Bull-free Weakly Chordal Perfectly Orderable Graphs*

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Abstract. Using doubly lexical orders and the notion of box partition due to de Figueiredo, Maffray, and Porto, we show that a certain subclass of bull-free weakly chordal graphs is perfectly orderable. This together with results of de Figueiredo, Maffray, and Porto confirms Chvátal's conjecture that bull-free graphs with no anti-hole and no odd hole are perfectly orderable; here hole means induced cycle with five or more vertices.

Keywords. bull-free graph, weakly chordal graph, perfectly orderable graph, doubly lexical order, box partition

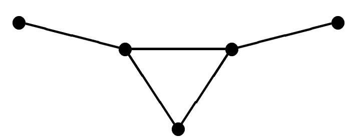

Fig. 1. The bull.

1 Overview

The *bull* is the graph shown in Figure 1. In this paper, a *hole* is an induced cycle with at least five vertices, and an *anti-hole* its complement. A graph is *weakly chordal* (also known as *weakly triangulated*) if it has no hole or anti-hole [Ha]. P_k is the induced path with k vertices.

In 1983, motivated by a desire to identify graphs that can be efficiently colored, Chvátal defined as *perfectly orderable* those graphs which admit a vertex ordering such that, for every induced subgraph, applying the greedy coloring algorithm (namely, coloring vertices with non-negative integers, each vertex being assigned the smallest color not already assigned to a neighbor) in the prescribed order yields an optimal coloring [C1]. While many classes of graphs are known to be perfectly orderable, recognizing perfectly orderable graphs is NP-complete [MP], and remains so even for weakly chordal graphs [Ho]. In 1989, Chvátal conjectured that [C2]

- *every weakly chordal graph with no P_5 is perfectly orderable, and*
- *every bull-free graph with no anti-hole and no odd hole is perfectly orderable.*

Recently, we confirmed the first conjecture [Ha2]. Here, we confirm the second. Our work builds on results of de Figueiredo, Maffray, and Porto, who made significant progress on the conjecture, reducing it to the conjecture that a certain subclass of bull-free weakly chordal

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graphs is perfectly orderable [dFMP, dFMP2]. Before explaining their results, we present some motivating background material.

2 Perfect orders and doubly lexical orders

Proofs of perfect orderability usually rely on the following characterization [C1] rather than the definition:

> *a linear vertex order $<$ is a perfect order if and only if there is no induced path (a, b, c, d) with $a < b$ and $d < c$.*

For example, a graph is *transitively orientable* (also called *comparability*) if its edges can be oriented so that whenever there are edges (u, v) and (v, w) oriented $u \leftarrow v$ and $v \leftarrow w$, then the edge (u, w) exists and is oriented $u \leftarrow w$. A transitive orientation is acyclic and so can be extended to a linear vertex order (for example, via topological sorting). Thus a graph is transitively orientable if and only if there is a linear vertex order $<$ with no induced path (u, v, w) satisfying $u < v < w$. Using the above characterization, it follows that every linear extension of a transitive orientation is a perfect order, so transitively orientable graphs are perfectly orderable [C1].

Another basic result concerns a *homogeneous set* of a graph, namely a proper vertex subset with size at least two, such that every vertex not in the set is adjacent to either all or none of the vertices in the set [HK]:

> *if every proper subgraph of a graph with a homogeneous set is perfectly orderable, then so is the graph.*

Thus to prove all graphs in a class are perfectly orderable, it is sufficient to consider graphs in the class with no homogeneous set.

Call a graph *co-bipartite* if it is the complement of a bipartite graph. Notice that co-bipartite graphs are bull-free. In the rest of this section we motivate our general approach by recalling how doubly lexical orders are used to prove that co-bipartite weakly chordal graphs are perfectly orderable.

With respect to comparison relations defined on vector scalars, *lexical* comparison relations are defined on vectors by comparing vectors scalar by scalar, starting with the most significant scalar, until two unequal scalars are found, and letting the result of that comparison determine the result of the vector comparison. In this paper, vectors will be written with scalars ordered in terms of increasing significance; thus the most significant scalar will be the rightmost for row vectors and the bottommost for column vectors. For example, [1001] $<$ [0101], since the vectors agree on the last two scalars, while the third-last scalar of the first vector is less than the third-last scalar of the second vector. An ordering of vectors is called *lexical* if the vectors are in non-decreasing lexical order. For example, [000] [100] [100] [010] [110] [001] are in lexical order.

A *doubly lexical order* of a matrix is a permutation of the rows and a permutation of the columns, such that both the rows (with scalars arranged according to the column permutation) and the columns (with scalars arranged according to the row permutation) are in lexical order. For example, (2,1) (1,2) is a doubly lexical order of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, yielding the doubly lexical ordered matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The symbol Γ denotes the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$; we say that a 0-1 matrix *has a* Γ if there are two rows and two columns such that the submatrix induced by these rows and columns (in the same relative order as in the matrix; neither the rows nor the columns need be consecutive) is Γ. There is a natural correspondence between 0-1 matrices and bipartite graphs: rows and columns correspond to vertices, and 1 entries to edges. A 0-1 matrix is *totally balanced* if and only if the corresponding bipartite graph has no hole. Our interest in doubly lexical orders is a consequence of the aforementioned correspondence and the following classical results on doubly lexical orders [AF,HSK,L]:

– *every matrix has a doubly lexical order,*
– *a doubly lexical order of a 0-1 matrix has no* Γ *if and only the matrix is totally balanced.*

These results imply that certain weakly chordal graphs are perfectly orderable; to explain how, we need more terminology.

A *non-trivial bipartition* of a graph G is a partition of the vertex set into two non-empty subsets. Given a bipartition X, Y of a graph G, define $G[X, Y]$ as the bipartite subgraph of G whose edge set consists of all edges with one end vertex in X and one end vertex in Y. By a doubly lexical order of $G[X, Y]$ (or of X, Y when G is clear from context) we mean a doubly lexical order of the corresponding 0-1 matrix. A *doubly lexical co-order* of $G[X, Y]$ is a doubly lexical order of $\overline{G}[X, Y]$. A *co-bipartite bipartition* of a graph G is a bipartition X, Y such that $G[X]$ and $G[Y]$ are cliques. Observe that a graph has a co-bipartite bipartition if and only if it is a co-bipartite graph (namely, the complement of a bipartite graph) and not a clique. The following theorem illustrates the natural correspondence between doubly lexical orders of totally balanced matrices and perfect orders of co-bipartite weakly chordal graphs, and motivates our main result; for this reason, we include the proof.

Theorem 1 [C2] *For every co-bipartite bipartition of a co-bipartite weakly chordal graph, every doubly lexical co-order extends to a perfect order.*

Corollary [C2] *Every co-bipartite weakly chordal graph is perfectly orderable.*

Proof of Theorem 1. Let $<$ be any extension of a doubly lexical co-order of a co-bipartite bipartition X, Y of a co-bipartite weakly chordal graph G to a total vertex order. $G[X]$ and $G[Y]$ are cliques and G has no anti-hole, so $\overline{G}[X, Y]$ has no hole. By the above result, the 0-1 matrix $\overline{G}[X, Y]$ has a doubly lexically ordering which has no Γ, so assuming such an ordering, the 0-1 matrix $G[X, Y]$ has no $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the graph G has no P_4 (x_1, x_2, y_2, y_1) with $x_1 < x_2$ and $y_1 < y_2$. $G[X]$ and $G[Y]$ are cliques, so every P_4 (a, b, c, d) in G can be labelled with a, b in X and c, d in Y. Thus G has no P_4 (a, b, c, d) with $a < b$ and $d < c$, so $<$ is a perfect order. □

Proof of Corollary. Consider any co-bipartite weakly chordal graph. If the graph is a clique, it has no P_4, and any vertex order is a perfect order. If the graph is not a clique, it has a co-bipartite bipartition, which has a doubly lexical co-order (since every matrix has a doubly lexical order), and the theorem applies. □

3 Box partitions and our main result

Box partitionable graphs generalize bipartite graphs, with vertices being replaced by connected subgraphs, and an adjacency constraint being imposed between boxes from opposite sides of the bipartition. The notion is due to de Figueiredo, Maffray, and Porto, who used box partitions to decompose bull-free graphs with no anti-hole or odd hole. Their results are summarized below; first we formally define box partition.

With respect to a set S of vertices in a graph G, a vertex x of $G-S$ is called *S-universal*, *S-partial*, or *S-null* if it is adjacent to respectively all, some but not all, or none, of the vertices of S; the set of all S-partial vertices is denoted $P(S)$. A *box partition* of a graph G is a partition of the vertices into non-empty sets X and Y such that, for each *box* (namely, each connected component of $G[X]$ or $G[Y]$) B, if the set $P(B)$ vertices is non-empty, then there are two adjacent vertices not in the box, one which is B-universal and $P(B)$-null, and the other $P(B)$-universal and B-null. Any two such vertices form a *B-auxiliary edge.* A *co-box partition* of a graph is a partition that induces a box partition of the complement of the graph, and an *auxiliary co-edge* is an auxiliary edge in the complement.

