Meyniel Weakly Triangulated Graphs I: Co-perfect orderability*

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Abstract. We show that Meyniel weakly triangulated graphs are co-perfectly orderable (equivalently, that P_5-free weakly triangulated graphs are perfectly orderable). Our proof is algorithmic, and relies on a notion concerning separating sets, a property of weakly triangulated graphs, and several properties of Meyniel weakly triangulated graphs.

Key Words. P_5-free graph, weakly triangulated graph, perfectly orderable graph, Meyniel graph

1. Introduction. Graph coloring is in general a hard problem. As such, it is interesting to investigate restrictions under which a graph's chromatic number might be easily determined.

One such investigation was initiated in 1984 by Chvátal [C1], who proposed to study those graphs on which a certain efficient coloring algorithm always returned an optimal coloring. The algorithm is this: given a linear order of the vertices of a graph, color the vertices in order, assigning to each the smallest positive integer (each integer representing a color) not assigned to any adjacent vertex already colored. A linear order of vertices of a graph is called a *perfect order* if for each induced ordered subgraph the number of colors used by the above algorithm equals the chromatic number of the subgraph; a graph is called *perfectly orderable* if it has a perfect order.

Having introduced a new class of graphs, it is natural to ask how the class relates to other previously studied classes of graphs. In [C1] Chvátal pointed out that

a linear order $<$ *is perfect if and only if there is*
no chordless path (a,b,c,d) *with* a $<$ b *and* d $<$ c

and that as a consequence *comparability graphs* (graphs whose edge sets can be transitively oriented), *triangulated graphs* (graphs with no chordless cycle of size four or more), and *co-triangulated graphs* (complements of triangulated graphs) are all perfectly orderable. While many other particular classes of graphs have since been shown to be perfectly orderable, Middendorf and Pfeiffer have shown that recognizing perfectly orderable graphs in general is *NP*-complete [MP].

In 1984, as a generalization of both triangulated graphs and co-triangulated graphs, we introduced the class of *weakly triangulated graphs* [H1], defined as those graphs with no chordless cycles of size five or more in the graph or its complement.

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Not all weakly triangulated graphs are perfectly orderable; in fact, Hoàng has shown that recognizing perfectly orderable graphs remains *NP*-complete for weakly triangulated inputs [Ho].

In 1990 Chvátal conjectured [C2] that

every P_5-free weakly triangulated graph is perfectly orderable,

where P_k is the chordless path with k vertices. This conjecture is known to hold for various subclasses of P_5-free weakly triangulated graphs, namely

- P_4-free graphs,
- co-triangulated graphs [C1],
- bull-free P_5-free weakly triangulated graphs [C2], where the *bull* is any copy of the graph with vertex set {1,2,3,4,5} and edge set {{1,2}, {2,3}, {3,4}, {2,5}, {3,5}},
- $\overline{D}_6$-free P_5-free weakly triangulated graphs [HK], where $\overline{D}_6$ is the complement of the *domino*, the domino being any copy of the graph with vertex set {1,2,3,4,5,6} and edge set {{1,2},{2,3},{3,4},{4,5},{5,6},{1,6},{1,4}}, and
- $\overline{P}_6$-free P_5-free weakly triangulated graphs [HMOP].

In this paper we prove the conjecture in its full generality by presenting a polynomial time algorithm to find a perfect order of any P_5-free weakly triangulated graph. The correctness of our algorithm relies on a result on separating sets, a result on weakly triangulated graphs, and several results on P_5-free weakly triangulated graphs.

Incidentally, a weakly triangulated graph is P_5-free if and only if it is the complement of a *Meyniel* graph, where a Meyniel graph is one in which every odd cycle with five or more vertices has at least two chords [M]. In other words, a Meyniel weakly triangulated graph is a $\overline{P}_5$-free weakly triangulated graph. A graph whose complement is perfectly orderable is called *co-perfectly orderable*, so the above conjecture can be restated as in our title, namely

every Meyniel weakly triangulated graph is co-perfectly orderable.

It is convenient to restate the conjecture in this way, since our main structural result is more naturally discussed in terms of $\overline{P}_5$-free weakly triangulated graphs (namely, Meyniel weakly triangulated graphs) than in terms of P_5-free weakly triangulated graphs.

2. Separating sets and handles. The key notion of this section is that of a handle. Before presenting its definition, we first present some basic terminology and notation. The terms *sees* and *misses* mean "is adjacent to" and "is not adjacent to" respectively. Given a vertex subset S of a graph G, the *neighborhood* $N(S)$ is the set of vertices of $G - S$ that see at least one vertex of S; whenever S consists of a single vertex v, we write $N(v)$ instead of $N(\{v\})$.

A *handle* of a graph G is a proper vertex subset H with size at least two, such that

- $G[H]$ is connected,
- some component J other than H of $G - N(H)$ satisfies $N(J) = N(H)$, and
- every vertex of $N(H)$ sees at least one vertex of every edge of $G[H]$.

