namely a subtree) of the embedded tree T then the *diameter* (or *size*) $|T'|$ of T' is the length of a longest path in T', namely

$$|T'|=\begin{cases}\max\{d(p_1,p_2)\colon p_1,p_2\in T'\} & \text{if } T'\neq\phi,\\ 0 & \text{if } T'=\phi.\end{cases}$$

Definition 1. A graph $G=(V,E)$ is a *neighborhood subtree tolerance* (NeST) *graph* if there exists an embedded tree T, a set $S=\{T(c_v,r_v)\colon v\in V\}$ of neighborhood subtrees of T and a set $\mathscr{T}=\{\tau_v\colon v\in V\}$ of positive numbers called *tolerances* such that

$$xy\in E \quad\Leftrightarrow\quad |T(c_x,r_x)\cap T(c_y,r_y)|\geqslant\min\{\tau_x,\tau_y\}.$$

The triple $(T,S,\mathscr{T})$ is called a *neighborhood subtree tolerance* (NeST) *representation* of G. G is called the graph *associated* with the NeST representation $(T,S,\mathscr{T})$.

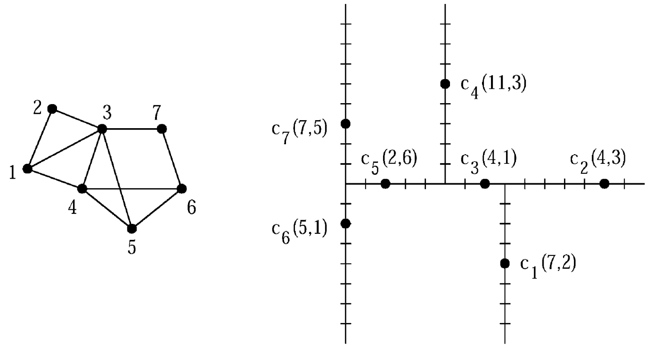

Fig. 2. A graph and its NeST representation. Pairs denote radius and tolerance values.

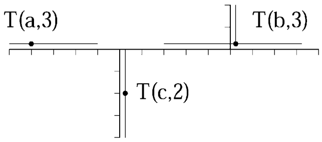

Fig. 3. A truncated, nonmaximal neighborhood subtree $T(a,3)$, a truncated, maximal neighborhood subtree $T(b,3)$ and an untruncated, maximal neighborhood subtree $T(c,2)$.

A NeST representation and its associated graph appear in Fig. 2.

We abbreviate $T(c_x,r_x)$ as T_x and $T_x \cap T_y$ as T_{xy}. With respect to a vertex x in a graph with NeST representation $(T,S,\mathscr{T})$, the terms T_x, c_x, r_x, τ_x always denote the associated neighborhood subtree, center, radius and tolerance, respectively.

A neighborhood subtree T_x of $(T,S,\mathscr{T})$ is *maximal* if $|T_x| = 2r_x$ and *truncated* if an endpoint $p \in T$ satisfies $d(p,c_x) < r_x$. Nonmaximal neighborhood subtrees are truncated, but the converse does not necessarily hold. Fig. 3 shows examples of maximal, nonmaximal and truncated neighborhood subtrees.

Maximality simplifies the analysis of NeST representations, since the diameter of a maximal neighborhood subtree depends only on its radius and not on the embedding tree. Having all pairwise neighborhood subtree intersections nonempty also simplifies analysis. Thus for the purposes of simplifying analysis, we make the following two assumptions throughout the paper (a proof that these assumptions do not restrict the generality of our results, namely that every NeST graph has a NeST representation in which all subtrees are maximal and untruncated, is outlined in [2] and given in complete detail in [16]):

1. All neighborhood subtrees can be made maximal in a NeST representation (by extending those edges of the embedding which are incident with leaves.)
2. All pairwise neighborhood subtree intersections can be made measurably non-empty in a NeST representation (by increasing the radius and tolerance for each neighborhood subtree.)

We now show how tolerances can be eliminated from a NeST representation. Such "tolerance-free" representations will be used repeatedly later. Let $G=(V,E)$ have NeST representation $(T,S,\mathscr{T})$. For each $x\in V$ define the set $B(x)$ by

$$B(x)=\{z\in M(x)\colon |T_{xz}|\geqslant|T_{xy}|,\ \text{for all } y\in M(x)\}.$$

$B(x)$ is the set of nonneighbors of x that maximize the size of their neighborhood subtree intersections with the neighborhood subtree of x.