Theorem 2 [dFMP] *Every bull-free graph with no anti-hole, no odd hole, and some even hole is transitively orientable or has a homogeneous set.*

Theorem 3 [dFMP2] *For every bull-free weakly chordal graph,*

1. *if the graph has a P_6, then it has a box partition,*
2. *if the graph has a box partition, then it has a homogeneous set or is transitively orientable.*
3. *if the graph has no box partition and at least two vertices, then it has a co-box partition.*

Let $\mathcal{G}$ be the class of P_6-free bull-free weakly chordal graphs. In light of the above theorems, to establish Chvátal's conjecture it is sufficient to do so for graphs in $\mathcal{G}$ with some co-box partition. Hence our main result, which parallels Chvátal's theorem of §2, and, with the above theorems of de Figueiredo, Maffray, and Porto, establishes Chvátal's conjecture:

Theorem 4 *For every co-box partition of a graph in $\mathcal{G}$, some doubly lexical co-order extends to a perfect order.*

Corollary *Every bull-free graph with no odd hole and no anti-hole is perfectly orderable.*

In §4, we reduce the above theorem to a simpler statement. An algorithmic proof of the reduced theorem is presented in §5. Proofs of lemmas and algorithm correctness are postponed to §6.

4 Reducing Theorem 4

In proving our main result, some difficulties are easily dealt with. One such difficulty is that co-boxes of co-box partitions may contain P_4's. We deal with this via the following lemma, a strengthening of a result from [dFMP] which allows us to restrict our attention to co-box partitions with P_4-free co-boxes. We call such co-box partitions *reduced.*

Lemma 5 *If a co-box of a co-box partition of a graph in $\mathcal{G}$ has a P_4, then the graph has a homogeneous set which is in the co-box and contains the P_4.*

Another difficulty is that vertex deleted subgraphs of co-box partitions may not themselves be co-box partitions. For example, the partition obtained by deleting $\{x_1, x_4, y_4\}$ from the co-box partition X, Y of the graph in Figure 3 is not a co-box partition, since co-box $\{x_2, x_3\}$ ends up with no auxiliary co-edge. Call a vertex subset S of a graph *comparable* if, for any pair of vertices in S, the neighborhood restricted to $G - S$ of one vertex is a subset or superset of the neighborhood restricted to $G - S$ of the other vertex. We introduce a new bipartition, closed under vertex deletion: a *quasi partition* is a bipartition X, Y of the vertex set V of a graph G, with possibly X or Y empty, such that, for every co-box Q (defined as in a co-box partition, namely any component of the complement of the graph induced by one of the parts of the bipartition), $G[Q]$ *is P_4-free, and*

1. *there is no $\overline{P}_3$ (q_1, q_2, q_3) of $G[Q]$ such that some vertex z of $G - Q$ is adjacent to q_1 but not q_2 or q_3,*
2. *there is no P_3 (q_1, q_2, q_3) of $G[Q]$ such that some vertex z of $G - Q$ is adjacent to q_1 and q_2 but not q_3,*
3. *there is no non-edge (q_1, q_2) of Q and non-edge (r_1, r_2) of $G - Q$, such that q_1 is adjacent to r_1 but not r_2, and q_2 is adjacent to neither r_1 nor r_2,*
4. *Q is comparable, namely, there are no vertices q_1, q_2 of Q and r_1, r_2 of $G - Q$ such that q_1 is adjacent to r_2 but not r_1, and q_2 is adjacent to r_1 but not r_2.*

The definition of a quasi partition is illustrated in Figure 2. The following lemma allows us to replace co-box partitions with quasi partitions in our proofs.

Lemma 6 *Every reduced co-box partition of a graph in $\mathcal{G}$ is a quasi partition.*

The essential difficulty in proving Theorem 4 is that there exist doubly lexical co-orders of quasi partitions of graphs in $\mathcal{G}$ which do not extend to perfect orders. As there is only one ordered subgraph which prevents such an extension, we give it a name, as follows. With respect to a bipartition X, Y of the vertices V of a graph G and a partial order $<$ on V which is total on each of X and Y, an *8-fig* is an ordered vertex subset $S = (x_1, \ldots, x_4, y_1, \ldots, y_4)$ such that $x_1 < x_2 < x_3 < x_4$, $y_1 < y_2 < y_3 < y_4$, and $G[S]$ is as shown in Figure 3. A related ordered subgraph, the *6-fig*, is defined exactly as an 8-fig, except that all references to vertices x_4 and y_4 are omitted. In other words, a 6-fig is an 8-fig restricted to the six vertices $x_1, x_2, x_3, y_1, y_2, y_3$, with the added proviso that vertices x_4 and y_4 need not exist.

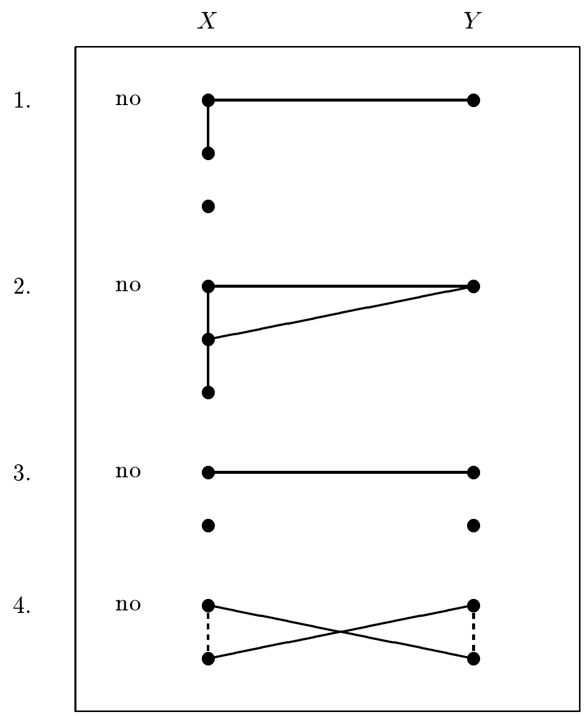

Fig. 2. Quasi partition conditions.

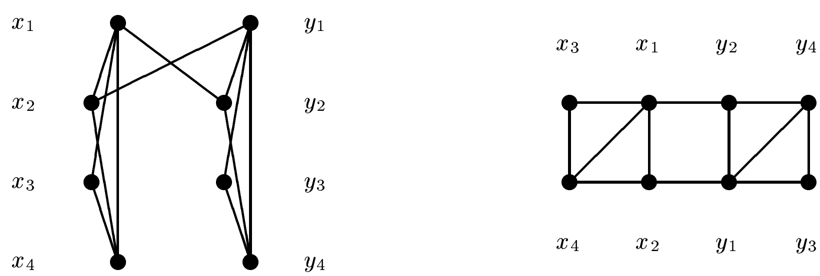

Fig. 3. Two drawings of an 8-fig.

Having introduced all necessary terminology, we end this section by stating and proving the reduction of our main theorem. The next section outlines the proof of the reduced theorem; proofs of all lemmas are postponed to the subsequent section.

Theorem 7 *For every quasi partition of a graph in $\mathcal{G}$,*

1. *a doubly lexical co-order extends to a perfect order if and only if the co-order induces no 8-fig, and*
2. *some doubly lexical co-order induces no 6-fig, and so no 8-fig.*

Proof of Theorem 4 from Theorem 7. Consider any co-box partition of a graph G in $\mathcal{G}$. If the partition is reduced, the result follows by Lemma 6 and Theorem 7, so suppose not. By Lemma 5, G has some homogeneous set H within some co-box. Let G' be the graph obtained by replacing H with a single vertex h. It is straightforward to verify that the original co-box partition induces a co-box partition when restricted to G'. By induction, we may assume that there is some doubly lexical co-order of this restricted co-box partition which extends to a perfect order of G'. Also, the graph $G[H]$ is perfectly orderable, either by induction with this theorem if $G[H]$ has a co-box partition, or by Theorem 3 if $G[H]$ has a box partition. Replacing vertices of H at the location of h in this doubly lexical co-order yields a doubly lexical co-order of G, since every vertex in H has the same neighborhood on the other side of the partition; this extends to a perfect order, since every P_4 of G is either entirely in H or includes at most one vertex of H. □

5 Proving Theorem 7

Although technically detailed, proving Theorem 7.1 is straightforward in that a natural candidate algorithm for combining the two total orders of an 8-fig-free doubly lexical co-order turns out to yield a perfect order. The algorithm is presented below; its correctness, which will be proved later, implies the correctness of Theorem 7.1.

Algorithm 8 *Input. An 8-fig-free doubly lexical co-order* $(x_1, x_2, \ldots, x_\alpha)$ $(y_1, y_2, \ldots, y_\beta)$ *of a quasi partition* X, Y *of a graph in* $\mathcal{G}$.
Output. A perfect order of the graph.