Handles will play a key role in our study of P_5-free weakly triangulated graphs; for now, we establish some results on handles in arbitrary graphs.

A vertex subset S of a graph G *separates* vertex subsets X and Y if every path from a vertex in X to a vertex in Y contains some vertex in S, and *minimally separates* X and Y if no proper subset of S separates X and Y.

THEOREM 2.1. *Let* S, X, *and* Y *be vertex subsets of a graph* G *such that* X *and* Y *are components of* G−S, S *minimally separates* X *and* Y, *and* X *has size at least two. Then some subset of* X *is a handle of* G.

Proof. Let G, S, X, and Y be as stated in the theorem. $N(X) = S = N(Y)$, since X and Y are components of $G - S$ minimally separated by S. If every vertex of S sees at least one vertex of every edge of $G[X]$, then X is a handle and the theorem holds. Suppose then that some vertex s in S misses both vertices of some edge $\{a, b\}$ of $G[X]$. Let Y_1 be the component of $G - N(\{a, b\})$ containing s, let S_1 be $N(Y_1)$, and let X_1 be the component of $G - S_1$ containing $\{a, b\}$. Observe that

$$Y \cup S - N(\{a, b\}) \subseteq Y_1$$

since s sees some vertex of Y, $G[Y]$ is connected, and every vertex in $S - N(\{a, b\})$ sees some vertex of Y. Also

$$S \cap N(\{a, b\}) \subseteq S_1$$

since Y_1 contains no vertex of $S \cap N(\{a, b\})$, and every vertex of S and so of $S \cap N(\{a, b\})$ sees some vertex of Y and so of Y_1. Now

$$X_1 \subseteq X$$

since every vertex of S is either in Y_1 or S_1, and furthermore

$$X_1 \subset X$$

since $X_s = X \cap N(s)$ is non-empty, and every vertex of $X_s \cap N(\{a, b\})$ is in in S_1 while every vertex of $X_s - N(\{a, b\})$ is in in Y_1. Finally observe that

$$N(X_1) = S_1$$

since X_1 is a component of $G - S_1$ and so $N(X_1) \subseteq S_1$, and since S_1 is a subset of $N(\{a, b\})$ and $\{a, b\}$ is a subset of X_1 and so $S_1 \subseteq N(X_1)$.

Now the conditions of the theorem hold with S_1, X_1 and Y_1 in place of S, X and Y respectively. If every vertex of S_1 sees at least one vertex of every edge of $G[X_1]$, then X_1 is a handle and the theorem holds. Otherwise, iterating the above process eventually produces a handle X_t which is a proper subset of X_{t-1} and so a proper subset of X. □

THEOREM 2.2. *A graph has a handle if and only if it contains* $\overline{\mathrm{P}}_3$.

Proof. Let G be a graph with a handle H. Then $G[H]$ contains some edge (h_1, h_2) and $G - (H \cup N(H))$ contains some vertex j and (h_1, j, h_2) is a $\overline{P}_3$ of G.

On the other hand, let G be a graph and let (p_1, p_2, p_3) be a $\overline{P}_3$ of G. Let X' be the edge (p_1, p_3), let S' be $N(X')$, let Y be the component of $G - S'$ containing p_2, let S be $N(Y)$, and let X be the component of $G - S$ containing X'. Then S minimally separates components X and Y of $G - S$, and X has at least two vertices. By the previous theorem, some subset of X is a handle of G. □

3. Weakly triangulated graphs. In this section we establish results on weakly triangulated graphs that will be useful later.

With respect to a set S of vertices in a graph G, a vertex x of $G - S$ is said to be *S-universal*, *S-partial*, or *S-null* if it sees respectively all, some but not all, or none of the vertices of S; a set X of $G - S$ is *S-universal* or *S-null* if every vertex in X is respectively S-universal or S-null.

With respect to a set S of vertices in a graph G, an edge e of $G - S$ is said to be *S-saturating* if, for each component S_j of $\overline{G}[S]$, at least one vertex of e is S_j-universal.

THEOREM 3.1. [H1] *Let* S, X, *and* Y *be vertex subsets of a weakly triangulated graph* G *such that* X *and* Y *are components of* G−S, S *minimally separates* X *and* Y, *and* $\overline{\mathrm{G}}$[S] *is connected. Then* X *and* Y *each contain some* S*-universal vertex.* □

THEOREM 3.2. *Let* S, X, *and* Y *be vertex subsets of a weakly triangulated graph* G *such that* X *and* Y *are components of* G−S, S *minimally separates* X *and* Y, *and* $\overline{\mathrm{G}}$[S] *is disconnected. Then* X *and* Y *each contain either some S-universal vertex or some* S*-saturating edge.*

Proof. Let G, S, X, and Y be as stated in the theorem. Argue by induction on $|X|$. Suppose that $|X| \leq 2$. For each component S_j of $\overline{G}[S]$, by considering $G - (S - S_j)$ it follows from Theorem 3.1 that some vertex of X is S_j-universal; now Theorem 3.2 holds since $G[X]$ is either a single vertex or a single edge. Suppose then that $|X| \geq 3$. Let $p = (x_1, x_2, \ldots, x_t)$ be a longest chordless path in $G[X]$.

