submitted March 2003 to Discrete Mathematics; revised January 2004

Notes on Hex

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Abstract

Inspired by Claude Berge's interest in and writings on Hex, we survey some results on the game.

Key words: game of Hex

1 Introduction

Claude Berge was a huge Hex fan. He played the game frequently and even wrote two articles on it: "*L'Art Subtil du Hex*" [5], an introduction intended for a general audience,[1] and "*Some Remarks about a Hex Problem*" [6], a puzzle commentary included in a book published in tribute to Martin Gardner. Being both an avid Hex player and a mathematician, Berge was interested in the mathematical aspects of the game. As he stated in [6],

> *It would be nice to solve some Hex problem by using nontrivial theorems about combinatorial properties of sets (the sets considered are groups of critical [locations on the board]). It is not possible to forget that a famous chess problem of Sam Loyd (the "comet"), involving parity, is easy to solve for a mathematician aware of the König theorem about bipartite graphs; also, in chess, the theory of conjugate squares of Marcel Duchamp and Alberstadt is a beautiful application of the algebraic theory of graph isomorphism (the two graphs are defined by the moves of the kings).*
>
> *The use of a mathematical tool may be unexpected and therefore adds some new interest to a game; but Hex exists as a most enjoyable game in its own right, for mathematician and layman alike.*

⋆ The first authour gratefully acknowledges the support of NSERC.

[1] Berge wrote the article so that it could be included in a commercial version of the game to be produced in France. Unfortunately, the venture did not come to fruition and the article was not widely distributed. A translated version will appear in [18].

Fig. 1. Hex on a 6×6 board: the empty board (left) and a game which White has won (right).

In this paper we survey some results on Hex, keeping the preceding comments in mind. Our paper is loosely modelled after Berge's introduction [5] and so includes sections on Hex rules, origins, fundamental properties, virtual-connections, openings, and puzzles; we have also added a section on computers.

2 Rules

The two-player game Hex takes its name from the playing board, which consists of a rhombus-shaped $n \times n$ array of hexagonal cells, where n can be any positive integer. Hex boards are shown in Figure 1. Berge's favourite board size was 14×14.

The players, say Black and White, are each assigned two opposite sides of the four boundary sides of the board, as indicated in the diagrams by the four stones placed off the board. Each player is also assigned a collection of coloured stones or playing pieces. Each player in turn places a stone onto a vacant cell. The goal is to form a single-coloured chain connecting the two correspondingly coloured sides. Each of the four corner cells is considered adjacent to both sides with which it is incident. The diagram on the right in Figure 1 shows the final state of a game won by White.

An unrestricted opening move is a considerable advantage for whomever plays first, so an additional rule which limits the opening is usually adopted. One such variation uses the *swap rule*: colours are assigned to the sides of the board; one player places a stone of either colour on any cell; the other player then chooses which colour to own. The game then continues as normal, with the next move played by whomever owns the opposite colour of the stone already on the board.

The swap rule effectively forces the first player to play a neutral opening, as the second player is likely to make the swap if the opening move seems advantageous. For example, the cell in the second row from each of two sides which meet at an acute corner is a common opening and was often played by Berge.

3 Origins and History

In [5], Berge uses both "invention" and "discovery" to refer to the origins of Hex, raising the question of whether the game is more an artefact of man or of mathematics. An argument in support of the latter is that the game was introduced on two separate occasions, and that on each occasion its genesis had a mathematical connection.

Hex was first invented in 1942 by Piet Hein (1905-1996), the well-known Danish mathematician, engineer, architect, and poet. According to Martin Gardner [15], Hein thought of the game while pondering the then-unsolved Four Colour Problem at the Institute for Theoretical Physics in Copenhagen. [2] Hein wrote a series of articles on the game, initially called "Polygon", which appeared in the Danish newpaper *Politiken* in December 1942 [20]. He soon afterwards formed the company Skjøde Skern, which marketed the game under the name of "Con-Tac-Tix".