1. *For all edges* $(x_a, y_{z'})$ *such that there is a* P_4 $(y_{z'}, x_a, x_c, x_b)$ *of* G *with* $a < b < c$*, orient the edge so that* $a < z'$*; similarly, for all edges* $(y_{a'}, x_z)$ *such that there is a* P_4 $(x_z, y_{a'}, y_{c'}, y_{b'})$ *of* G *with* $a' < b' < c'$*, orient the edge so that* $a' < z$*.*
2. *Extend the edge orientation defined by the doubly lexical co-order and the previous step to a total vertex order, and return this order.*

We thus turn our attention to the proof of Theorem 7.2. Call a 6-fig *maximal* if there is no other 6-fig (with respect to the given bipartition and the given vertex order, which is total on each of the two parts) in which every vertex is greater than or equal to the corresponding vertex of the original 6-fig. Call a bipartition X, Y *distinguished* if every vertex misses at least one vertex in the opposite part and every two vertices in the same part have different neighborhoods in the opposite part. We shall prove Theorem 7.2 via an algorithm whose essential subroutine takes as input a maximal 6-fig of a doubly lexical co-order of a distinguished quasi partition (or *dq-partition*) of a graph in $\mathcal{G}$ and reorders the vertices to obtain a "better" doubly lexical co-order. In order to motivate the algorithm and to develop the notation necessary to describe it, we present several lemmas. The first part of the first of these lemmas is described pictorially in Figure 4, which shows the 19 different neighbourhoods (represented by the black vertices; the white vertices are the remaining vertices of the 6-fig) a vertex can have on a 6-fig.

Lemma 9 *Let* $S = (1, 2, 3, 4, 5, 6)$ *be an unordered 6-fig of a quasi partition* X, Y *of a graph in* $\mathcal{G}$*. Then the vertices not in* S *can be partitioned into the 19 sets whose* S*-neighborhoods*

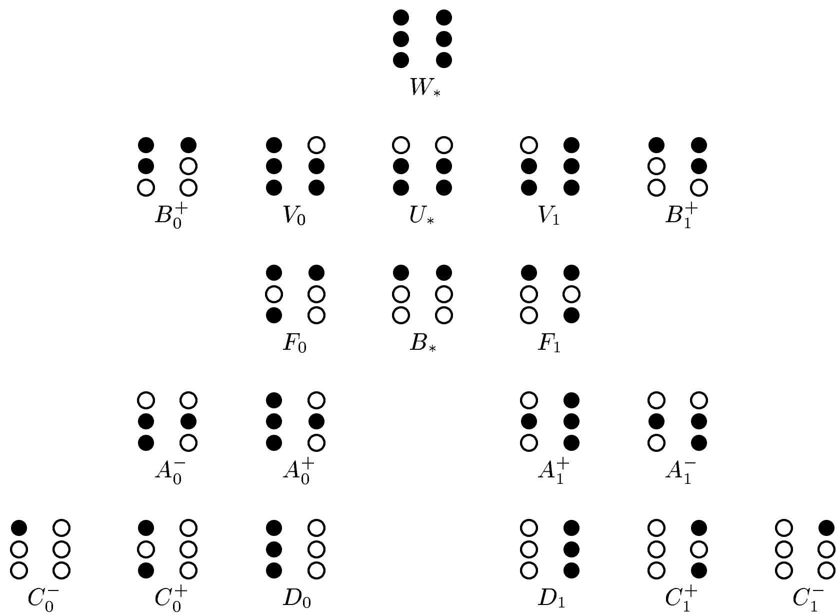

Fig. 4. Possible neighborhoods on a 6-fig; black vertices are neighbours, white vertices are non-neighbours. For example, with respect to the 6-fig $S = (1, 2, 3, 4, 5, 6)$, B_0^+ is the set of vertices not in S which are adjacent to $1, 2, 4$ and non-adjacent to 3,5,6.

are as indicated below (and also pictorially in Figure 4) such that adjacency among the 19 sets is as indicated in Figure 5.

A_0^- $\{2,3,5\}$	B_0^+ $\{1,2,4\}$	C_1^+ $\{4,6\}$	U_* $\{2,3,5,6\}$
A_0^+ $\{1,2,3,5\}$	B_1^+ $\{1,4,5\}$	D_0 $\{1,2,3\}$	V_0 $\{1,2,3,5,6\}$
A_1^- $\{2,5,6\}$	C_0^- $\{1\}$	D_1 $\{4,5,6\}$	V_1 $\{2,3,4,5,6\}$
A_1^+ $\{2,4,5,6\}$	C_0^+ $\{1,3\}$	F_0 $\{1,3,4\}$	W_* $\{1,2,3,4,5,6\}$
B_* $\{1,4\}$	C_1^- $\{4\}$	F_1 $\{1,4,6\}$	

Lemma 10 *Let X, Y be a dq-partition of a graph in $\mathcal{G}$ whose complement is connected, let $S = (a_X, b_X, c_X, a_Y, b_Y, c_Y)$ be a maximal 6-fig with respect to a doubly lexical co-order $<$ of the quasi partition, and let $A_0^-, \ldots, W_*$ be the 19 sets which partition the rest of the vertices of the graph as defined in Lemma 9, with $a_X, \ldots, c_Y$ corresponding to $1, \ldots, 6$ respectively. Let R be $U_* \cup V_0 \cup V_1 \cup W_*$, and let B_0, B_1, R_0, R_1 be $B_* \cap X$, $B_* \cap Y$, $R \cap X$, $R \cap Y$ respectively. Then the following hold, as do the symmetric conditions obtained by interchanging X with Y and subscripts 0 with 1:*

1. *$(A_0^{\pm} \cup B_0 \cup C_0^{\pm} \cup D_0 \cup F_0) \subset X$,*
2. *B_0^+ is empty, B_0 is an independent set, D_0 is a clique, B_0 is $(B_1 \cup C_1^{\pm} \cup F_1)$-null, and R is $(B_* \cup C_0^{\pm} \cup C_1^{\pm} \cup F_0^{\pm} \cup F_1^{\pm})$-universal,*
3. *$B_0 < b_X < F_0 < C_0^{\pm} < c_X$,*
4. *$R_0 < (\{a_X\} \cup A_0^{\pm}) < (\{c_X\} \cup C_0 \cup D_0)$,*
5. *$F_0 \cup F_1$ is non-empty if and only if the following all hold: both F_0 and F_1 are non-empty, some a_0 in $A_0^{\pm}$ satisfies $b_X < a_0$, and some a_1 in $A_1^{\pm}$ satisfies $b_Y < a_1$ (and so also $b_X < D_0$ and $b_Y < D_1$).*

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
A = U*	{2356}		0	0	1		1		0		1	1	1	1	1	0		1		
B = V0	{12356}	0				1	1	1	1		1	1	1	1	1	0				
C = V1	{23456}	0							0		1	1	1	1	1	1		1	1	1
D = W*	{123456}	1				1	1	1	1		1	1	1	1	1	1		1	1	1
E = D0	{123}		1		1		1	1	1	1	1	1		0	0	0	0	0	0	0
F = C0-	{1}	1	1		1	1				1	1	0	0	0	0	0		0	0	0
G = C0+	{13}		1		1	1			1	1		0		0	0			0	0	0
H = A0-	{235}	0	1	0	1	1		1			1	1	1	1		0	0	0		0
I = A0+	{1235}					1	1	1			1	1		1		0	0			0
J = F0	{134}	1	1	1	1	1	1		1	1		0	0	0	0			0	0	0
K = B0+	{124}	1	1	1	1	1	0	0	1	1	0				0	1	1	0	0	0
L = B*	{14}	1	1	1	1		0		1		0				0	1		0		
M = B1+	{145}	1	1	1	1	0	0	0	1	1	0				0	1	1	0	0	1
N = F1	{146}	1	1	1	1	0	0	0			0	0	0	0		1	1	1		1
O = A1-	{256}	0	0	1	1	0	0		0	0		1	1	1	1				1	1
P = A1+	{2456}					0			0	0		1		1	1			1	1	1
Q = C1-	{4}	1		1	1	0	0	0	0		0	0	0	0	1		1			1
R = C1+	{46}			1	1	0	0	0			0	0		0		1	1			1
S = D1	{456}			1	1	0	0	0	0	0	0	0		1	1	1	1	1	1	

Fig. 5. Adjacency among vertices partitioned by neighborhood on 6-fig (123456). 0 means non-adjacent, 1 means adjacent, blank means forbidden. See Figure 8 for justification.

Lemma 11 *Let X, Y, $<$, S, and all associated sets be as in Lemma 10, let C_0^* be $\{c_X\}$ and all vertices d_0 such that $c_X < d_0$, let A_0^* be $\{a_X\} \cup A_0^{\pm} \cup (D_0 - C_0^*)$, let B_0^* be $\{b_X\} \cup F_0 \cup C_0^{\pm}$ and all vertices b_0 of B_0^- such that for some a_0 in A_0^*, $a_0 < b_0$, let R_0^* be $R_0 \cup (B_0^- - B_0^*)$, let $C_1^*, A_1^*, B_1^*, R_1^*$ be defined symmetrically, and let $<_*$ be the vertex order defined with respect to X, Y, $<$, S so that*

- *for fixed j = 0,1, $<_*$ is the same as $<$ on A_j^* and on $X - A_j^*$,*
- *for fixed j = 0,1, and for all a_j, b_j, c_j in A_j^*, B_j^*, C_j^*, $b_j <_* a_j <_* c_j$.*

Then

1. *$<_*$ is a doubly lexical co-order of X, Y,*
2. *if S is chosen so that c_X is maximum and b_X is maximum for this c_X, then there is no 6-fig $(x_1, x_2, x_3, y_1, y_2, y_3)$ of $<_*$ with $c_X = x_3$ or $c_X < x_3$.*

Having presented all necessary lemmas, we now present our final algorithm. The correctness of this algorithm, presented in the next section, establishes Theorem 7.2 and Theorem 4.