Definition 2. The pair (T,S), where T is an embedded tree and S is a set of neighborhood subtrees of T, is a *tolerance-free NeST representation* of $G=(V,E)$ if

$$xy\in E \quad\Leftrightarrow\quad |T_{xy}|>\min\{|T_{x\dot{x}}|,|T_{y\dot{y}}|\},$$

where for a vertex z, $\dot{z}$ is any element of $B(z)$. In the case $B(z)=\emptyset$, we define $T_{z\dot{z}}=\emptyset$.

Before stating and proving the main result of this section, we define the *perturbation number* of a tolerence-free NeST representation of a graph G. The perturbation number provides a measure of how much a tolerance-free NeST representation can be altered and yet remain a representation for G. With this definition the theorem is easily established.

Definition 3. The *perturbation number* $\rho(R,G)$ of a tolerence-free NeST representation $R=(T,S)$ of $G=(V,E)$ is defined by

$$\rho(R,G)=\begin{cases}\min(D) & \text{if } D\neq\emptyset,\\ 0 & \text{if } D=\emptyset,\end{cases}$$

where $D=\{|T_{xy}|-|T_{x\dot{x}}|\colon |T_{xy}|>|T_{x\dot{x}}|,\ xy\in E\}$.

Theorem 1. *There exists $\mathscr{T}$ such that $(T,S,\mathscr{T})$ is a NeST representation of G if and only if (T,S) is a tolerence-free NeST representation of G.*

Proof. ($\Leftarrow$) Let $G=(V,E)$ have a tolerence-free NeST representation (T,S). Define a NeST representation $(T,S,\mathscr{T})$ where each tolerance $\tau_x\in\mathscr{T}$ is defined by $\tau_x=|T_{x\dot{x}}|+\varepsilon$ and ε is given by

$$\varepsilon=\begin{cases}\rho((T,S),G)/2 & \text{if } \rho((T,S),G)\neq 0,\\ 1 & \text{otherwise.}\end{cases}$$

To prove $(T,S,\mathscr{T})$ is a NeST representation of G we must verify that

$$xy\in E \quad\Leftrightarrow\quad |T_{xy}|\geqslant\min\{\tau_x,\tau_y\}$$

for all $x,y\in V$. First suppose that $xy\in E$. Thus, either $|T_{xy}|>|T_{x\dot{x}}|$ or $|T_{xy}|>|T_{y\dot{y}}|$; assume the former without loss of generality. Since $|T_{xy}|>|T_{x\dot{x}}|$ it follows that $\rho((T,S),G)>0$ and $|T_{xy}|-|T_{x\dot{x}}|\geqslant\rho((T,S),G)>\varepsilon$, hence, $|T_{xy}|>|T_{x\dot{x}}|+\varepsilon=\tau_x$, and we are done. Next suppose that $xy \notin E$. Thus $|T_{xy}| \leqslant |T_{x\overset{\bullet}{x}}|$ and $|T_{xy}| \leqslant |T_{y\overset{\bullet}{y}}|$. Since $\varepsilon > 0$, $|T_{xy}| < |T_{x\overset{\bullet}{x}}| + \varepsilon = \tau_x$ and $|T_{xy}| < |T_{y\overset{\bullet}{y}}| + \varepsilon = \tau_y$, and again we are done.

($\Rightarrow$) Let $G=(V,E)$ have a NeST representation $(T,S,\mathscr{T})$. To prove (T,S) is a tolerence-free NeST representation of G we must verify that

$$xy \in E \quad \Leftrightarrow \quad |T_{xy}| > \min\{|T_{x\overset{\bullet}{x}}|, |T_{y\overset{\bullet}{y}}|\},$$

for all $x, y \in V$. First suppose that $xy \in E$. Thus, either $|T_{xy}| \geqslant \tau_x$ or $|T_{xy}| \geqslant \tau_y$; assume the former without loss of generality. Observe that $\tau_x > |T_{xz}|$, for all $z \in M(x)$, hence, $|T_{xy}| \geqslant \tau_x > |T_{x\overset{\bullet}{x}}|$, and we are done. Next suppose that $xy \notin E$. Thus $|T_{xy}| \leqslant |T_{x\overset{\bullet}{x}}|$ and $|T_{xy}| \leqslant |T_{y\overset{\bullet}{y}}|$ by definition of $\overset{\bullet}{x}$ and $\overset{\bullet}{y}$. $\square$

Different graphs may share the same tolerence-free NeST representation. For example, the two non-isomorphic graphs with two vertices each have a tolerence-free NeST representation in which the embedding tree is a line segment, and the two subtrees are a pair of intersecting segments.