Case 1. Each vertex of S is $(X_1 = X - x_1)$-partial or X_1-universal.
Observe that in $G' = G - x_1$, S minimally separates components X_1 and Y of $G' - S$. Thus by inductive assumption, X_1 contains either a S-universal vertex or a S-saturating edge; since X_1 is a proper subset of X, the theorem holds.

Case 2. Each vertex of S is $(X_t = X - x_t)$-partial or X_t-universal.
Argue as in the previous case, with x_t and X_t replacing x_1 and X_1 respectively.

Case 3. Some s_1 in S is X_1-null, and some s_t in S is X_t-null.
Since S minimally separates X and Y, each vertex of S sees some vertex of X. Thus $N(s_j) \cap X = \{x_j\}$, for $j = 1$ and t. Since $G[X]$ is connected and $|X| \geq 3$, $t \geq 2$, so $x_1 \neq x_t$, and $s_1 \neq s_t$. Also

$$s_1 \text{ sees } s_t.$$

(Suppose not. Then in the graph $G' = G - (S - S')$, $S' = \{s_1, s_t\}$ minimally separates components X and Y of $G' - S'$, $\overline{G'}[S']$ is connected, and X contains no S'-universal vertex, contradicting Theorem 3.1.) Thus $(s_1, x_1, \ldots, x_t, s_t)$ is a chordless cycle with $t + 2$ vertices, and G being weakly triangulated implies that $t = 2$, and $G[X]$ is a clique. Now observe that

$$\text{each vertex of } S \text{ sees at least one vertex of } \{x_1, x_2\}.$$

(If not, some s_3 in S is $\{x_1, x_2\}$-null and sees some x_3 in X. Arguing as above, by considering Theorem 3.1 on $G - (X - \{x_1, x_3\}) - (S - \{s_1, s_3\})$ it follows that s_3 sees s_1. Similarly, s_3 sees s_2. But then $(s_1, x_2, s_3, x_1, s_2, x_3)$ is the complement of a chordless cycle, contradiction.) Finally, let $X' = \{x_1, x_2\}$ and observe that in $G' = G - (X - X')$, S minimally separates components X' and Y of $G' - S$. Inductive hypothesis and neither x_1 nor x_2 being S-universal imply that $\{x_1, x_2\}$ is S-saturating in G', so

$$\{x_1, x_2\} \text{ is } S\text{-saturating in } G$$

and the theorem holds. □

4. Handles in $\overline{P}_5$-free weakly triangulated graphs. Our key theorems for P_5-free weakly triangulated graphs are obtained by focusing on handles in the complements of such graphs. In order to avoid repeatedly referring both to a graph and its complement, we find it convenient to restrict our attention in this section to $\overline{P}_5$-free weakly triangulated graphs. At the end of the section we state as corollaries those results on P_5-free weakly triangulated graphs we shall need later.

A handle H of a graph G will be called *sharp* if some vertex of H is $N(H)$-universal, and *blunt* otherwise.

THEOREM 4.1. *Let* H *be a blunt handle of a* $\overline{\mathrm{P}}_5$*-free weakly triangulated graph* $\mathrm{G}=(\mathrm{V},\mathrm{E})$*, and let* $\mathrm{S}=\mathrm{N}(\mathrm{H})$*. Then there is a partition* $(\mathrm{H}_1,\mathrm{H}_2)$ *of* H *and a partition* $(\mathrm{S}_1,\mathrm{S}_2)$ *or* $(\mathrm{S}_0,\mathrm{S}_1,\mathrm{S}_2)$ *of* S *such that*

1. $\mathrm{G}[\mathrm{H}]$ *is bipartite, with parts* H_1 *and* H_2*,*
2. H_k *is* S_k*-universal and* $\mathrm{S}_{3-\mathrm{k}}$*-null, for* $\mathrm{k}=1,2$*,*
3. S_1 *is* S_2*-universal,*
4. S_0 *is* $(\mathrm{H}\cup\mathrm{S}-\mathrm{S}_0)$*-universal,*
5. *each vertex of* $\mathrm{R}=\mathrm{V}-(\mathrm{H}\cup\mathrm{S})$ *is* S_1*-null or* S_2*-null, and*
6. *each vertex of* R *that is partial on a component of* $\overline{\mathrm{G}}[\mathrm{S}_0]$ *is* $(\mathrm{S}_1\cup\mathrm{S}_2)$*-null.*

Proof. Let G, H, and S be as stated in the theorem, and let J be any component of $G-S$ other than H such that $N(J)=S$. S minimally separates components H and J of $G-S$, so Theorems 3.1 and 3.2 and H being blunt imply that H contains an S-saturating edge $\{h_1^*,h_2^*\}$ and that neither vertex of the edge is S-universal. Define $S_0=S\cap N(h_1^*)\cap N(h_2^*)$, $S_1=S\cap N(h_1^*)-N(h_2^*)$, and $S_2=S\cap N(h_2^*)-N(h_1^*)$. S_0 may be empty, but S_1 and S_2 are not, so (S_1,S_2) or (S_0,S_1,S_2) is a partition of S as stated in the theorem.