Algorithm 12 *Input. A quasi partition X, Y of a graph G in $\mathcal{G}$.*
Output. A perfect order $<$ of G.

0. *Find any doubly lexical co-order $<$ of G with respect to X, Y.*
1. *Let X' be all vertices of X which are not Y-universal, and let Y' be all vertices of Y which are not X-universal. Define $<_\sim$ as any partial vertex order which is total on $X - X'$ (in any order), total on $Y - Y'$ (in any order), and for which $(X - X') <_\sim X'$ and $(Y - Y') <_\sim Y'$.*

2. Let $G_1, \ldots, G_t$ be the subgraphs of G induced by the connected components of $\overline{G}[X' \cup Y']$; the order of indexing is arbitrary. Let $X_1, \ldots, X_t$ and $Y_1, \ldots, Y_t$ be the associated subsets of X' and Y'. Extend the definition of $<_\sim$ so that $X_1 <_\sim \ldots <_\sim X_t$ and $Y_1 <_\sim \ldots <_\sim Y_t$.

3. For each G_j of the previous step:

(a) Define equivalence classes of vertices within X_j and Y_j according to their neighborhoods on the other part. Let X_j^* and Y_j^* be subsets of X_j and Y_j obtained by taking one representative from each class.

(b) Extend the definition of $<_\sim$ to X_j^*, Y_j^* as follows: if there is no 6-fig of $G[X_j^* \cup Y_j^*]$ with respect to $(X_j^*, Y_j^*, <)$, then define $<_\sim$ on X_j^*, Y_j^* as $<$; otherwise, find a 6-fig $(a_X, b_X, c_X, a_Y, b_Y, c_Y)$ with a_X, b_X, c_X in X_j^* and a_Y, b_Y, c_Y in Y_j^* for which c_X is maximum and for which b_X is maximum for this choice of c_X, redefine $<$ on X_j^*, Y_j^* as the $<_*$ described in Lemma 11, and repeat this step.

(c) Extend $<_\sim$ from X_j^*, Y_j^* to X_j, Y_j by ordering different equivalence classes according to the ordering of their representatives, ordering vertices within a class arbitrarily.

4. Using Algorithm 8, extend the 6-fig-free doubly lexical co-order $<_\sim$ of X, Y to a perfect order $<$ of G.

6 Proofs of lemmas and correctness of algorithms

A *P_4-partial labelling* $(s_1, \ldots, s_t)$ of a vertex subset S of a graph is a labelling of the vertices of S, such that $\{s_1, \ldots, s_4\}$ is a P_4, and for $5 \leq j \leq t$, s_j is partial on $S_{j-1} = \{s_1, \ldots, s_{j-1}\}$. A P_4-partial labelling is *maximal* if it cannot be extended by adding another vertex. Any maximal P_4-partial labelling which does not contain all the vertices of the graph is a homogeneous set.

Lemma 13 *For every P_4-partial labelling $(s_1, \ldots, s_t)$ in a bull-free graph,*

1. for $5 \leq k \leq t$, vertex s_k is partial on some P_4 with four smaller labels,

2. for $4 \leq k \leq t$, vertex s_k is in some P_4 with three vertices with smaller labels.

Sees and *misses* are synonyms for "is adjacent to" and "is non-adjacent to" respectively.

Proof of Lemma 13. Up to symmetry, there are eight ways a vertex can be partial on a P_4; in one case, the resulting graph is a bull; in every other case, the vertex forms a P_4 with three of the vertices of the original P_4. Thus, for $k \geq 5$, 2 follows from 1 (for $k = 4$, 2 follows from the definition of P_4-partial labelling).

To prove 1, argue by induction: assume 1 and 2 hold up to a fixed value of k. Let j be the smallest label greater than three such that s_{k+1} is partial on S_j. By 2 for j, s_j is in some P_4 Q with three previous vertices. Now s_{k+1} is partial on Q (for $j = 4$, $S_4 = Q$; for $j \geq 5$, s_{k+1} is partial on S_j but not S_{j-1}, so s_{k+1} either sees or misses exactly one of the vertices of Q). □

Lemma 14 *Let S be a maximal P_4-partial labelling whose initial P_4 is contained in some box B of a box partition of a C_5-free bull-free graph. Then S is contained in B and is a homogeneous set of the graph.*

Proof of Lemma 14. Consider the smallest index j such that s_j is not in B. By Lemma 13, s_j is partial on some P_4 $(v_1, v_2, v_3, v_4) = P$ of S_{j-1} and so of B. By the definition of box partition, there is some auxiliary edge (x, y) such that x is P-null and sees s_j, and y is P-universal and misses s_j. Up to symmetry, there are eight different ways s_j can be P-partial. In each case, the graph induced by $\{x, y, s_j\} \cup P$ contains either a bull or C_5, contradiction. Thus S is contained in B. S is maximal and does not contain all the vertices of the graph (every box partition has at least two boxes), so S is a homogeneous set. □

Proof of Lemma 5. A P_4 of a graph is a P_4 of the complement of the graph; the analogous statement holds for homogeneous sets. Now apply Lemma 14. □

Proof of Lemma 6. Let Q be any co-box of the co-box partition X, Y. We first claim that some Q-auxiliary co-edge has one vertex in X and one in Y. To see why, suppose Q is a subset of X; the argument is symmetric for Y. Let (a, b) be a Q-auxiliary co-edge, with b Q-null. Then b is in Y, since it misses a vertex in Q but is not in Q. If a is in X we are done, so suppose a is in Y; let T be the co-box containing a. Thus T contains b and every Q-partial vertex of $G - Q$. Let (c, d) be a T-auxiliary co-edge, with d T-null. Then d is in X, and (b, d) is a Q-auxiliary co-edge with one vertex in each of X and Y, and the claim holds.

To prove 1 through 4, assume Q is in X, and let (x, y) be a Q-auxiliary co-edge with x in X and y in Y. Argue by contradiction. If 1 holds, $\{z, q_1, q_2, q_3, x\}$ induces a bull. If 2 holds, $\{z, q_1, q_2, q_3, y\}$ induces a bull. If 3 holds, $\{q_1, q_2, r_1, r_2, x, y\}$ induces P_6. If 4 holds, $\{q_1, q_2, r_1, r_2, x, y\}$ induces C_6 or a graph with a bull or C_5, unless q_1 sees q_2, and r_1 sees r_2, in which case, for any vertex q_3 of Q which is $\{q_1, q_2\}$-null (such a vertex exists, since $\overline{G}[Q]$ is connected and P_4-free), $\{q_1, q_2, r_1, r_2, x, y\}$ induces $\overline{C}_6$ or a graph with a bull. □

With respect to a doubly lexical order or co-order of some $G[X, Y]$, the *neighborhood vector* of a vertex is the 0-1 vector associated with that vertex in the doubly lexical order or co-order.

Lemma 15 *Let $(x_1, x_2, \ldots, x_\alpha)$ $(y_1, y_2, \ldots, y_\beta)$ be a doubly lexical co-order of a quasi partition X, Y of a graph in $\mathcal{G}$. Then*

0. *if $j < k$ then the neighborhood vector of x_j is lexically greater than or equal to the neighborhood vector of x_k,*
1. *there are no x_j, x_k, y_p such that $j < k$, y_p sees x_k but not x_j, and x_j and x_k are in the same co-box,*
2. *if there are x_j, x_k, y_p such that $j < k$, y_p sees x_k but not x_j, and x_j and x_k are in different co-boxes (so x_j sees x_k), then there is some $q > p$ such that y_q sees x_j but not x_k, and for every such q, y_q and y_p are in different co-boxes,*
3. *there is no $j < k$ and $p < q$ such that x_j misses y_p and y_q, and x_k misses y_p but sees y_q,*

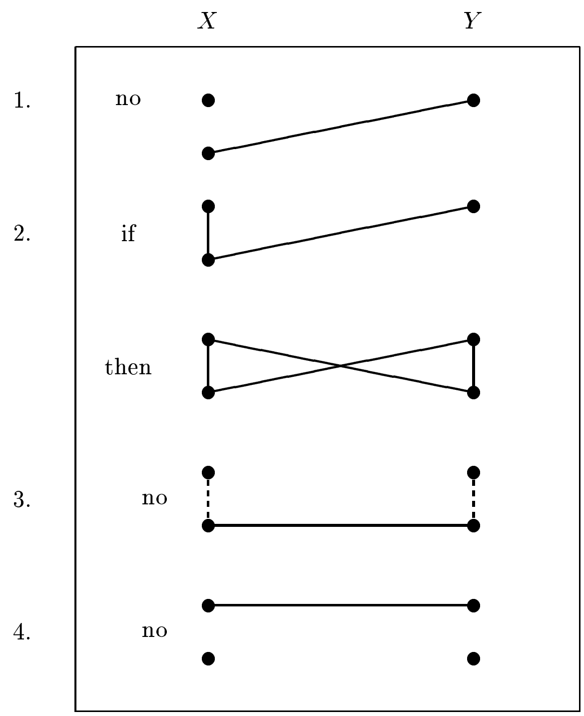

Fig. 6. Lemma 15 conditions.