Hex was reinvented by Nobel laureate John Nash (1928-) sometime around the fall of 1948, when he arrived at Princeton as a mathematics student [28]. Although not recalling the details surrounding the invention, he does remember that it was "a matter of connecting topology and game theory" [29]. Nash's original version of the game was abstract; he was apparently interested in it more for its game-theoretic properties than because he thought it might be interesting to play. In the spring of 1949 Nash described the game to David Gale, a Princeton instructor who briefly served as Nash's stand-in supervisor. Nash's description on that occasion was of a game which could be played on a checkerboard, with squares considered adjacent if they shared an edge or an upper-right/lower-left corner. Gale realized that a rhombus-shaped array of hexagons would make a more suitable board, and built such a board that very day. Gale subsequently donated the board to the Princeton math department common room, where the game, called "Nash", became immediately popular [14,28].

From these two starting points the game spread, especially via the world's mathematics communities. Parker Brothers produced a version of the game under the name of "Hex" in the early 1950s, but the game did not reach a wide audience until 1957, when Martin Garder introduced it to the readers of his *Scientific American* columns [15]. Berge most likely learned of the game in the fall of 1957, when he arrived at Princeton for the start of a yearlong academic visit.

Hex continues to be popular to this day. A 9×9 board version has been commercially available since 1991 [30]. Hex is frequently mentioned in books ranging from general mathematics [3,4] to advanced graph theory [21], while the first book dedicated solely to the game appeared in 2000 [8]. There is also a thriving online community; for example, the 10×10, 14×14, and 18×18 games can be played at `www.playsite.com`, while `www.gamerz.net/pbmserv/` offers a game of any size. Hex has also been played at recent Computer Games Olympiads [1,25].

4 Fundamental Properties

As Berge remarked in [5], the appeal of Hex to mathematicians seems paradoxical:

> *Pieces never move once played; the rules are simple and the game can be played at any age. One might therefore wonder why Hex seems so interesting to mathematicians.*

Hex is played on a graph and so of interest to graph theorists [21]. A reason for its broader

[2] We would be interested to learn of additional references concerning Hein's invention of Hex.

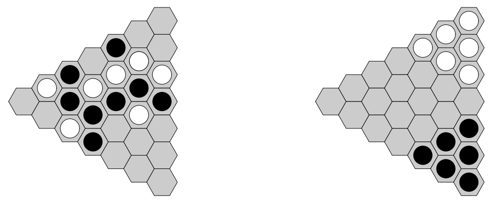

Fig. 2. The Game of Y: a game won by Black (left) and Hex as a special case of Y (right).

appeal is reflected in the following well-known results, which indicate that for Hex, simplicity is an indication of elegance and complexity rather than triviality.

Theorem 1 *For the game of Hex played on an $n \times n$ board,*

(1) (Hein [20], Nash [15]) the game cannot end in a draw;
(2) (Nash [15]) at any point in a game, the placement of an extra stone for a player is not disadvantageous for that player;
(3) (Nash [15]) there exists a winning strategy for the first player;
(4) (Reisch [31]) starting from an arbitrary placement of black and white stones, determining the winner is PSPACE-complete.

A proof of (1) follows from the fact that *an $n \times m$ Hex board with all cells occupied has a winning chain for at least one of the two colours.* $(*)$

As with many topological results, $(*)$ requires some care to prove. The earliest rigorous proof we are aware of, given by Beck in 1969 [3], uses induction and the Jordan Curve Theorem. Gale later gave an algorithmic proof [13] using only elementary graph theory and the fact that the edges of three mutually intersecting hexagons form the complete bipartite graph $K_{1,3}$. Gale further showed that $(*)$ is equivalent to the Brouwer fixed point theorem.[3] We present here an unpublished proof due to Ea Ea, formerly known as Craige Schensted, which involves a game closely related to Hex, namely the game of Y.[4] Ea Ea's proof actually implies the strengthening of $(*)$ obtained by replacing "at least one" with "exactly one".

Like Hex, Y was discovered on more than one occasion, including by Claude Shannon [15] and independently Ea Ea [32] in the early 1950s. A Y board consists of a triangular configuration of hexagons. As in Hex, each of the corner cells belongs to both of its bordering sides. The goal is to build a chain which connects all three sides, as shown in Figure 2.