3. Unit and proper NeST graphs

Unit tolerance and unit interval graphs have interval representations where all intervals have unit length. Proper tolerance and proper interval graphs have interval representations where no interval is properly contained in another. Proper interval graphs are exactly unit interval graphs [18], yet unit tolerance graphs form a strict subclass of proper tolerance graphs [3]. Thus, it is natural to ask whether unit NeST graphs are exactly proper NeST graphs. The answer is yes, as Theorem 2 shows.

Define *unit* (or *fixed diameter*) *NeST graphs* to be those graphs having a NeST representation in which all neighborhood subtrees have the same diameter. By scaling, one may assume that all neighborhood subtrees in a unit NeST representation have unit diameter; we call such representations *unit NeST representations*. Define *proper NeST graphs* to be those graphs with a NeST representation in which no neighborhood subtree is properly contained in another. Note that the two assumptions made in Section 2 apply to fixed diameter and proper NeST representations.

The following lemma gives a closed formula for neighborhood subtree intersection size:

Lemma 1 (Bibelnieks and Dearing [2]). *If $T_x \not\subset T_y$, $T_y \not\subset T_x$ and $T_{xy} \neq \emptyset$ then $|T_{xy}| = r_x + r_y - d(c_x, c_y)$.*

Theorem 2. *G is a proper NeST graph if and only if G is a unit NeST graphs.*

Proof. Fixed diameter NeST representations are obviously proper, so it suffices to show that proper NeST graphs are unit NeST graphs. Let $(T,S,\mathscr{T})$ be a proper NeST representation of $G=(V,E)$ where all neighborhood subtrees are maximal. Let

$r = \max\{r_x\colon x \in V\}$ and define a new NeST representation $(T', S', \mathcal{T}')$ from $(T, S, \mathcal{T})$ as follows:

- $T' = (\bigcup_{x\in V} L_x) \cup T$ where L_x is a line segment of length $2r$ attached[1] to T at c_x, for each $x \in V$.
- For all $x \in V$, c'_x is located on L_x such that $d(c'_x, c_x) = r - r_x$.
- For all $x \in V$, $r'_x = r$ and $\tau'_x = \tau_x$.

Claim 1. *$(T', S', \mathcal{T}')$ is a unit NeST representation.*

Since $r'_x = r$ for all $x \in V$, it suffices to show all neighborhood subtrees in R' are maximal. Let $x \in V$ and $p \in L_x$ such that $d(c'_x, p) = r$. Since T_x is maximal there is a point $q \in T$ such that $d(q, c_x) = r_x$. It follows that in T', $P(p, q) = P(q, c_x) \cup P(c_x, c'_x) \cup P(c'_x, p)$, hence, $d(p, q) = r_x + (r - r_x) + r = 2r$ in T'. This proves the maximality of T'_x, so the claim holds.

Claim 2. *$(T', S', \mathcal{T}')$ is a NeST representation of G.*

It must be shown that for all $x, y \in V$ $xy \in E \Leftrightarrow |T'_{xy}| \geqslant \min\{\tau'_x, \tau'_y\}$. It follows from Lemma 1 that $|T'_{xy}| = r'_x + r'_y - d(c'_x, c'_y) = 2r - d(c'_x, c'_y)$, but $d(c'_x, c'_y) = d(c_x, c'_x) + d(c_x, c_y) + d(c_y, c'_y) = (r - r_x) + d(c_x, c_y) + (r - r_y)$. It follows that, $|T'_{xy}| = 2r - (2r - r_x - r_y + d(c_x, c_y)) = r_x + r_y - d(c_x, c_y)$, and so $|T'_{xy}| = |T_{xy}|$ since the original representation is proper. Since $\tau'_x = \tau_x$, for all $x \in V$, it follows from $xy \in E \Leftrightarrow |T_{xy}| \geqslant \min\{\tau_x, \tau_y\}$ that $xy \in E \Leftrightarrow |T'_{xy}| \geqslant \min\{\tau'_x, \tau'_y\}$, so the claim holds, and so the theorem holds. □

Theorem 3 yields a simple characterization of proper NeST graphs.

Definition 4. (T, X) is called a *phylogeny*[2] if T is an edge weighted tree with leaf set X.

Theorem 3. *$G = (V, E)$ is a proper NeST graph if and only if there exists a phylogeny (T, V), such that*

$$xy \in E \quad \Leftrightarrow \quad \begin{cases} d(x, y) < d(x, p) & \text{for all } p \in M(x) \\ & \text{or} \\ d(x, y) < d(y, q) & \text{for all } q \in M(y). \end{cases}$$

Proof. ($\Rightarrow$) By Theorem 2, let $(T, S, \mathcal{T})$ be a unit NeST representation of $G = (V, E)$ in which all neighborhood subtrees are maximal and all neighborhood subtree intersections

[1] It may be necessary to "rearrange" the embedded tree T in the plane to accommodate the insertion of L_x.