4. *there is no* $j < k$ *and* $p < q$ *such that* x_j *sees* y_p *and misses* y_q *and* x_k*,* x_k *misses* y_p *and* y_q*, and* y_p *misses* y_q*,*
5. *if* $(x_b, x_c, x_a, y_{z'})$ *and* $(y_{b'}, y_{c'}, y_{a'}, x_z)$ *are* P_4*'s and* $a < b < c$ *and* $a' < b' < c'$*,*
   (a) $a' < z'$ *if* $a = z$*,*
   (b) $(x_z, x_a, x_b, x_c, y_{z'}, y_{a'}, y_{b'}, y_{c'})$ *is an 8-fig if* $z < a$ *and* $z' < a'$*.*

Proof of Lemma 15. Recall that a doubly lexical co-order of $G[X, Y]$ is a doubly lexical order of $\overline{G}[X, Y]$; this and the fact that a neighborhood vector in $\overline{G}[X, Y]$ is the 0-1 complement of the corresponding vector in $G[X, Y]$ implies 0. 1 follows from 0 and the fact that co-boxes of quasi partitions are comparable. 2 follows from 0 and 1. 4 follows from part 3 of the definition of quasi partition.

To see 3, first observe that $\overline{G}[X, Y]$ has no hole. (Argue by contradiction: let $(a_1, b_1, \ldots, a_z, b_z)$ be such a hole, with $A = \{a_1, \ldots, a_z\}$ a subset of X and $B = \{b_1, \ldots, b_z\}$ a subset of Y. Since z is at least three, no two vertices of A or of B are in a comparable subset of X or Y, so by 1 no two such vertices are in the same co-box of $G[X, Y]$. Thus every two such vertices are non-adjacent in $\overline{G}$, and the hole is a hole in $\overline{G}$, a contradiction since G is weakly chordal.) So $\overline{G}[X, Y]$'s adjacency matrix has no Γ, so $G[X, Y]$'s adjacency matrix has no $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Consider 5. Relabel vertices using only their subscripts, namely a, b, c, z in X and a', b', c', z' in Y. First consider 5(a). Suppose $z = a$. Then b sees a' (otherwise, $\{b, a', b'\}$ contradicts 1 if b sees b'; $\{a, b, a', b'\}$ contradicts 2 if b misses b'), so $z' \neq a'$. Also, b misses b' by 1 with $\{a, b, b'\}$, so z' misses at least one of a', b' (otherwise $\{a, b, a', b', z'\}$ induces a bull), so z' is in the co-box of Y containing a' and b', and $a' < z'$ by 1 with $\{b, a', z'\}$. Thus 5(a) holds. Finally, consider 5(b). We first claim that

z is in a different co-box from a and b, so z sees a and b, and similarly z' is in a different co-box from a' and b', so z' sees a' and b'.

To see why, argue by contradiction: suppose z is in the same co-box as a and b. By 1, z sees z', and a and b miss b' and c'. Thus a sees a' (if not, b misses a', by 1 with $\{a, b, a'\}$; z sees a, by 4 with $\{z, a, a', b'\}$; z sees b, by 4 with $\{z, b, a', b'\}$; and $\{z, a, b, z'\}$ contradicts the definition of quasi partition). So b sees a' by 4 with $\{a, b, a', b'\}$. Since a misses b', interchanging z, a, b, c with z', a', b', c' and repeating the argument so far implies that z' is in a different box from a', b'. Thus z' sees a' and b', and $\{z', a', b', a, b\}$ induces a bull, contradiction. Thus our first claim holds. Next we claim that

b misses b'.

Argue by contradiction: suppose b sees b'. Then

- b sees a', b' sees a, and a sees a' (all by 1),
- z sees z' (by 3 with b, b'),
- c sees z (if not, c misses b' by 1 with x, c sees a' by 4 with z, b', and $\{z, c, z', a', b'\}$ is a bull),
- c' sees z' (symmetric to previous),
- c sees a' (if not, c misses b' by 1 with a', and $\{z, c, z', a', b'\}$ is a bull),
- c' sees a (symmetric to previous),
- the largest (with respect to $<$) $\{a', z'\}$-partial vertex d in X sees c, z' and misses a', and $c < d$ (by 2),
- d misses b' (by 1 with a'),
- d sees z (by 4 with a', b'),
- d misses c' (by 3 with z, b'),
- d sees a, b (by 1 with z'),
- the largest (with respect to $<$) $\{a, z\}$-partial vertex d' in Y sees c', z and misses a, and $c' < d'$ (by 2),
- d' in Y with $c' < d'$ sees z, z', a', b', c' and misses a, b, c (symmetric to d),
- d sees d' (else $\{a, b, d, d', z'\}$ is a bull),
- c sees c', c sees b', and b sees c' (in each case by 3 with d, d'),
- the largest (with respect to $<$) $\{c', d'\}$-partial vertex e in X sees d, c' and misses d', and $d < e$ (by 2),
- e sees z (by 1 with c'),
- e sees a', b' (by 3 with d, c'),
- e sees z' (else e contradicts the choice of d)
- e sees a, b, c (by 1 with z'),

but then $\{a, b, e, z', d'\}$ is a bull, so the claim holds. Next, we claim that

b misses a',

for if not, then

- a sees a' (by 1 with b),
- a sees b' (else a, b, z', a', b' induce a bull),

– some d in X with $b < d$ misses a' and sees b, z' (by 1),
– d misses b' (by 1 with a'),
– d sees a (else a, b, d are in the same co-box, contradicting 1 with b, d, z'),
– d sees z (by 4 with a', b'),
– some d' in Y with $b' < d'$ sees z, a', b', z' and misses a, b (symmetric to above)
– d misses d' (by 3 with b, b')

and $\{d, d', a, b, z'\}$ is a bull. To complete the proof of 5(b) and Lemma 15, observe that a misses b' (symmetric to the previous claim), a misses a' (by 4 with b, b'), c misses a', b' (by 3 with b, z'), c' misses a, b (symmetric to previous), c misses c' (by 3 with b, b'), c sees z (by 4 with a', b'), c' sees z' (symmetric to previous), z misses z' (else $\{z, a, b, z'b'\}$ is a bull), and finally $\{z, a, b, c, z', a', b', c'\}$ is a 8-fig. □

Proof of correctness of Algorithm 8. By Lemma 15.5(a), an edge is oriented in step 1 in at most one direction, so step 1 is well-defined. A cycle $(c_1, c_2, \ldots, c_k)$ in an oriented graph is *cyclically oriented* if either $c_1 < c_2 < \ldots < c_k < c_1$ or $c_1 < c_k < c_{k-1} \ldots < c_1$, otherwise it is *acyclically oriented*. We claim that

the edge orientation resulting from step 1 is acyclic.

Argue by contradiction. Consider a (not necessarily induced) cycle $C = (c_1, c_2, \ldots, c_t)$ of G, with cyclic orientation $c_1 < c_2 < \ldots < c_t < c_1$ and labelled so that c_1 is the lexically smallest vertex of C in X. Since X and Y are each totally ordered by the doubly lexical order, the cycle includes vertices from both X and Y, so the labelling is well defined. By the choice of c_1, vertex c_t is not in X, so (c_t, c_1) is a cross edge (namely an edge with an endpoint in X and one in Y). Also, c_{t-1} is not in X (otherwise, $c_{t-1} < c_1$ contradicts the choice of c_1, while $c_1 < c_{t-1}$ contradicts Lemma 15.5(a)). Let s be the largest index less than t such that c_{s-1} is in X; s is well defined, since there are at least two cycle cross edges. Thus (c_{s-1}, c_s) is a cycle cross edge; since Y is totally ordered, $c_s < c_{s+1} < \cdots < c_t$, so $c_s < c_t$. Now $c_{s-1} < c_1$ contradicts the choice of c_1, while $c_{s-1} = c_1$ contradicts Lemma 15.5(a) (with c_1, c_t, c_s), and $c_1 < c_{s-1}$ contradicts Lemma 15.5(b) (with (c_t, c_1) and (c_{s-1}, c_s)). Thus the claim holds, and since any acyclic partial vertex order can be extended into a total order (for example, via topological sorting), step 2 is well-defined.

It remains only to show that the vertex order is a perfect order. Consider any P_4 $P = (v_a, v_b, v_c, v_d)$; we wish to show that P is "well oriented", namely at least one of $v_b < v_a$ or $v_c < v_d$ holds. By relabelling X and Y if necessary, we may assume v_a is in X. By assumption, $G[X]$ and $G[Y]$ are P_4-free, so P contains at least one vertex from Y. Also, co-boxes of a quasi partition are comparable, so P does not have both end vertices in X and both middle vertices in Y. There are four cases to consider.

Case: $P = (x_a, y_b, x_c, y_d)$. In this case, $x_c < x_a$ by Lemma 15.1 with x_a, x_c, y_d. Similarly, $y_b < y_d$. Thus x_c or y_b is the least vertex with respect to $<$ in P, so P is well oriented.

Case: $P = (x_a, x_b, y_c, y_d)$. In this case P is well oriented by Lemma 15.3.

Case: $P = (x_a, x_b, x_c, y_d)$. In this case, $x_c < x_a$ by Lemma 15.1 with x_a, x_c, y_d. If $x_b < x_a$, then P is well oriented; if not, then $x_c < x_a < x_b$, and P is well oriented by step 1 of the algorithm.