Y is a generalization of Hex, as illustrated in Figure 2: playing Y from the state shown in the right diagram is equivalent to playing Hex in the unoccupied cells of this state, since a chain wins for Y if and only if the chain restricted to the originally unoccupied cells wins for Hex. Thus to prove $(*)$, it suffices to prove that *a Y board with all cells occupied has a winning chain for exactly one of the two colours.* $(**)$

[3] This theorem says that any continuous mapping of an n-dimensional ball onto itself must have a point which maps to itself. The Nash equilibrium theorem in game theory uses this fact. See [27,22].

[4] We are grateful to Steven Meyers for communicating this result to us. Meyers independently found an alternate proof of $(**)$: a Y k-board with all cells occupied has a winning chain for colour c if and only if at least two of the three $k-1$ boards obtained by removing the k cells along a side have a winning chain for c [26].

Fig. 3. The triangle-cell correspondence used in Schensted's Y-reduction: cell (x, y) of the smaller board corresponds to triangle $\{(x, y), (x{+}1, y), (x, y{+}1)\}$ of the larger board.

Proof of (1). As mentioned, it suffices to prove $(**)$, since $(**)$ implies $(*)$ which in turn implies (1). Ea Ea's proof of $(**)$ uses a process called *Y-reduction* illustrated in Figure 4.

A *k-board* is a board on which each side has k-cells. In one Y-reduction step, a completely filled k-board is reduced to a completely filled $(k-1)$-board by replacing each "triangle" consisting of three pairwise adjacent hexagons oriented the same way as the board with a single hexagon whose colour is that of the majority of the colours of the triangle. For example, for the Cartesian coordinate cell labelling shown in Figure 3, triangle $\{(x, y), (x + 1, y), (x, y + 1)\}$ of the k-board corresponds to cell (x, y) of the $(k - 1)$-board.

The Y-reduction process is repeated until the original k-board has been reduced to a 1-board as shown in Figure 4. Now $(**)$ follows from the easily verified fact that among the sequence of boards in the reduction, one board has a winning chain with a particular colour if and only if the next board has a winning chain with that colour. □

A *board-state* B is defined by specifying for each board cell whether it is unoccupied, occupied by a black stone, or occupied by a white stone. Once stone sets have been assigned to players, we refer to a *P-stone* as a stone belonging to player P. A *game-state* $G = (B, P, Q)$ is defined by specifying a board-state B, the player with the next move P, and P's opponent Q.

Proof of (2). Let $S = (B, P, Q)$ be a game-state for which P has a winning strategy W, let c be an unoccupied cell in S, let B' be the board configuration obtained by adding a P-stone at c, and let $S' = (B', P, Q)$. Let W' be the P-strategy obtained from W by avoiding c: the response to each Q-move is exactly as in W, except that if the response cell is already occupied by a P-stone, then move to any unoccupied cell. It is easy to show by induction on the number of unoccupied cells that W' is well-defined and a winning strategy for S'. In the base case, there are no unoccupied cells in S'; since P wins S, Q has no winning chain in S and so also no winning chain in S', and so (1) implies that P has a winning chain in S'. □

Proof of (3). Nash's classic proof of this result, which holds for any game satisfying (2), uses a form of proof by contradiction now known as *strategy stealing*. Suppose that the second player Q has a winning strategy W. The first player P can then "steal" W, by treating the opening move as an extra P-stone placed on the board and then following the "extra-stone" strategy W' obtained from W as described in the proof of (2). Since W is a winning strategy for Q, it follows from (2) that W' is a winning strategy for P. But then both players have a winning strategy, contradicting (1). Thus (3) holds. □

Observe that Nash's proof holds for any board which "looks the same" for each player, namely with a board automorphism which preserves cell adjacency and maps one player's sides to