[2] The term *phylogeny* has appeared recently in both biological and mathematical contexts. In biology a phylogeny is usually defined as the evolutionary history of a collection of objects (species or sequences), whereas in mathematics a phylogeny is a leaf labeled tree. We use the term phylogeny for its mathematical meaning, not because there is any evolutionary significance.

have positive measure. Let (T,S) be the corresponding tolerence-free NeST representation of G. (T,S) satisfies the edge condition $xy \in E \Leftrightarrow |T_{xy}| > \min\{|T_{x\overset{\bullet}{x}}|, |T_{y\overset{\bullet}{y}}|\}$ so by Lemma 1

$$xy \in E \quad \Leftrightarrow \quad r_x + r_y - d(c_x, c_y) > \min\{r_x + r_{\overset{\bullet}{x}} - d(c_x, c_{\overset{\bullet}{x}}), r_y + r_{\overset{\bullet}{y}} - d(c_y, c_{\overset{\bullet}{y}})\}. \tag{*}$$

Since all neighborhood subtree diameters are equal and all neighborhood subtrees are maximal, for all $x \in V$, $|T_x| = 2r$ for some r, so

$$xy \in E \quad \Leftrightarrow \quad 2r - d(c_x, c_y) > \min\{2r - d(c_x, c_{\overset{\bullet}{x}}), 2r - d(c_y, c_{\overset{\bullet}{y}})\}$$

and so

$$xy \in E \quad \Leftrightarrow \quad d(c_x, c_y) < \max\{d(c_x, c_{\overset{\bullet}{x}}), d(c_y, c_{\overset{\bullet}{y}})\}$$

and so

$$xy \in E \quad \Leftrightarrow \quad \begin{cases} d(c_x, c_y) < d(c_x, c_p) \quad \text{for all } p \in M_x \\ \qquad\qquad\qquad \text{or} \\ d(c_x, c_y) < d(c_y, c_q) \quad \text{for all } q \in M_y. \end{cases}$$

It follows that a phylogeny (T', V) can be defined which satisfies the theorem.

($\Leftarrow$) Given (T, V), let $(T', S', \mathscr{T}')$ be a NeST representation with $S' = \{T(c_v, r): v \in V\}$ such that T' is an embedding of T, $d(c_x, c_y)$ is the same in T' as in T, $2r > \max\{d(c_x, c_y): x, y \in V\}$ (so each two subtrees intersect), and $\tau'_v = \min\{2r - d(c_v, c_w): w \in M_v\}$. Now (*) holds (reverse the steps in the preceding argument) with $\tau'_v = |T_{v\overset{\bullet}{v}}|$ and (T', S', τ') is a unit NeST representation, as desired. $\square$

The above description of proper NeST graphs is tolerance and radius-free: a proper NeST graph is characterized solely by the location of neighborhood subtree centers within an embedded tree. A phylogeny which satisfies Theorem 3 is a *proper phylogeny*.

4. Fixed distance NeST graphs and threshold tolerance graphs

Monma et al. [18] introduced *threshold tolerance graphs* (graphs $G = (V, E)$ for which there exist positive weights $\{w_v \mid v \in V\}$ and tolerances $\{t_v \mid v \in V\}$ such that $xy \in E$ if and only if $w_x + w_y \geqslant \min\{t_x, t_y\}$) and asked whether there is a characterization of complements of threshold tolerance graphs as intersection graphs of some restricted form of subtrees of a tree. In this section, we respond to this question by showing that there is such a characterization for threshold tolerance graphs themselves (not their complements): they are exactly *fixed distance NeST graphs*, namely graphs with a NeST representation in which all pairs of neighborhood subtree centers are equidistant.

A *star* is a tree in which a vertex, called the *center vertex*, is adjacent to all other vertices in the tree. An *embedded star* is an embedding of a star. The *origin* of an embedded star is the point corresponding to the center vertex of the associated star.

Definition 5. A *star NeST representation* is a NeST representation $(T, S, \mathscr{I})$ in which

- T is an embedded star,
- no two neighborhood subtree centers are located at the same point in T and
- all neighborhood subtree centers are equidistant from the origin of T.