Case: $P = (x_a, x_b, y_c, x_d)$. In this case, $x_b < x_a$ by Lemma 15.1 with x_a, x_b, y_c, since x_a, x_b, x_d are all in the same co-box, and P is well oriented. □

Proof of Lemma 9. There are $2^6 = 64$ different neighborhoods a vertex can have on the six vertices of the unordered 6-fig $S = (1, 2, 3, 4, 5, 6)$. In 38 of these cases, the resulting graph induces a bull, P_6, a hole, or an anti-hole. This is illustrated in Figure 7, which shows for each possible neighborhood on S the corresponding forbidden induced subgraph. The remaining 26 cases include the 19 listed in the lemma, plus the following 7, which we now rule out.

Relabel S as $(x_1, x_2, x_3, y_1, y_2, y_3)$. Since x_2 misses x_3, and since x_1 and x_2 have incomparable neighborhoods on Y, x_2 and x_3 are in the same co-box, and x_1 is in a different co-box from x_2 and x_3, and the corresponding results hold for y_1, y_2, y_3. Thus an S-neighborhood is not

$$\{\}, \{x_2\}, \{y_2\}, \text{ or } \{x_2, y_2\}$$

(in each case, v misses x_1, x_3, y_1, y_3, so either x_1 and x_3 or y_1 and y_3 are in the same co-box, contradiction)

$$\text{or } \{x_1, x_3, y_2\} \text{ or } \{x_2, y_1, y_3\}$$

(for the first set, if v is in Y, v, y_1, y_3 are in the same co-box, contradiction; if v is in X, v, x_2, x_3, y_2 contradict 1 of the definition of quasi partition; the second set is symmetric)

$$\text{or } \{x_1, x_2, y_1, y_2\}$$

(assume v is in X, respectively Y; then v, x_2, x_3, y_2, respectively v, y_2, y_3, x_2, contradict 1 of the definition of quasi partition).

Thus we have established the 19 sets listed in the lemma as the only possible neighborhoods on a 6-fig. The adjacency relationships among these sets, as described in Figure 5, follows by forbidding the bull, P_6, holes and antiholes as induced subgraphs. See Figure 8 for further details. □

{}			{34}	C5	{1234}	{234}	C5	{1345}	{1345}	Bu	{2345}
{1}			{35}	P6	{12346}	{235}			{1346}	Bu	{1236}
{2}			{36}	C6	{12346}	{236}	C6	{13456}	{1356}	C5	{1246}
{3}	P6	{12346}	{45}	Bu	{1456}	{245}	Bu	{1246}	{1456}	Bu	{1256}
{4}			{46}			{246}			{2345}	Bu	{1246}
{5}			{56}	P6	{12346}	{256}			{2346}	C5	{1345}
{6}	P6	{12346}	{123}			{345}	C5	{1234}	{2356}		
{12}	Bu	{1234}	{124}			{346}	C5	{1234}	{2456}		
{13}			{125}	Bu	{1234}	{356}	C6	{12346}	{3456}	C5	{1234}
{14}			{126}	C5	{1456}	{456}			{12345}	Bu	{2346}
{15}	Bu	{1345}	{134}			{1234}	Bu	{2345}	{12346}	Bu	{1256}
{16}	C5	{1246}	{135}			{1235}			{12356}		
{23}	P6	{13456}	{136}	C5	{1246}	{1236}	C5	{1456}	{12456}	Bu	{1236}
{24}	Bu	{1246}	{145}			{1245}			{13456}	Bu	{1236}
{25}			{146}			{1246}	Bu	{1236}	{23456}		
{26}	P6	{13456}	{156}	C5	{1246}	{1256}	Bu	{1234}	{123456}		

Fig. 7. Forbidden subgraphs arising from neighborhoods on a 6-fig. For example, the entry {26} P6 {13456} means: if a vertex v is adjacent to only vertices 2,6 of an unordered 6-fig $(1, 2, 3, 4, 5, 6)$, then $\{v\} \cup \{1, 3, 4, 5, 6\}$ induces P_6.

```
                    A B C D E F G H I J K L M N O P Q R S

A = U*  {2356}      . . . b . p . . . c m m m c . . p . .
B = V0  {12356}     . . . . b b b b . c c c c w . . . . .
C = V1  {23456}     . . . . . . . . . w c c c c b . b b b
D = W*  {123456}    b . . . b b b b . b b b b b b . b b b
E = D0  {123}       . b . b . b b p b c b . . . . . . . .
F = C0- {1}         p b . b b . . . b b . . . . . . . . .
G = C0+ {13}        . b . b b . . p b . . . . . . . . . .
H = A0- {235}       . b . b p . p . . c b p b . . . . . .
I = A0+ {1235}      . . . . b b b . . c b . b . . . . . .
J = F0  {134}       c c w b c b . c c . . . . . . . . . .
K = B0+ {124}       m c c b b . . b b . . . . . b b . . .
L = B*  {14}        m c c b . . . p . . . . . . p . . . .
M = B1+ {145}       m c c b . . . b b . . . . . b b . . b
N = F1  {146}       c w c b . . . . . . . . . . c c b . c
O = A1- {256}       . . b b . . . . . . b p b c . . . p p
P = A1+ {2456}      . . . . . . . . . . b . b c . . b b b
Q = C1- {4}         p . b b . . . . . . . . . b . b . . b
R = C1+ {46}        . . b b . . . . . . . . . . p b . . b
S = D1  {456}       . . b b . . . . . . . . b c p b b b .

                    A B C D E F G H I J K L M N O P Q R S

A = U*  {2356}      . b b . . . . b . . . . . . b . . . .
B = V0  {12356}     b . . . . . . . . . . . . . b . . . .
C = V1  {23456}     b . . . . . . b . . . . . . . . . . .
D = W*  {123456}    . . . . . . . . . . . . . . . . . . .
E = D0  {123}       . . . . . . . . . . . . b b b b c c w
F = C0- {1}         . . . . . . . . . . b b b b p . c c c
G = C0+ {13}        . . . . . . . . . . b . b b . . c c c
H = A0- {235}       b . b . . . . . . . . . . . b b p . b
I = A0+ {1235}      . . . . . . . . . . . . . . b b . . b
J = F0  {134}       . . . . . . . . . . b b b b . . b b b
K = B0+ {124}       . . . . . b b . . b . . . b . . b b b
L = B*  {14}        . . . . . b . . . b . . . b . . b . .
M = B1+ {145}       . . . . b b b . . b . . . b . . b b .
N = F1  {146}       . . . . b b b . . b b b b . . . . . .
O = A1- {256}       b b . . b p . b b . . . . . . . . . .
P = A1+ {2456}      . . . . b . . b b . . . . . . . . . .
Q = C1- {4}         . . . . c c c p . b b b b . . . . . .
R = C1+ {46}        . . . . c c c . . b b . b . . . . . .
S = D1  {456}       . . . . w c c b b b b . . . . . . . .
```

Fig. 8. Subgraphs induced by two vertices and an unordered 6-fig (1,2,3,4,5,6). In the upper (lower) matrix, the two vertices are adjacent (non-adjacent). In each row, the set name and vertex neighborhood on the 6-fig is listed; for example, $A = U*$ is the set of vertices whose neighborhood on the 6-fig is $\{2,3,5,6\}$. b=bull, c=C_5, m=C_6, w=$\overline{C_6}$, p=P_6. For example, entry `w` in row B and column N of the upper matrix means that if a vertex $b \in V_0$ is adjacent to a vertex $n \in F_1$, then $\{a,n\} \cup \{1,\ldots,6\}$ induces a subgraph which contains $\overline{C}_6$; entry `b` in row C and column H of the lower matrix means that if a vertex $c \in V_1$ is non-adjacent to a vertex $h \in A_0^-$, then $\{c,h\} \cup \{1,\ldots,6\}$ induces a subgraph which contains a bull.

Lemma 16 *Let Q be any co-box in a dq-partition X, Y of a graph in $\mathcal{G}$. Then*

1. *any vertex partial on a $\overline{P}_3$ (q_1, q_2, q_3) of $G[Q]$ sees q_2,*
2. *any vertex partial on a P_3 (q_1, q_2, q_3) of $G[Q]$ misses q_2,*

and for any doubly lexical co-order $<$ of X, Y,

3. *for every $\overline{P}_3$ (q_1, q_2, q_3) of $G[Q]$, $q_2 < q_1$ and $q_2 < q_3$,*
4. *for every P_3 (q_1, q_2, q_3) of $G[Q]$, $q_1 < q_2$ and $q_3 < q_2$.*

Proof of Lemma 16. Argue 1 by contradiction. Consider a $\{q_1, q_2, q_3\}$-partial vertex z which misses q_2. By the definition of quasi partition, z sees both q_1 and q_3; also by this definition, the co-box containing $\overline{P}$ is comparable, so any $\{q_1, q_3\}$-partial vertex y not in the co-box misses q_2. Such a y exists, since the quasi partition is distinguished. But now y contradicts the first part of the definition of quasi partition. A similar argument establishes 2.