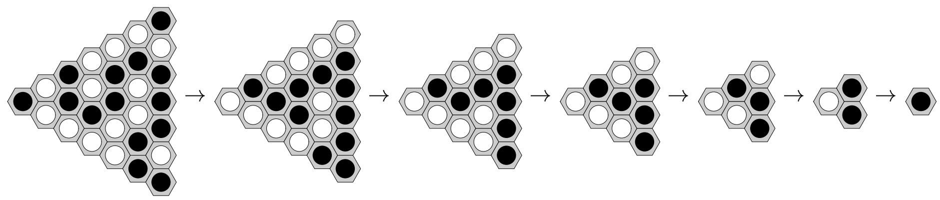

Fig. 4. Schensted's Y reduction. Each board has a winning black chain.

the other player's sides. For example, the proof holds for a hexagon-shaped board whose edges in cyclic order belong respectively to Black, Black, White, White, Black, White. On the other hand, the proof does not hold for asymmetric boards, and (3) does not necessarily hold either. For example, for $n \times (n-1)$ boards, the player whose sides are closer together has a winning pairing strategy regardless of which player starts: the player P with the longer sides can play so that any side-to-side Q-chain implies the existence of a reflected side-to-side P-chain, contradicting the strengthening of $(*)$ mentioned earlier [15].

Nash's proof is tantalizing: other than showing existence, the proof reveals nothing about any first-player-win strategy. From the time that Nash's proof was first known, people have wondered: how difficult is it to find such a strategy?

In order to describe three results which answer this question, we introduce two more games closely related to Hex. The *Shannon-colouring game*, also known as the *Shannon-switching game*, is played on a graph. Two vertices are initially coloured black; players then alternately colour an edge. Black wins by completing a black path between the original two vertices, while White wins by blocking all such paths. The *Shannon-vertex-colouring game* is defined similarly, except that players colour vertices instead of edges. Notice that the Shannon-vertex-colouring game is a generalization of Hex.

In 1976 [12] Even and Tarjan showed that the Shannon-vertex-colouring game is PSPACE-complete. In 1981 [31] Reisch showed that the same conclusion holds for Hex itself, namely that (4) holds; as the proof is long and complicated, we will not reproduce it here.

Reisch's result suggests that the answer to the preceding question is "probably very difficult", with the caveat that it might be easier to find a winning strategy for opening Hex game-states, namely those in which there are only a few stones on the board, than for arbitrary Hex game-states. In any event, there is at present no evidence to suggest that finding a winning Hex strategy is "easy".

Easily solved games are usually not interesting to play, so Reisch's result was greeted with relief by many Hex players, especially since explicit winning strategies had been discovered earlier for Bridg-It, a special case of the Shannon-colouring game invented by Gale, and for the Shannon-colouring game itself: Gross found a pairing strategy for Bridg-It in 1961 [16], while Lehman used matroid theory to describe a winning strategy for the Shannon-colouring game in 1964 [24].

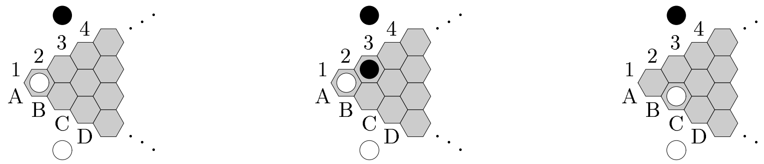

Fig. 5. On a board of size 3×3 or larger: a losing opening for White (left), a losing reply for Black (center), and another losing opening for White (right).

5 Openings

The *value* of a game-state is the winner assuming perfect play. To *solve* a game-state is to determine its value, namely whether there is a winning strategy for the player who is about to move. The fact that Hex is PSPACE-complete suggests that there is likely to be no polynomial time algorithm to solve arbitrary Hex game-states. Even for the initially empty $n{\times}n$ board, no polynomially computable winning strategy which holds for arbitrary n is known. In light of the swap rule, there is considerable interest in knowing which opening move sequences lead to wins and which do not.

A few results are known which hold for boards of arbitrary size. Using a strategy-stealing argument, Beck showed [3] that opening in the acute corner, as shown in Figure 5, is a loss on any board size: if White had a winning strategy for opening in the acute corner, say as shown as cell A1 in the left diagram in Figure 5, then Black could steal the same strategy to win after replying at A2 as in the center diagram, proving that White has no such winning strategy. Notice that it does not follow from Beck's argument that A2 is a winning Black reply to A1. In fact, Beck and Holland later showed that this is a losing response for Black; they also showed that opening at B1 as in the right diagram of Figure 5 loses for White [4].