For any s_1 in S_1 and s_2 in S_2, neither h_1^* nor h_2^* sees both s_1 and s_2; since $\{h_1^*,h_2^*\}$ is S-saturating, this implies that s_1 and s_2 are in different components of $\overline{G}[S]$, so

$$S_1 \text{ is } S_2\text{-universal.}$$

Thus (3) holds. Also

$$\text{every vertex of } V-(H\cup S) \text{ is } S_1\text{-null or } S_2\text{-null .} \tag{$*$}$$

(Suppose not. Then some r in $V-(H\cup S)$ sees some s_1 in S_1 and some s_2 in S_2, and (s_1,h_2^*,r,h_1^*,s_2) is a $\overline{P}_5$ in G, contradiction.) Thus (5) holds.

Theorems 3.1 and 3.2 with G, S, H and J imply that J contains either an S-universal vertex or an S-saturating edge; $(*)$ rules out the former, so some edge J^* of J is S-saturating. Now $(*)$ implies that one vertex of J^*, say j_1^*, is S_1-universal, and the other, say j_2^*, is S_2-universal. Arguing as for $(*)$, interchanging H with J, and (h_1^*,h_2^*) with (j_1^*,j_2^*), we have

$$\text{every vertex of } H \text{ is } S_1\text{-null or } S_2\text{-null.}$$

For $k=1$ and 2, define H_k as the vertices of H that are S_{3-k}-null. Since H is a handle and H_1 is S_2-null, any vertex in H that sees some h_1 in H_1 must be S_2-universal, and so in H_2. Similarly, any vertex in H that sees some h_2 in H_2 must be S_1-universal, and so in H_1. Since $G[H]$ is connected, it follows that $G[H]$ is bipartite, with parts H_1 and H_2 universal on S_1 and S_2 respectively. Thus (1) and (2) hold, and it remains only to show (4) and (6).

Suppose that S_0 is non-empty. Observe that

$$S_0 \text{ is } (S - S_0)\text{-universal.}$$

(Suppose not. Then some s_0 in S_0 misses some s' in $S_1 \cup S_2$. By relabelling if necessary, we may assume that s' is in S_1. Let S' be the component of $\overline{G}[S]$ containing $\{s', s_0\}$. By Theorem 3.1 on $G[H \cup S' \cup J]$, some vertex j in J sees s_0 and s'. Thus $(s_0, s', h_2^*, j, h_1^*)$ is a $\overline{P}_5$, contradiction.) Also,

$$S_0 \text{ is } (J^* = \{j_1^*, j_2^*\})\text{-universal.}$$

(If any s_0 in S_0 is J^*-null, then $(s_1, j_2^*, s_0, j_1^*, s_2)$ is a $\overline{P}_5$ for any s_1 in S_1 and s_2 in S_2, contradiction. If any s_0 in S_0 misses exactly one of J^*, by relabelling if necessary say j_1^*, then $(s_0, j_1^*, h_1^*, j_2^*, s_1)$ is a $\overline{P}_5$ for any s_1 in S_1, contradiction.) This implies that

$$S_0 \text{ is } H\text{-universal.}$$

(If not, then since H is a handle, there is some s_0 in S_0 and some edge (h_u, h_v) of $G[H]$, such that s_0 sees h_u but not h_v. By relabelling h_u and h_v if necessary, we may assume that h_u is in H_1 and h_v is in H_2. Then $(s_0, h_v, j_2^*, h_u, s_2)$ is a $\overline{P}_5$ for any s_2 in S_2, contradiction.) Thus (4) holds. Finally, we claim that

$$\text{each vertex of } V - (H \cup S) \text{ partial on a component of } \overline{G}[S_0] \text{ is } (S_1 \cup S_2)\text{-null.}$$

(Suppose not. Then some r in $V - (H \cup S)$ is partial on some component of $\overline{G}[S_0]$, and also sees some vertex in $S_1 \cup S_2$. By relabelling if necessary, we may assume that r sees some s_1 in S_1. The partiality of r implies the existence of a pair of non-adjacent vertices s_0 and s_0' in S_0, such that r sees the former and not the latter. But now $(s_0, s_0', r, h_2^*, s_1)$ is a $\overline{P}_5$, contradiction.) Thus (6) holds. □

THEOREM 4.2. *Let* H *be a sharp handle in a* $\overline{\mathrm{P}}_5$*-free weakly triangulated graph* G *and let* S=N(H). *Let* H_1 *be those vertices of* H *that are* S*-universal, and* H_0 *those that are not. Then*

1. $\mathrm{G}[\mathrm{H}_0]$ *is an independent set, and*
2. *for each component* S′ *of* $\overline{\mathrm{G}}$[S], *every vertex of* H_0 *is* S′*-null or* S′*-universal.*

Proof. Let G, H, S, H_1 and H_0 be as stated in the lemma. H_1 is non-empty since H is sharp. Let J be any component of $G - S$ other than H such that $N(J) = S$.

