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A Note on the Circle Containment Problem

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Abstract. We extend a result due to Bárány *et al.* and prove the following theorem: given any set S of n points in the plane, there are points x and y in S, such that every circle that contains x and y contains at least $\lceil \frac{5}{84}(n-2) \rceil$ other points of S.

What is the largest integer Π_n such that in every configuration S of n points in the plane there are points x and y, such that every circle that contains x and y contains at least Π_n other points of S? This problem was introduced by Newmann-Lara and Urrutia [1] and has also been studied by Hayward *et al.* [2] and Baárány *et al.* [3]. In this note we establish an improved lower bound for Π_n. It is known [2] that $\Pi_n \leq \lceil n/4 \rceil - 1$.

Theorem. $\Pi_n \geq \lceil \frac{5}{84}(n-2) \rceil$.

Before proving the theorem, we present a preliminary result.

Lemma. *Among any set T of k points in the plane, there are at least $\binom{k-3}{2}$ pairs of points $\{x,y\}$ that satisfy the following property:*

every circle that contains x and y contains at least one of the other $k-2$ points of T. $(*)$

Proof of the Lemma. Let U be the set of pairs of points of T that do NOT satisfy property $(*)$. For each pair of points $\{v, w\}$ of U, join the points v, w in T with a straight line segment. No two such segments can cross. (Suppose that they did; then the segments would be diagonals of a convex quadrilateral. But in this case it can be shown [1],[2] that one of the diagonals has the property that every circle that contains both endpoints contains at least one of the endpoints of the other diagonals, contradicting the assumption that both segments are associated with a pair in U.) Thus the segments are the edges of a planar graph with at most k vertices. Since a planar graph with n vertices has at most $3n-6$ edges, U contains at most $3k-6$ pairs, and so T contains at least $\binom{k}{2} - (3k-6) = \binom{k-3}{2}$ pairs that satisfy $(*)$. □

A parenthetic comment: let D be the drawing obtained by joining each pair of points of T with a straight line segment. Define the crossing graph of D to be the graph whose vertices are the segments, and whose edges are the pairs of crossing segments. The lemma can be restated as this: any vertex cover of the crossing graph of D consists of at least $\binom{k-3}{2}$ segments.

Proof of the Theorem. We use a counting argument from [3]. Let T be a subset of k points of S, and let x and y be points in T. We shall say that $\{x, y\}$ dominates T if property $(*)$ is satisfied. The lemma implies that some pair of points $\{u, v\}$ of S dominates at least $\left[\binom{k-3}{2}\binom{n}{k}\right] / \binom{n}{2}$ subsets.

For each of these subsets, every circle that contains u and v contains at least one of the other $k-2$ points of the subset. Since a point of $S - \{u, v\}$ could be in at most $\binom{n-3}{k-3}$ subsets that include u and v, every circle that contains u and v contains at least

$$\frac{\binom{k-3}{2}\binom{n}{k}}{\binom{n}{2}\binom{n-3}{k-3}} = (n-2)\frac{(k-3)(k-4)}{k(k-1)(k-2)}$$

other points of S. Setting $k = 8$ or 9 gives the largest coefficient of $n-2$, namely $\frac{5}{84}$. □

References

[1] V. Neumann-Lara and J. Urrutia, A combinatorial result on points and circles in the plane, Technical Report TR-85-15, University of Ottawa, November 1985.

[2] R. Hayward, D. Rappaport, and R. Wenger, Some extremal results on circles containing points, *Discrete Comput. Geom.*, this issue, pp. 253-258.

[3] I. Bárány, J. H. Schmerl, S. J. Sidney, and J. Urrutia, A combinatorial result about points and balls in Euclidean space, *Discrete Comput. Geom.*, this issue, pp. 259-262.

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