Results such as these can be proved for Rex, also known as Reverse Hex or Misère Hex, where connecting the two sides *loses* the game. Gardner mentions that Robert Winder showed that Rex is a first player win if and only if the board size is even [15]; a proof is given by Lagarias and Sleator [23], who show that the eventual loser can force the entire board to be filled. Evans proved that the opening in the acute corner is a Rex win on even boards [11].

For boards of fixed size, many results are known. Solving game-states on boards up to size 5×5 is within human capacity. Enderton announced solutions to all 6×6 single-move openings [10]. Using a computer, Van Rijswijck independently verified Enderton's results and extended them by solving many 6×6 multiple-move openings [34]; an interactive library of these solutions is available online [33]. By hand, Yang solved several 7×7 single-move openings [37–39] and the centermost 8×8 and 9×9 single-move openings [36]. Recently, Hayward et. al. determined the outcome all 7×7 single-move openings by computer [19]. These results are summarized in Figure 11, which shows all winning and losing opening moves on all board sizes up to 7×7.

In the next three sections we describe the ideas behind some of these recent opening results.

6 Strategy: virtual-connections

As Berge remarked in [5], Hex is a subtle game with considerable strategic depth. Our discussion in this section, though, will be limited to one fundamental aspect of Hex strategy, namely the notion of a virtual-connection. For a further discussion of strategy, see Cameron Browne's book *Hex Strategy: Making the Right Connections* [8].

At some point during a Hex game, a pair of cells might be "virtually connected": regardless of how the opponent plays, the player can connect the pair. For example, in Figure 6 the two black stones are virtually connected via the two unoccupied cells; if at any time White plays in one of the unoccupied cells, Black can play in the other and so form the connection. A virtual-connection of this form, consisting of two unoccupied cells mutually adjacent to the two destination cells, is called a *bridge*.

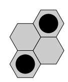

Fig. 6. A bridge virtual-connection.

Roughly speaking, a *virtual-connection* for a player is a cell-to-cell second-player-win subgame for that player: the player can connect the two destination cells even if the opponent plays first. Analogously, a *(virtual) semi-connection* is a first-player-win subgame for a player: the player can connect the two cells if allowed the first move.

As Berge observed, two virtual-connections with a common occupied destination combine to form a larger virtual-connection if the associated sets of connecting cells do not interesect; for example, the six Black virtual-connections in Figure 7 whose unoccupied cell sets are marked u, v, w, x, y, z respectively combine to form a side-to-side virtual-connection for Black [5].

Anshelevich gave a more extensive set of rules for combining virtual- and semi-connections [2]. Before describing these rules, we first present formal definitions for such connections. Although our notation is slightly different, the definitions given here are essentially those given in [2].

Let S, X, Y be pairwise non-intersecting cell sets with X and Y connected and all cells

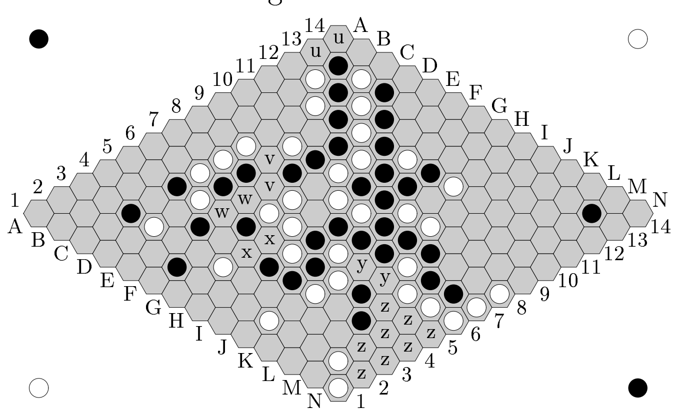

Fig. 7. From [5]. The Black virtual-connections with respective carrriers v, w, x, y, z combine to form a side-to-side, and so winning, virtual-connection.

in S unoccupied. For a fixed board-state B and a player P, the *subgame* $P(B{:}X,S,Y)$ is the game of Hex restricted to playing in S, where P wins by forming a chain of P-stones connecting X and Y. $P(B{:}X,S,Y)$ is a *semi-connection* if there is a winning strategy for this subgame assuming P plays first, and a *virtual-connection* if there is a winning strategy for this subgame even if P's opponent plays first. For a virtual- or semi-connection $P(B{:}X,S,Y)$, S and X, Y are usually referred to as the *carrier* and *ends* respectively.