Argue 3 by contradiction; then q_2 is not the $<$-largest vertex of $\{q_1, q_2, q_3\}$. Since the co-box is comparable and distinguished, some vertex in the other part sees only the $<$-largest of any vertex subset of the co-box, contradicting 1. A similar argument establishes 4. □

Proof of Lemma 10. First consider 1. B_0 is in X by definition. $A_0^\pm \cup C_0^\pm \cup D_0$ is in X by the definition of quasi partition, since otherwise a_Y and c_Y and so also b_Y are in the same co-box although a_Y and b_Y have incomparable X-neighborhoods. F_0 is in X by the definition of quasi partition with b_X, c_X, c_Y. Thus 1 holds.

Next consider 2 and 3. First observe that $B_0^+ \cap Y$ is empty by the definition of quasi partition with b_X, c_X, c_Y. All vertices of $B_0 \cup B_0^+ \cup \{b_X\} \cup F_0 \cup C_0^\pm \cup \{c_X\}$ are in the same co-box of X, and so have pairwise comparable Y-neighborhoods. Since the order is doubly lexical, it follows that $x < x^*$ for two vertices in the same co-box of X if and only if x's Y-neighborhood contains x^*'s Y-neighborhood. In particular, this implies that $(B_0 \cup B_0^+ \cup \{b_X\} \cup F_0) < (C_0^\pm \cup \{c_X\})$. Since the 6-fig S is maximal, $C_0^\pm < c_X$ and $(B_0 \cup B_0^+) < b_X$. Since (f, b_X, c_X) is a $\overline{P}_3$ for every f in F_0, $b_X < F_0$ by Lemma 16. Thus 3 holds.

We now finish the proof of 2, having proved 1 and 3 and that $B_0^+ \cap Y$ is empty. $B_0^+ \cap X$ is empty, for otherwise the $\overline{P}_3$ (b_0^+, c_X, b_X), contradicts Lemma 16, since $b_0^+ < c_X$ and $b_X < c_X$; thus B_0^+ is empty. B_0 is an independent set, for otherwise the $\overline{P}_3$ (b_0, c_X, b_0^*) formed with two adjacent vertices b_0, b_0^* of B_0 contradicts Lemma 16. Let d and d^* be any two vertices of D_0. Since the partition is distinguished, there is some $\{d, d^*\}$-partial vertex y in Y; by Lemma 9, y is in U_*, V_1 or B_1. It follows that d and d^* are adjacent, for otherwise y, d, d^* induces a bull with b_X, b_Y if y is in U_* or V_1, or with a_X, a_Y if y is in B_1. Thus D_0 is a clique. B_0 is $(B_1 \cup F_1 \cup C_1^\pm)$-null by part 3 of the definition of quasi partition with c_X, c_Y, a vertex in B_0, and a vertex in $B_1 \cup F_1 \cup C_1^\pm$. R_0 is $(B_1 \cup F_1 \cup C_1^\pm)$-universal since co-boxes are comparable and R_0 sees c_Y. Also, R_0 is $(B_0 \cup F_0 \cup C_0^\pm)$-universal; assuming the contrary, let r be a vertex of R_0 which misses a vertex γ_X of $B_0 \cup F_0 \cup C_0^\pm$; r and γ_X and so c_X are in the same co-box, so $r < (B_0 \cup \{b_X\} \cup F_0 \cup C_0^\pm \cup \{c_X\})$, since co-boxes are comparable and r sees c_Y; now (r, γ_X, c_X) or (r, γ_X, b_X) is a $\overline{P_3}$, contradicting Lemma 16. Thus 2 holds.

Next consider 4. Let α_X be any vertex in $\{a_X\} \cup A_0^\pm$. For any γ_X in $\{c_X\} \cup C_0 \cup D_0$, $\alpha_X < \gamma_X$ by Lemma 15.3 with $\alpha_X, \gamma_X, a_Y, b_Y$. It remains to show that $R_0 < (\{a_X\} \cup A_0^\pm)$.

For any u in $X \cap (U_* \cup V_0)$, $u < \alpha_X$ by Lemma 15.3 with α_X, u, a_Y, c_Y. Let α_X^* be any vertex in $\{a_X\} \cup A_0^-$. For any v in $X \cap V_1$, $v < \alpha_X^*$ by Lemma 15.1 with α_X^*, v, a_Y since v and α_X are in the same co-box; it follows that $v < \alpha_X$ (assuming the contrary, $\alpha_X < v$ and by the argument so far α_X is in A_0^+ and $\alpha_X < v < a_X$; since $\alpha_X < v$, the lexically largest $\{v, \alpha_X\}$-partial vertex of Y, say w, misses v and sees α_X and is larger than c_Y; by the argument so far, the only vertices in Y larger than c_Y are in D_1 or W_*; since w sees α_X, w is not in D_1 and so is in W_*; but now v, a_X, w contradict Lemma 15.1).

Let w_0 be any vertex in $X \cap W_*$ such that $\alpha_X < w_0$. Let W_0^+ be the vertices w_0^+ of $W_* \cap X$ satisfying $(\{c_X\} \cup D_0) < w_0^+$, and let W_1^+ be the corresponding vertices of $W_* \cap Y$. We claim that

- w_0 is in W_0^+,
- every w_0^+ in W_0^+ misses some vertex in W_1^+,
- every vertex in Y which misses some w_0^+ in W_0^+ is in W_1^+.

To prove the claim, first observe that by Lemma 15.2 with α_X, w_0, c_Y there is a lexically largest $\{\alpha_X, w_0\}$-partial vertex z in Y which sees α_X but not w_0 and is larger than c_Y. Then z is in W_*, since by the argument so far the only vertices in Y larger than c_Y are in D_1 or W_*, and z sees α_X whereas D_1 is $\{a_X\} \cup A_0^{\pm}$-null. Relabel z as w_1. Now the first part of the claim holds (otherwise, there is some δ_X in $\{c_X\} \cup D_0$ such that $w_0 < \delta_X$, and the largest $\{w_0, \delta_X\}$-partial vertex y in Y sees w_0 but not δ_X and is larger than w_1 by Lemma 15.2 with w_0, w_1, δ_X; by our choice of w_1, y is not $\{\alpha_X, w_0\}$-partial, so sees α_X, a contradiction since the only vertices in Y larger than c_Y are in W_* or D_1, and no such vertex sees α_X and misses c_X). Since $c_Y < w_1$, repeating the above argument implies that w_1 is in W_1^+, so the second part of the claim holds. The third part of the claim follows by observing that $\{c_Y\} \cup D_1$ is W_*-universal, so any vertex y in Y which misses a vertex of W_0^+ must be lexically larger than $\{c_Y\} \cup D_1$; by the argument so far, any such vertices are in W_1^+. Thus the claim holds.

Let W^+ be $W_0^+ \cup W_1^+$ and let W^- be $W_* - W^+$. We complete the proof of 4 by showing W^+ is empty; suppose not. Since the graph is co-connected, for every w in W^+ there is a co-path, namely a path in the complement of the graph, connecting w with $\{a_X, a_Y\}$; over all possible w, consider a shortest such co-path $P = (p_0, \ldots, p_k)$ with $p_0 = w$ and p_k in $\{a_X, a_Y\}$. Since P is shortest, $P \cap W^+ = \{p_0\}$ and $P \cap \{a_X, a_Y\} = \{p_k\}$. By relabelling X and Y if necessary, we may assume that p_0 is in X; then p_1 is in X, for otherwise p_1 is in W_1^+ by the claim and $P - \{p_0\}$ contradicts the choice of P. By the claim, there is a vertex p_{-1} in W_1^+ which misses p_0; by the claim, p_{-1} sees p_1, for otherwise p_1 is in W_0^+. Let j be the largest index such that $p_0, \ldots, p_j$ is in X; if $j = k$ then $p_j = a_X$ and we define p_{j+1} as a_Y. Observe that p_0 and p_j are in the same co-box but are incomparable on the subset $\{p_{-1}, p_{j+1}\}$ of Y. This contradiction of the definition of quasi partition completes the proof of 4.

Next consider 5. Let f_0 be a vertex in F_0. Since the partition is distinguished, there is some $\{b_X, f_0\}$-partial vertex in Y; since $b_X < f_0$, the largest such vertex sees b_X but not f_0; this and Lemma 9 implies that the vertex, say a_1, is in $A_1^{\pm}$ and so misses a_X. Notice that $b_Y < a_1$ (otherwise the fact that the 6-fig is maximal implies $a_1 < a_Y$, and a_X, f_0, a_1, a_Y contradict Lemma 15). Since $a_X < b_X$, the largest $\{a_X, b_X\}$-partial vertex in Y sees a_X but not b_X, and so is larger that a_1. By Lemma 9, such a vertex must be in B_1 or F_1; the former case is ruled out by the maximality of the 6-fig and the fact that $B_1 < c_Y$. Thus there is some vertex f_1 in F_1. Now 5 follows by repeating the above argument, and the fact that $A_j^{\pm} < D_j$ for $j = 0$ and 1. □

Proof of Lemma 11. Given two vertices in the same part of a bipartition and a vertex order complete on each part of the bipartition, define the *largest distinguisher* of the two vertices to be the largest (with respect to the order) vertex of the other part which is partial on the two vertices. Let $\mathcal{D}(x_1, x_2, <)$ denote the largest $\{x_1, x_2\}$ distinguisher with respect to order $<$. To prove 1, we wish to show that $x_1 <_* x_2$ if and only if $\mathcal{D}(x_1, x_2, <_*)$ misses x_2. Lemma 10 and the fact that $<$ is doubly lexical imply that C_0^* is null on $C_1^* \cup A_1^* \cup B_1^*$, that A_0^* is null on $C_1^* \cup A_1^*$ but not B_1^*, that B_0^* is null on C_1^* but not A_1^*, and that R_0^* is not null on C_1^*, from which it follows that, for fixed $j = 0$ or 1,

$$R_j^* \;<_*\; B_j^* \;<_*\; A_j^* \;<_* C_j^* \,.$$

To conclude the proof of 1, it is sufficient to show that $\mathcal{D}(x_1, x_2, <) = \mathcal{D}(x_1, x_2, <_*)$ for x_1, x_2 both in C_0^* or A_0^* or B_0^* or R_0^*. By the definition of $<_*$, the orders $<$ and $<_*$ agree on $Y - A_1^*$; since they also agree on A_1^*, and since $R_1^* < A_1^* < C_1^*$, it follows that $<$ and $<_*$ agree on $Y - B_1^*$. Thus to show the above equality, it is sufficient to show that $\mathcal{D}(x_1, x_2, <)$ and $\mathcal{D}(x_1, x_2, <_*)$ are both in $Y - A_1^*$ or both in $Y - B_1^*$.