Lemma 2. *A graph G is a fixed distance NeST graph if and only if it has a star NeST representation.*

Proof. If $G=(V,E)$ has a star NeST representation then the distance between any two distinct neighborhood subtree centers is twice the distance from any neighborhood subtree center to the origin of the embedded star, so G is a fixed distance NeST graph. A straightforward inductive argument based upon the observation that $\bigcup_{x,y\in V} P(c_x, c_y)$ is an embedded star shows that any fixed distance NeST graph has a star NeST representation. $\square$

By the lemma we may assume that fixed distance NeST representations are always star NeST representations.

Definition 6. A *radius-only representation* of a graph $G=(V,E)$ is a set of nonnegative numbers $R=\{s_v\colon v\in V\}$ such that

$$xy\in E \quad\Leftrightarrow\quad \begin{cases} s_x > s_p & \text{for all } p\in M(y) \\ & \text{or} \\ s_y > s_q & \text{for all } q\in M(x). \end{cases}$$

Theorem 4. *A graph is a fixed distance NeST graph if and only if it has a radius-only representation.*

Proof. ($\Rightarrow$) Assumption 2 from Section 2 holds for star NeST representations, so let $G=(V,E)$ have a tolerance-free, star NeST representation (T,S) with all neighborhood subtree intersections nonempty, and let $R=\{r_x\colon x\in V\}$. We wish to show that

$$xy\in E \quad\Leftrightarrow\quad \begin{cases} r_x > r_p & \text{for all } p\in M(y) \\ & \text{or} \\ r_y > r_q & \text{for all } q\in M(x), \end{cases}$$

and thus that R is a radius-only representation of G. Since neighborhood subtree centers are equidistant

$$|T_{ab}| > |T_{ac}| \Rightarrow r_b > r_c \quad\text{and}\quad |T_{ab}| \geqslant |T_{ac}| \Rightarrow r_b \geqslant r_c.$$

If $xy \in E$ then $|T_{xy}| > |T_{x\overset{\bullet}{x}}|$ or $|T_{xy}| > |T_{y\overset{\bullet}{y}}|$; assume the former without loss of generality. As a consequence, $r_y > r_{\overset{\bullet}{x}}$, hence $r_y > r_q$ for all $q \in M(x)$. If $xy \notin E$ then $x \in M(y)$ and $y \in M(x)$, hence $r_x \leqslant r_p$, for some $p \in M(y)$, is true simply because $r_x \leqslant r_x$. Similarly, $r_y \leqslant r_q$, for some $q \in M(x)$, is true.

($\Leftarrow$) Let $G=(V,E)$ have a radius-only representation R'. We may assume that $r'_x \neq r'_y$ for all $x \neq y$. For simplicity, let $V=\{1,\ldots,|V|\}$ and R' be indexed such that $r'_x > r'_y \Leftrightarrow x > y$. Pick some $m > 0$ and assign values to the radii $\{r_x : x \in V\}$ by the formula $r_k = m + km/|V|$. Hence $r'_x > r'_y \Leftrightarrow r_x > r_y$.

Let T be an embedded star with $|V|$ line segments each of length $3m$. Locate each neighborhood subtree center a distance m from the origin of T such that no two neighborhood subtrees are located at the same point of T. Hence (T,S) is a tolerance-free star NeST representation. By our assignment of values to the neighborhood subtree radii we have $|T_{xy}| = r_x + r_y - 2m$, for all $x,y \in V$. Note that $r_x \geqslant r_y \Leftrightarrow |T_{xz}| \geqslant |T_{yz}|$ for all $z \in V$.

We now show that (T,S) is a tolerance-free, star NeST representation of G. If $xy \in E$ then $r_x > r_p$, for all $p \in M(y)$ or $r_y > r_q$, for all $q \in M(x)$, hence $|T_{xy}| > |T_{yp}|$, for all $p \in M(y)$ or $|T_{xy}| > |T_{xq}|$, for all $q \in M(x)$. If $xy \notin E$ then $r_x \leqslant r_p$, for some $p \in M(y)$, and $r_y \leqslant r_q$, for some $q \in M(x)$, hence $|T_{xy}| \leqslant |T_{yp}|$, for some $p \in M(y)$ and $|T_{xy}| \leqslant |T_{xq}|$, for some $q \in M(x)$. $\square$

Notice that in the tolerance-free star NeST representation constructed in the second-half of the preceding proof, no neighborhood subtree properly contains another (since neighborhood subtree centers are distance $2m$ apart while neighborhood subtree radii have lower and upper bounds of m and $2m$). Thus the proof of Theorem 4 implies the following strengthening of Lemma 2:

Theorem 5. *A graph G is a fixed distance NeST graph if and only if it has a proper star NeST representation* (*and so G is a proper fixed distance NeST graph*). $\square$

Monma et al. asked whether threshold tolerance graphs can be characterized as intersection graphs of subtrees in a tree [17]. In response, we show that threshold tolerance graphs are exactly fixed distance NeST graphs, using a theorem of Saks (Theorem 2.5 in [17]) which can be restated as follows:

Theorem 6. *A graph $G=(V,E)$ is a threshold tolerance graph if and only if there exists a total order $>$ of V such that*

$$xy \in E \quad \Leftrightarrow \quad \begin{cases} x > p & \text{for all } p \in M(y) \\ & \text{or} \\ y > q & \text{for all } q \in M(x). \end{cases}$$

Theorem 7. *G is a fixed distance NeST graph if and only if G is a threshold tolerance graph.*

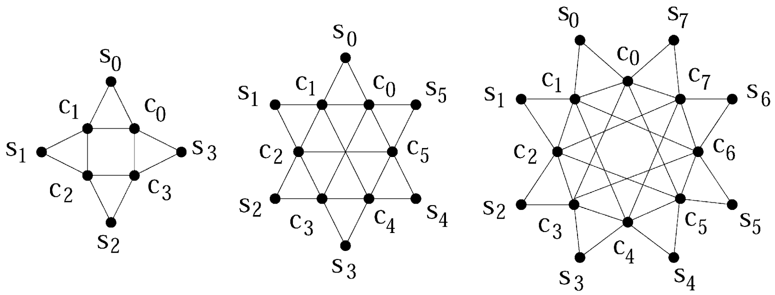

Fig. 4. The 2-star, 3-star and 4-star: graphs which are weakly chordal but not NeST.

Proof. By Saks' theorem, it follows that G is threshold tolerance if and only if G admits a radius-only representation. Now use Theorem 4. □

In [17] a polynomial time recognition algorithm is presented for threshold tolerance graphs. This with Theorem 7 implies polynomial recognition of fixed distance NeST graphs.

5. NeST graphs and weakly chordal graphs

Bibelnieks and Dearing [2] showed that NeST graphs are weakly chordal and asked whether this containment is strict. (Recall that a graph is weakly chordal if neither the graph nor its complement contains an induced cycle with five or more vertices.) We answer this question by exhibiting a class of weakly chordal graphs which are not NeST.

Definition 7. For $m \geqslant 2$, an *m-star* is a tripartite graph G with vertex set $C_{\text{odd}} \cup C_{\text{even}} \cup S$ such that

- $S = \{s_0, \ldots, s_{2m-1}\}$, $C_{\text{odd}} = \{c_1, c_3, \ldots, c_{2m-1}\}$ and $C_{\text{even}} = \{c_0, c_2, \ldots, c_{2m-2}\}$,
- s_i is adjacent to c_i and $c_{i+1 \bmod 2m}$ and
- $C_{\text{odd}} \cup C_{\text{even}}$ induces a complete bipartite graph with bipartition $(C_{\text{odd}}, C_{\text{even}})$.

The 2-star, 3-star and 4-star appear in Fig. 4.

Lemma 3. *If $G = (V, E)$ is a NeST graph, but not a proper NeST graph, then there exist $x, y \in V$ such that $xy \notin E$ and $N(x) \subseteq N(y)$.*

Proof. Let $G = (V, E)$ be a NeST graph but not a proper NeST graph and let (T, S) be a tolerance-free NeST representation of G. For each $x \in V$, define $P(x) = \{z \in V : T_x \subset T_z\}$. Since (T, S) is not a proper NeST representation there is some $w \in V$ such that $P(w) \neq \emptyset$.

Suppose for all x such that $P(x) \neq \emptyset$, $P(x) \subseteq N(x)$. A new tolerance-free NeST representation (T', S') for G is derived as follows:

- $T' = T \cup L$ where L is a line segment of length $\max\{2r_v : v \in V\}$ attached to T at c_w
- $S' = S$, except c'_w is located on L such that $d(c_w, c'_w) = \delta$ where $\delta > \max\{r_z - d(c_w, c_z) : z \in P(w)\}$ and $r'_w = r_w + \delta$.