Theorem 2 *(Anshelevich [2])*

- AND-*rule: Let $P(B{:}X,S,Y)$ and $P(B{:}Y,T,Z)$ be virtual-connections with X and S each disjoint from T and Z. Then (i) if each cell of Y has a P-stone then $P(B{:}X,S \cup T,Z)$ is a virtual-connection; (ii) if Y consists of a single cell then $P(B{:}X,S \cup T,Z)$ is a semi-connection.*
- OR-*rule: Let $P(B{:}X,S_j,Y)$ be a semi-connection for $j = 1,\ldots,t$ such that the intersection of the sets S_j is empty. Then $P(B{:}X,S,Y)$ is a virtual-connection, where S is the union of the sets S_j.* □

Notice that part (i) of the AND-rule is essentially a restatement of the previous Berge observation.

Figure 8 illustrates the OR-rule. The marked cells in the leftmost diagram form the carrier of a semi-connection connecting the top and bottom white cell sets, since a white stone played at cell a virtually connects the white cell sets. The marked cells in the middle diagram form a similar semi-connection. Since the carriers of these two semi-connections do not intersect, their union forms the carrier of a virtual-connection between the two white cell sets, as shown in the rightmost diagram.

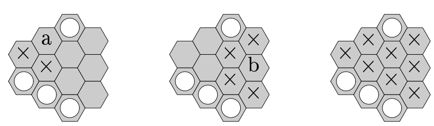

Fig. 8. Two semi-connections (left, middle) which form a virtual-connection (right).

Figure 9 illustrates why the carriers in the OR-rule must be non-intersecting. Since the carriers of the semi-connections in the first two diagrams intersect, a black stone played in their intersection, as shown in the third diagram, can prevent the two white cell sets from being joined.

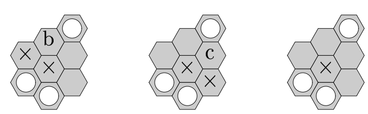

Fig. 9. Two semi-connections (left, middle) which do not form a virtual-connection. Black can separate the white pieces by playing at the intersection of the two semi-connections' carriers (right).

The most important virtual- and semi-connections are those which connect a player's two sides, since such a connection occurs by definition if and only if there is an associated winning strategy. We call such subgames *win-links* and *win-semilinks* respectively. We use $P(B{:}{-}S{-})$ to denote such a connection joining P's two sides.

As Berge remarked in [5], to play Hex well it is crucial to be able to recognize virtual-connections. This is especially true near the end of a game, when each player is likely to have several win-semilinks. In particular, Berge observed that a move to a cell which is not in the carrier of some opponent win-semilink is a losing move, since on the next move the opponent can turn the semi-connection into a win-link.

We call the intersection of the set of all unoccupied board cells with all known opponent win-semilink carriers a player's *must-play*. For an unoccupied cell c of a board-state B, we define $B + c_P$ as the state obtained from B by adding a P-stone at c. Berge's remark can be restated as follows:

Theorem 3 *(Berge [5]) Q has a winning strategy for $(B + c_P, Q, P)$ if c is not in P's must-play for (B, P, Q).*

Must-play analysis can significantly reduce the number of possible moves under consideration. For example, suppose the diagrams in Figure 9 are subgames in which White's goal is to connect the two sets of white stones; the first two diagrams show White win-semilinks and the third diagram shows the corresponding Black must-play, so Black has only one possible move to consider.