Now some vertex j^* of J is S-universal. (Suppose not. Then Theorems 3.1 and 3.2 on $G[H \cup S \cup J]$ imply that some edge $\{j_1, j_2\}$ of $G[J]$ is S-saturating and that j_1 and j_2 miss some s_2 and s_1 in S respectively, and $(s_1, j_2, h_1, j_1, s_2)$ is the complement of a chordless cycle or path for every h_1 in H_1, contradiction.)

Now (1) holds. (Suppose not. Let h_0 and h_0' be adjacent vertices of H_0. Since h_0 is not S-universal, h_0 misses some s' in S; since H is a handle, s' sees h_0'. Similarly, h_0' misses some s in S which sees h_0. But now (s, h_0', j^*, h_0, s') is the complement of a chordless cycle or path, contradiction.)

Finally, (2) holds. (Suppose not. Then some h_0 in H_0 is partial on some component of $\overline{G}[S]$. Thus there are non-adjacent vertices s and s' in S such that h_0 sees s but not s'. H_1 is non-empty, $G[H]$ is connected, and $G[H_0]$ has no edges, so h_0 has a neighbor h_1 in H. Now (s, s', h_0, j^*, h_1) is a $\overline{P}_5$, contradiction.) □

We close this section by stating results for P_5-free weakly triangulated graphs which correspond to the previous theorems. We first need a bit more notation. Given a subset S of the vertices of a graph G, the *non-neighborhood of S*, denoted $M(S)$, is the set of vertices of $G - S$ that miss at least one vertex of S. An *anti-handle* of a graph is a vertex subset that is a handle in the complement; an anti-handle H is *sharp* if some vertex of H is $M(H)$-null, and *blunt* otherwise. Observe that H is a sharp (respectively blunt) handle of G if and only if H is a sharp (respectively blunt) anti-handle of $\overline{G}$.

The following corollaries follow immediately from the preceding definitions and Theorems 4.1 and 4.2.

COROLLARY 4.3. *Let* H *be a blunt anti-handle of a* P_5*-free weakly triangulated graph* G=(V,E), *and let* S=M(H). *Then there is a partition* $(\mathrm{H}_1,\mathrm{H}_2)$ *of* H *and a partition* $(\mathrm{S}_1,\mathrm{S}_2)$ *or* $(\mathrm{S}_0,\mathrm{S}_1,\mathrm{S}_2)$ *of* S *such that*

1. $\overline{\mathrm{G}}[\mathrm{H}]$ *is bipartite, with parts* H_1 *and* H_2,
2. H_k *is* S_k*-null and* $\mathrm{S}_{3-\mathrm{k}}$*-universal, for* $\mathrm{k}= 1, 2$,
3. S_1 *is* S_2*-null,*
4. S_0 *is* $(\mathrm{H} \cup \mathrm{S}-\mathrm{S}_0)$*-null,*
5. *each vertex of* R=V$-$(H $\cup$ S) *is* S_1*-universal or* S_2*-universal, and*
6. *each vertex of* R *that is partial on a component of* $\mathrm{G}[\mathrm{S}_0]$ *is* $(\mathrm{S}_1 \cup \mathrm{S}_2)$*-universal.*

□

COROLLARY 4.4. *Let* H *be a sharp anti-handle in a* P_5*-free weakly triangulated graph* G *and let* S=M(H). *Let* H_1 *be those vertices of* H *that are* S*-null, and* H_0 *those that are not. Then*

1. $\mathrm{G}[\mathrm{H}_0]$ *is a clique, and*
2. *for each component* S$'$ *of* G[S], *every vertex of* H_0 *is* S$'$*-universal or* S$'$*-null.*

□

5. The algorithm. Having established sufficient structural properties, we now show that P_5-free weakly triangulated graphs are perfectly orderable by presenting a polynomial time ordering algorithm. In the algorithm and its proof we refer to the relative position of a vertex in an order as its *rank*.

Algorithm Perfect Order

Input: A P_5-free weakly triangulated graph $G = (V, E)$.

Output: A perfect order $<_G$ of G.