If $v \notin P(w)$ then $|T'_{vw}| = |T_{vw}|$. If $v \in P(w)$ then $|T'_{vw}| > |T_{vw}|$ but $vw \in E$. Thus

$$xy \in E \quad \Leftrightarrow \quad |T_{xy}| > \min\{|T_{\overset{\bullet}{xx}}|, |T_{\overset{\bullet}{yy}}|\}$$

and so (T', S') is a tolerance-free NeST representation of G in which $P(w) = \emptyset$. This can be repeated for all x contradicting the assumption that G is not a proper NeST graph. Thus there exist $x, y \in V$ such that $y \in P(x)$ and $xy \notin E$. If $z \in N(x)$ then $|T_{\overset{\bullet}{xx}}| = |T_{xy}| = |T_x| \geqslant |T_{xz}|$ which implies $|T_{xz}| > |T_{\overset{\bullet}{zz}}|$. Since $T_x \subset T_y$, $|T_{yz}| \geqslant |T_{xz}| > |T_{\overset{\bullet}{zz}}|$, hence, $z \in N(y)$ and so $N(x) \subseteq N(y)$. $\square$

The above lemma also follows from standard tolerance graph arguments (e.g. as in [18]) and the fact that proper NeST graphs are exactly bounded NeST graphs [2].

Since m-stars do not satisfy the conclusion of Lemma 3, either m-stars are proper NeST graphs or they are not NeST graphs. We now use results on proper NeST graphs from Section 3 to show m-stars cannot have proper phylogenies and so are not proper NeST graphs.

Definition 8. A graph isomorphic to the graph with vertex set $\{1,2,3,4\}$ and edge set $\{12, 34\}$ is called a $2K_2$. We use $(12, 34)$ to denote a $2K_2$ with edges 12 and 34.

Lemma 4. *If $G = (V, E)$ is a proper NeST graph with proper phylogeny (T, V) and (ab, cd) is a $2K_2$ in G then $P(a,b) \cap P(c,d) = \emptyset$ in (T, V).*

Proof. Since (T, S) is a proper phylogeny for G, $ab \in E$ implies $d(a,b) < d(a,c), d(a,d)$ or $d(a,b) < d(b,c), d(b,d)$. Without loss of generality, assume the former. Similarly, $cd \in E$ implies $d(c,d) < d(a,c)$, $d(b,c)$ or $d(c,d) < d(a,d), d(b,d)$. In the former case $P(a,b) \cap P(c,d) = \emptyset$ since a is closer to b than d in (T, V) but c is closer to d than b. In the latter case the same result holds since a is closer to b than to c but d is closer to c than to b. $\square$

Lemma 4 indicates that the $2K_2$-structure of a proper NeST graph imposes constraints upon any proper phylogeny for that graph. This leads to the following result:

Lemma 5. *No m-star has a proper phylogeny.*

Proof. Let $G = (V, E)$ be an m-star with vertices labelled as in Definition 7. Observe that $(c_i s_i, c_j s_j)$, $(c_i s_i, c_j s_{j-1})$, $(c_i s_{i-1}, c_j s_j)$ and $(c_i s_{i-1}, c_j s_{j-1})$ are each $2K_2$'s in G whenever i and j have the same parity. By Lemma 4, $P(c_i, s_i) \cap P(c_j, s_j)$, $P(c_i, s_i) \cap P(c_j, s_{j-1})$, $P(c_i, s_{i-1}) \cap P(c_j, s_j)$ and $P(c_i, s_{i-1}) \cap P(c_j, s_{j-1})$ are all empty, and so, $P(s_i, s_{i-1}) \cap P(s_j, s_{j-1}) = \emptyset$. In particular, we have the following chain of disjoint paths: $P(s_0, s_1) \cap P(s_2, s_3) = \emptyset$, $P(s_1, s_2) \cap P(s_3, s_4) = \emptyset$, $P(s_2, s_3) \cap P(s_4, s_5) = \emptyset, \ldots,$ $P(s_{2m-4}, s_{2m-3}) \cap P(s_{2m-2}, s_{2m-1}) = \emptyset$, $P(s_{2m-3}, s_{2m-2}) \cap P(s_{2m-1}, s_0) = \emptyset$. The lemma follows since no phylogeny is consistent with such a chain. □

It remains to show that m-stars are weakly chordal. Let G be an m-star with vertices labelled as in Definition 7. Now observe that $N(s_i)$ is a clique (so no s_i is in a cycle or the complement of a cycle with five or more vertices) and that the remaining vertices induce a complete bipartite graph (which induces no cycle with four or more vertices in the graph or in its complement). Thus we have shown the following:

Lemma 6. *NeST graphs form a proper subclass of weakly chordal graphs.* □

6. Fixed tolerance fixed distance NeST graphs and threshold graphs

A *fixed tolerance NeST graph* is a graph with a NeST representation in which all tolerances are the same. The ideas used in the previous section to construct a family of graphs that are weakly chordal but not NeST can be used to construct a family of graphs which are strongly chordal but not fixed tolerance NeST [16]. This extends a result of Bibelnieks and Dearing [2] who gave one separating example.