Suppose x_1, x_2 are both in R_0^*. If both are in R_0 or in B_0, then $\mathcal{D}(x_1, x_2, <)$ and $\mathcal{D}(x_1, x_2, <_*)$ are both in $Y - B_1^*$, since B_1^* is R_0-universal. If exactly one is in R_0, then $\mathcal{D}(x_1, x_2, <)$ and $\mathcal{D}(x_1, x_2, <_*)$ are both in C_1^* so both in $Y - B_1^*$, since c_Y is $\{x_1, x_2\}$-partial, and $Y - C_1^*$ precedes C_1^* in both $<$ and $<_*$. Suppose x_1, x_2 are both in B_0^*. Then $\mathcal{D}(x_1, x_2, <)$ and $\mathcal{D}(x_1, x_2, <_*)$ are both in $Y - B_1^*$, since B_0^* is B_1^*-null. Suppose x_1, x_2 are both in A_0^* or in C_0^*. Then $\mathcal{D}(x_1, x_2, <)$ and $\mathcal{D}(x_1, x_2, <_*)$ are both in $Y - A_1^*$, since $A_0^* \cup C_0^*$ is A_1^*-null. Thus 1 holds.

To prove 2, argue by contradiction. Let $(x_1, x_2, x_3, y_1, y_2, y_3)$ be as stated in the lemma. Let β_0 and β_1 be $\{b_X\} \cup B_0 \cup F_0 \cup \cup \{c_X\} \cup C_0^{\pm}$ and $\{b_Y\} \cup B_1 \cup F_1 \cup \cup \{c_Y\} \cup C_1^{\pm}$ respectively.

First suppose that $x_3 = c_X$. Since x_2 misses x_3, x_2 is in $\beta_0 - \{c_X\}$. Since $x_1 <_* x_2$ and x_1 and x_2 are in different co-boxes, x_1 is in R_0. Since y_1 sees x_2 but not x_3, y_1 is β_0-partial and so in $A_1 \cup D_1$. Since $y_1 <_* y_2$ and $y_1 <_* y_3$ and y_2 misses y_3, y_1 is not in D_1 (in $<_*$, the only vertices after some d_1 in D_1 are in $c_Y \cup D_1$, which induces a clique), so is in A_1. In $<_*$, the only vertices after some a_1 in A_1 are in A_1^* or C_1^*, so y_2 and y_3 are in $A_1^* \cup C_1^*$. This and the fact that y_2 misses y_3 implies that y_2 and y_3 are both in A_1. Now x_2 in β_0 misses y_1 in A_1; since b_X is A_1-universal, and since co-boxes are comparable, the Y-neighborhood of b_X contains the Y-neighborhood of x_2, so $b_X <_* x_2$. Since x_1, x_2, x_3 are all in $X - A_0^*$ and y_1, y_2, y_3 are all in $Y - A_1^*$, $(x_1, x_2, x_3, y_1, y_2, y_3)$ is a 6-fig for $<$, with $b_X < x_2$ and $c_X = x_3$, contradicting the lemma hypothesis that b_X is maximum for c_X.

Next suppose that $c_X <_* x_3$. Thus $c_X < x_3$ and x_3 is in D_0. Since x_2 misses x_3 and $x_2 <_* x_3$, x_2 is in B_0 or R_0; since $x_1 <_* x_2$, x_1 is in R_0. Thus x_1, x_2, x_3 are all in $X - A_0^*$, so $x_1 < x_2 < x_3$. Since y_1 misses x_1 in R_0, y_1 is not in β_1; arguing as in the previous case, y_1 is not in D_1. If y_1 is in A_1, then y_2 and y_3 are in A_1 (again arguing as in the previous case). If y_1 is in R_0, then $y_1 <_* y_2$ and $y_1 <_* y_3$ imply $y_1 < y_2$ and $y_1 < y_3$. Also, $y_2 <_* y_3$ and y_2 misses y_3 implies $y_2 < y_3$ (since co-boxes are comparable). Thus $y_1 < y_2 < y_3$, and

$(x_1, x_2, x_3, y_1, y_2, y_3)$ is a 6-fig for $<$, with $c_X < x_3$, contradicting the lemma hypothesis that c_X is maximum. □

Proof of correctness of Algorithm 12. By Theorem 7.1 and the correctness of Algorithm 8, it suffices to show that the order $<_\sim$ returned by steps 0 through 3 is a doubly lexical co-order with no 6-fig. Step 0 is well-defined, since any matrix has a doubly lexical co-order. Consider step 1. Assume that steps 2 and 3 are correct, namely that $(X', Y', <_\sim)$ is a 6-fig-free doubly lexical co-order of $G[X' \cup Y']$. Then $(X, Y, <_\sim)$ is doubly lexical co-order of G (the lexical vectors of each added vertex is $[0, 0, \ldots, 0]$) which induces no 6-fig (every vertex in a 6-fig is null or partial on the other part), so step 1 is correct. Next consider step 2. Assume that step 3 is correct, namely that for every j, $(X_j, Y_j, <_\sim)$ is a 6-fig-free doubly lexical co-order of $G[X_j \cup Y_j]$. Then $(X', Y', <_\sim)$ is a doubly lexical co-order of $G[X' \cup Y']$ (since for all j X_j is $(Y - Y_j)$-universaland not Y_j-universal, and Y_j is $(X - X_j)$-universal and not $X - j$-universal) and 6-fig-free (since the subgraph induced by a 6-fig is co-connected), so step 2 is correct.

Next consider step 3. There is nothing to prove for step 3a, so consider step 3b. For every j, the induced partition X_j^*, Y_j^* of G_j is a quasi partition (every induced subgraph of a quasi partition is a quasi partition) and distinguished (by step 1, every x_j^* in X_j^* misses some vertex y of Y; by the definition of G_j, every such y is in Y_j and so at least one is in Y_j^*; by the definition of the equivalence classes, no two vertices of X_j^* have the same Y_j^*-neighborhoods), and $G[X_j^* \cup Y_j^*]$ is co-connected. To see this, let x_j and x_j' be two vertices of X_j in the same equivalence class. Then $G' = G[X_j - \{x_j\} \cup Y_j]$ is co-connected. (If not, let S be vertex set of the co-component of G' containing x_j' let T be the vertex set of any other co-component of G', and relabel x_j' as x_s. Since $G[X_j \cup Y_j]$ is co-connected and G' is not, x_j misses at least one vertex in each of S and T. Since x_s is T-universal, x_j is $T \cup Y$universal, so x_j misses some x_t in $T \cup X$. Every vertex of X_j misses at least one vertex of Y_j, so x_s misses some y_s in Y and so in $Y \cup S$, and x_t misses some y_t in Y and so in $Y \cup T$. But then x_j and x_t are incomparable with respect to $\{y_s, y_t\}$, a contradiction since x_j are non-adjacent and so in the same co-box.) Deleting vertices one at a time, and repeating the above argument, it follows that $G[X_j^* \cup Y_j^*]$ is co-connected. Thus the hypotheses of Lemma 11 apply to $G_j[X_j^* \cup Y_j^*]$, and $(X_j^*, Y_j^*, <_*)$ is a doubly lexical co-order with no 6-fig $(x_1, x_2, x_3, y_1, y_2, y_3)$ such that the rank of x_3 in $<_*$ is the same as or greater than the rank of c_X in $<$. Thus, step 3b will be repeated at most $|X_j| - 2$ times (once for each possible rank of c_X) terminating with $(X_j, Y_j, <_\sim)$ a doubly lexical co-order with no 6-fig. Finally, the correctness of step 3c follows by observing that the extension preserves the doubly lexical co-order (since vector added is added at the same position as an identical vector) and creates no 6-fig (no two vertices in a 6-fig have the same neighborhood on the other part). □

7 Conclusions

We have given an algorithmic proof that P_6-free bull-free weakly chordal graphs which admit a co-box partition are perfectly orderable. This together with results of de Figueiredo, Maffray, and Porto implies that all bull-free graphs with no odd hole and no anti-hole are perfectly orderable. An interesting open problem posed by Chinh Hoàng [HT] is whether all P_6-free graphs with no anti-hole are perfectly orderable.

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