1. Search for an anti-handle H in G. If none is found, let $<_G$ be any permutation of V, and stop. If H is blunt, define H_1, H_2, S_1, S_2, h_1^* and h_2^* as in Corollary 4.3, and go to step 2; if H is sharp, define H_1 and H_0 as in Corollary 4.4, and go to step 3.
2. Define $V' = V - (H - \{h_1^*, h_2^*\})$, $E' = E[V'] \cup \{(h_1^*, h_2^*)\}$, and $G' = (V', E')$. Recursively obtain perfect orders $<_{G'}$ and $<_H$ of G' and $G[H]$ respectively. To obtain $<_G$, start with $<_{G'}$ and
   - for $k = 1$ and then 2, decrement the rank of h_k^* by the least amount necessary so that $h_k^* < s$ for every s in S_{3-k},
   - for $k = 1$ and 2, place each vertex of H_k at the rank of h_k^*, with the relative order of vertices of H_k determined by $<_H$.
3. Pick any vertex h_1^* in H_1, and define $V' = V - (H - h_1^*)$. Recursively obtain perfect orders $<_{V'}$ and $<_H$ of $G[V']$ and $G[H]$ respectively. To obtain $<_G$, start with $<_{V'}$ and
   - add vertices of $H_1 - h_1^*$ to the order at the rank of h_1^*, with the relative order of vertices of H_1 determined by $<_H$,
   - by increasing rank according to $<_H$ add each vertex h_0 of H_0 to the order, assigning h_0 the largest rank less than the ranks of all vertices of $H_1 \cup (N(h_0) \cap S))$.

THEOREM 5.1. *Algorithm Perfect Order is correct.*

LEMMA 5.2. *Let* G=(V,E) *be a* P_5*-free weakly triangulated graph, let* H *be a blunt anti-handle of* G*, and let* H_1, H_2, S_1, S_2, h_1^* *and* h_2^* *be as defined in Corollary 4.3.*

1. *Let* $|H| = 2$*. Then the graph obtained by adding the edge* (h_1^*, h_2^*) *to* G *is* P_5*-free and weakly triangulated.*
2. *Let* h_k *be in* H_k*, let* $\{a,b,c\} \subseteq V - H$*, and let* (a,b,c,h_k) *be a* P_4 *of* G*. Then* c *is in* $V-(H \cup S)$*, and* (a,b,c,s_{3-k}) *is a* P_4 *of* G*, for all* s_{3-k} *in* S_{3-k}*.*
3. *Let* P *be a* P_4 *of* G *including at least two vertices of* H*. Then* P *has the form*

(s_k, h_{3-k}, x, h_k), $(s_k, h_{3-k}, h_k, s_{3-k})$, $(h_k, h'_k, h_{3-k}, s_k)$, *or* $(h_k, h'_k, h_{3-k}, h'_{3-k})$, *where* $k=1$ *or* 2, $\{h_j, h'_j\}$ *are in* H_j, s_j *is in* S_j, *and* x *is in* $V-(H\cup S)$.

Proof. The results follow directly from Corollary 4.3. □

LEMMA 5.3. *Let* G *be a* P_5*-free weakly triangulated graph, let* H *be a sharp anti-handle of* G*, let* H_1, H_0*, and* S *be as defined in Corollary 4.4, and let* P *be any* P_4 *of* G*. Then* $|P \cap H_0| \leq 2$*, and either*

- $|P\cap H_0| = 0$*, and* $|P \cap H_1| \leq 1$ *or* $|P\cap H_1| = 4$,
- $|P \cap H_0| = 1$*, and* P *has the form* (h_1, h_0, h'_1, h''_1), (s, h_0, r, s'), (s, h_0, h_1, h'_1), (s, h_0, r, h_1), (h_0, r, s, s')*, or* (r, h_0, r', s),
- $|P \cap H_0| = 2$*, and* P *has the form* (s, h_0, h'_0, h_1), (s, h_0, h'_0, s'), (h_1, h_0, h'_0, h'_1)*, or* (h_0, h'_0, h_1, h'_1),

where $\{h_0, h'_0\}$ *are in* H_0, $\{h_1, h'_1, h''_1\}$ *are in* H_1, $\{s, s'\}$ *are in* S*, and* $\{r, r'\}$ *are in* $V-(H\cup S)$.

Proof. The results follow directly from Corollary 4.4, except for the following observation: G does not contain any chordless path $P = (h_0, h, h', h'')$ with h_0 in H_0 and $\{h, h', h''\}$ all in H_1, for otherwise P and any neighbor of h_0 in S forms a P_5. □

Proof of theorem. It is routine to verify that the order $<_G$ returned by the algorithm is well-defined; we need only show that $<_G$ is perfect. With respect to a given linear order $<$, call a P_4 (a, b, c, d) *bad* if $a < b$ and $d < c$ and *good* otherwise. To show $<_G$ is perfect we need only show that

$$\text{every } P_4 \text{ of } G \text{ is good under } <_G. \qquad (*)$$

Consider step 1: G has no anti-handle. Theorem 2.2 implies that G is P_3-free, and so P_4-free and $(*)$ holds trivially.

Next consider step 2: G has a blunt anti-handle H. Since G is P_5-free and weakly triangulated, so is $G[V']$, and therefore by Lemma 5.2 so is G', since G' is obtained from $G[V']$ by adding the edge (h_1^*, h_2^*). We may assume that $<_{G'}$ and $<_H$ are perfect orders of G' and $G[H]$ respectively; from this we wish to show $(*)$. Let P be a P_4 of G. There are four cases to consider.