Chvátal and Hammer [5] introduced *threshold graphs* as those graphs G for which there is a set of positive weights $\{w_v \mid v \in V\}$ and a positive threshold t, such that $xy \in E$ if and only if $w_x + w_y \geqslant t$. Observe that fixed-tolerance threshold tolerance graphs are exactly threshold graphs. Threshold graphs can be characterized as a certain kind of NeST graph, as we now show.

Theorem 8. *A graph is a threshold graph if and only if it has a NeST representation which is both fixed tolerance and fixed distance.*

Proof. ($\Rightarrow$) Let G be a threshold graph with positive values $S = \{w_v \mid v \in V\} \cup \{t\}$ so that

$$xy \in E(G) \;\Leftrightarrow\; w_x + w_y \geqslant t.$$

Let $r_v = w_v$ for each v, let ε be the minimum value in S, let $\tau = \varepsilon/2$, let $c = (t - \tau)/2$, and let G' be the graph with this star NeST representation (the tolerance is τ, and each subtree T_v has radius r_v and is distance c from the star center). Then τ, c, r_v are all positive, $t = 2c + \tau$, and by the construction of G'

$$xy \in E(G') \;\Leftrightarrow\; |T_{xy}| \geqslant \tau.$$

Furthermore,

$$xy \in E(G') \;\Leftrightarrow\; r_x + r_y - 2c \geqslant \tau,$$

since if $T_x \nsubseteq T_y$ and $T_y \nsubseteq T_x$ then

$$r_x + r_y - 2c = |T_{xy}|,$$

whereas if $T_x \subseteq T_y$ or $T_y \subseteq T_x$ then $xy \in E(G')$ and

$$r_x + r_y - 2c \geqslant |T_{xy}| = \min\{|T_x|, |T_y|\} \geqslant \varepsilon > \tau.$$

Thus,

$$xy \in E(G') \quad \Leftrightarrow \quad xy \in E(G)$$

and so G has a fixed tolerance fixed distance NeST representation.

($\Leftarrow$) Let G be a graph with a fixed tolerance, fixed distance NeST representation. Since the proof of Lemma 2 does not alter tolerances, G has a fixed tolerance star NeST representation. If this representation is also proper, then the argument used in the previous case can be reversed, and the proof is finished. It is not difficult to show that we can make this assumption (namely that any graph with a fixed tolerance, fixed distance NeST representation has a fixed tolerance star NeST representation which is also proper), but it is easier to finish the proof without making this assumption.

Thus consider any (not necessarily proper) fixed tolerance star NeST representation of G. Let T_a, T_z be subtrees with smallest and largest radii, respectively. If a and z are adjacent, then z is adjacent to every vertex in $G - z$ (since $|T_{az}| \geqslant |T_{vz}|$ for all v in $G - z$), whereas if a and z are non-adjacent, then a is adjacent to no vertex in G (since $|T_{av}| \leqslant |T_{az}|$ for all v in $G - a$). This argument also holds if G is replaced with any vertex induced subgraph. Thus, every vertex induced subgraph of G contains either a universal or isolated vertex, and so, by a theorem due to Chvátal and Hammer [5], G is a threshold graph. $\square$

7. Conclusions and open problems

We have established several results on NeST graphs, including

- the equivalence of proper NeST graphs and unit NeST graphs,
- the equivalence of fixed distance NeST graphs and threshold tolerance graphs,
- the proper inclusion of NeST graphs in weakly chordal graphs,
- the inclusion of fixed tolerance and fixed distance NeST graphs in proper NeST graphs,
- the equivalence of threshold graphs and fixed tolerance, fixed distance NeST graphs.

The second result answers a question of Monma et al. [17]; the third answers a question of Bibelnieks and Dearing [2]. Three open problems are

- to determine whether every NeST graph is a proper NeST graph,
- to establish purely graph theoretic (that is, tolerance-free) characterizations for NeST graphs, and
- to determine the recognition complexity for NeST graphs.

A related open problem is to determine the recognition complexity for tolerance graphs. This might be easier than the above problem for NeST graphs, since the underlying geometry of a NeST representation (an embedded tree) is not unique, whereas the underlying geometry of a tolerance representation (a line) is.

In closing, we point out that many papers have appeared recently on topics concerning tolerance graphs. One paper which addresses problems similar to those discussed in this paper is [15]. For a comprehensive survey on tolerance graphs, see [13].

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