Case: $|P \cap H| = 0$.
Here P is contained in $G - H$, so the relative order of P under $<_G$ is the same as under $<_{G'}$, so $(*)$ holds for P.

Case: $|P \cap H| = 1$.
By relabelling if necessary, we may assume that P contains a vertex of H_1. Replacing this vertex with h_1^* yields another P_4 P^* with the same relative order under $<_G$ as that of P; thus it suffices to show that $(*)$ holds for P^*.

If the relative order of the end edges of P^* is the same under $<_G$ as under $<_{G'}$ then P^* is a P_4 in G' and good under $<_{G'}$. Suppose then that the relative order of an end edge of P^* is different under $<_G$ from under $<_{G'}$. Let $P^* = (a, b, c, d)$; by relabelling if necessary we may assume that c and d are vertices of P^* whose relative order differs in the two orders. If $c <_G d$ then $(*)$ holds for P^*, so assume that $d <_G c$, and so $c <_{G'} d$. In going from $<_{G'}$ to $<_G$, the only vertices whose relative rank is decremented are vertices of H. It follows that d is in H, namely $d = h_1^*$. Thus $P^* = (a, b, c, h_1^*)$, where none of a, b, or c is in H. By Lemma 5.2, c is in $V - (H \cup S)$ and (a, b, c, s_2) is a P_4, for all s_2 in S_2.

Thus c is not in S, $h_1^* <_G c$, and $c <_{G'} h_1^*$; so h_1^* was decremented below c in $<_G$ because of some s_2 in S_2 such that $s_2 <_{G'} c$. Since the relative order of vertices of $G - H$ is the same under $<_G$ as under $<_{G'}$, thus $s_2 <_G c$. Since (a, b, c, s_2) is a P_4 in G' as well as G, it is good under $<_{G'}$, so $b <_{G'} a$. Again, the relative order of vertices of $G - H$ is the same under $<_G$ as under $<_{G'}$, so $b <_G a$, and P^* is good under $<_G$.

Case: $|P \cap H| \geq 2$.
We need consider the four forms for P described in Lemma 5.2. If $P = (s_k, h_{3-k}, x, h_k)$ or $P = (s_k, h_{3-k}, h_k, s_{3-k})$ or $P = (h_k, h'_k, h_{3-k}, s_k)$, then $h_{3-k} <_G s_k$. If $P = (h_k, h'_k, h_{3-k}, h'_{3-k})$, then P is a P_4 in $G[H]$ and good under $<_H$. Thus $(*)$ holds for P in this case and so for step 2.

Now consider step 3: G has a sharp anti-handle H. Again let P be any P_4 of G; we wish to show that $(*)$ holds. Lemma 5.3 describes the possible forms of P; these are considered in the following cases.

Case: $|P \cap H_0| = 0$.
If P contains at most one vertex of H_1 then replacing this vertex with h_1^* yields a P_4 with the same relative vertex order as P which is good under $<_{V'}$ and so under $<_G$. If P contains two or more vertices of H_1 then by Lemma 5.3 P is contained in $G[H_1]$ and is good under $<_H$ and so under $<_G$.

Case: $|P \cap H_0| = 1$.
Subcase: $P = (h_1, h_0, h'_1, h''_1)$. Here $(*)$ holds for P, since $h_0 <_G h_1$.
Subcase: $P = (s, h_0, r, s')$ or (s, h_0, h_1, h'_1) or (s, h_0, r, h_1). Here $h_0 <_G s$.
Subcase: $P = (h_0, r, s, s')$.
If $r <_G h_0$ then $(*)$ holds for P, so assume the contrary. Let x be the smallest vertex under $<_G$ of $V - h_0$ such that $h_0 <_G x$; from the algorithm it follows that x is in H_1 or $S \cap N(h_0)$. Observe that $x <_G r$.

First suppose that x is in $S \cap N(h_0)$. Since x sees h_0, x is different from s and s'. By Theorem 4.2, x misses s and s'. Thus x sees r, for otherwise (x, h_0, r, s, s') is a P_5, contradiction. Thus $P' = (x, r, s, s')$ is a P_4 and $x <_G r$ so $x <_{V'} r$. Since P' is good under $<_{V'}$, $s <_{V'} s'$, so $s <_G s'$, so $(*)$ holds for P. Next suppose that x is in H_1. By Theorem 4.2 (x, r, s, s') is a P_4 and arguing as above $(*)$ holds for P.

Subcase: $P = (r, h_0, r', s)$.

If $h_0 <_G r$ then $(*)$ holds for P so assume the contrary. By Theorem 4.2 $P' = (r, h_1^*, r', s)$ is a P_4; also $h_0 <_G h_1^*$ and $r <_G h_0$, so $r <_G h_1^*$. Since P' is good under $<_{V'}$, $r <_{V'} s$ so $r <_G s$ so $(*)$ holds for P.

Case: $|P \cap H_0| = 2$.

By Lemma 5.3, there are four possible forms for P. With the first two, $(*)$ holds for P since $h_0 <_G s$; with the third, $(*)$ holds for P since $h_0 <_G h_1$. Consider then the fourth form, namely $P = (h_0, h_0', h_1, h_1')$.

If $h_1 <_G h_1'$ then $(*)$ holds for P. Suppose then that $h_1' <_G h_1$. Since the relative order of h_1 and h_1' is the same under $<_H$ as under $<_G$ and since P is good under $<_H$, it follows that $h_0' <_H h_0$. Thus h_0 was added to $<_G$ after h_0'. Now we claim that $h_0' <_G h_0$. Suppose not: then $h_0 <_G h_0'$ and from the algorithm it follows that h_0 sees some s in S that misses h_0'. But then $(s, h_0, h_0', h_1, h_1')$ is a P_5, contradiction. Thus the claim holds, so $(*)$ holds for P and the proof of Theorem 5.1 is complete. □

To conclude this section, we sketch an argument that our algorithm is polynomial. Let n be the number of vertices of the input graph G. The anti-handle search of step 1 can be performed in $O(n^3)$ time, as follows.

First search for a P_3, say by traversing edges and finding a vertex not adjacent to all previous vertices in the traversal sequence. This takes $O(n^2)$ time. Next trace the steps described in Theorem 2.2 with the graph $\overline{G}$ in place of G. Finding each of the initial sets X, S, and Y involves identifying a particular component of a particular induced subgraph, and takes $O(n^2)$ time. Next trace the proof of Theorem 2.1, again with $\overline{G}$ in place of G; finding each of X_k, S_k, and Y_k takes $O(n^2)$ time, and k is less than n. Thus the total time for the anti-handle search is $O(n^3)$.

In steps 2 and 3 the combining of orders found by recursive calls can be done in $O(n^2)$ time. Thus the amount of work that occurs during one call of the algorithm is $O(n^3)$, exclusive of any recursive calls.

Now consider the recursion tree associated with a call of the algorithm. Let X be any internal node (namely, corresponding to an algorithm call resulting in a recursive call), and let m be the number of edges in the complement of the graph associated with X. It is routinely verified that the sum of the number of edges in the complements of the graphs associated with all children of X that are themselves internal is less than m. Now using standard recurrence relation arguments, it can be shown that the time of our algorithm is $O(mn^3) = O(n^5)$.

The efficiency of our algorithm can be improved. For example, in step 2 the subgraph $G[H]$ is the complement of a bipartite graph; this can be exploited. Also, our complexity analysis can probably be tightened. However, for the sake of brevity, we have kept the algorithm and the analysis simple.

6. Conclusions. Via results on separating sets in arbitrary graphs and cutsets in weakly triangulated graphs, we have established sufficient structural results on $\overline{P}_5$-free weakly triangulated graphs (Meyniel weakly triangulated graphs) to show that such graphs are co-perfectly orderable.

In light of the *NP*-completeness of recognizing perfectly orderable weakly triangulated graphs, it is unlikely that any "nice" characterizations of these graphs exist. However, it is possible that further perfectly orderable subclasses of weakly triangulated graphs can be identified. For example, we know of no P_6 and $\overline{P}_6$-free weakly triangulated graph, nor any bull-free weakly triangulated graph, that is not perfectly orderable.

On the other hand, it is known that $\overline{P}_5$-free weakly triangulated graphs are not necessarily perfectly orderable. The graph obtained from a triangle and three dominoes (as defined in §1) by joining each triangle vertex to a domino vertex of degree two is weakly triangulated and $\overline{P}_5$-free but not perfectly orderable.

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REFERENCES

[AF] R.P. Anstee & M. Farber, Characterizations of totally balanced matrices, *J. Algorithms* **5** (1984) 215-230.

[C1] V. Chvátal, Perfectly orderable graphs, in "Topics on Perfect Graphs" (C. Berge & V. Chvátal, eds.), Ann. Discrete Math. **21**, North-Holland (1984) 253-257.

[C2] V. Chvátal, A class of perfectly orderable graphs, Rpt. 89573-OR, Forschungsinstitut für Diskrete Mathematik, Bonn (May 1989)

[HK] C.T. Hoàng & N. Khouzam, On brittle graphs, *J. Graph Theory* **12** (1988) 391-404.

[H1] R.B. Hayward, Weakly triangulated graphs, *J. Comb. Theory, Series B* **39** (1985) 200-209.

[Ho] C.T. Hoàng, On the complexity of recognizing a class of perfectly orderable graphs, to appear in *Discrete App. Math.*.

[HMOP] C.T. Hoàng, F. Maffray, S. Olariu & M. Preissman, A charming class of perfectly orderable graphs, *Discrete Math.* **102** (1992) 67-74.

[M] H. Meyniel, On the perfect graph conjecture, *Discrete Math.* **16** (1976) 334-342.

[MP] M. Middendorf & F. Pfeiffer, On the complexity of recognizing perfectly orderable graphs, *Discrete Math.* **80** (1990) 327-333.

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