7 Strategy: dead cells

Another form of analysis useful in reducing the number of moves under consideration is based on the notion of a "dead cell"; informally, a cell is dead if the colour of any stone on it is irrelevant to the outcome of the game. The ideas we discuss here evolved from [17] and [19]; a more detailed treatment will appear in [7].

For a board-state B and players P and Q, a set S of unoccupied cells of B is a *P-win-set* if adding a P-stone to each of the cells of S yields a board-state with a winning P-chain. A P-win-set is *minimal* if no proper subset is a P-win-set. A cell is *P-live* if it is in some minimal P-win-set and *P-dead* otherwise. In the first diagram of Figure 10, the marked cell is dead while all other unoccupied cells are live.

Theorem 4 *For board-state B and players P, Q, a cell is P-dead if and only if it is Q-dead.*

Proof. Let x be a cell of B in some minimal P-win-set W. Let B' and $B' + x$ be the board-states obtained from B by adding a P-stone to each of the cells of $W - x$ and W respectively. Since W is minimal, $B' + x$ has a winning P-chain but B' does not. Thus every minimal Q-win-set of B' includes x, so x is Q-live in B' and so also in B. □

We thus call a cell *live* if it is live for both players and *dead* otherwise. As the following results show, dead cells may be ignored when considering possible moves.

Theorem 5 *Let (B, P, Q) be a game-state, let x be a dead cell of B, and let B^*_P and B^*_Q be the states obtained from B by adding at x a P-stone and a Q-stone respectively. Then (B, P, Q), (B^*_P, P, Q), (B^*_Q, P, Q) all have the same value.*

Fig. 10. From the left: a dead cell; two Black-captured sets; Black a dominates each x and y; each Black a dominates the two adjacent y.

Proof. Let T be one of $\{P, Q\}$, let U be the opponent of T, let S be a winning T-strategy for (B, P, Q), and let S' be the strategy obtained from S by replacing any move to x with a T-move to any live cell. Let Z be any final game-state reached from B, and let Z^*_P and Z^*_Q be the states obtained from B by setting the status of x to have a P-stone and a Q-stone respectively. Since x is not in any T-win-set of B, there is a winning T-chain in one of $\{Z, Z^*_P, Z^*_Q\}$ if and only if there such a chain in all of these states. It follows that S' is a T-winning strategy for (B, P, Q) and so also for (B^*_P, P, Q) and (B^*_Q, P, Q). □

Corollary 6 *If player P has a winning strategy for a game-state, then P has a winning strategy in which every P-move is to a live cell.*

Several relevant concepts build on the notion of dead cell. For a board-state B, players P, Q, and live cells w and z, z *P-harms* w, and w is *P-harmed* by z, if w is dead in the state B' obtained by adding a P-stone at z. For example, each of the marked cells in the second diagram of Figure 10 is Black-harmed by a cell with the same mark. Notice that the preceding corollary can be strengthened by replacing "live cell" with "live and not currently P-harmed cell".

A pair of cells is *P-captured* if each P-harms the other. More generally, a set S of P-harmed cells is *P-captured* if for each Q-move q in S there is a P-reply p which leaves q dead and every cell of $S-p-q$ P-captured. For example, the y-cells in the second diagram of Figure 10 form a Black-captured pair, while the x-cells form a larger Black-captured set.

For a fixed game-state, we say that one P-move dominates another if the former is a winning move or the latter a losing move. For example, the Black move to a in the third diagram of Figure 10 dominates moves to x and y; the next theorem explains why.

Theorem 7 *Let S be a P-captured set of a board-state B and let B' be the state obtained from B by placing a P-stone at any subset of cells of S. Then (B, P, Q) and (B', P, Q) have the same value. Also, for a cell p occupied by a P-stone of B and the state B^* obtained by removing this P-stone, the P-move to p dominates each P-move to S.*

Proof. The result follows by induction on the size of S, since a winning strategy for (B', X, Y) can be transformed into a winning strategy for (B, X, Y) by adding a response for each Q-move in S which kills the cell to which Q just played; we omit the details. □

Using dead cell analysis, many moves can often be omitted from consideration, greatly simplifying the analysis of a given board-state. For example, for any opening move m which creates a bridge to a player's boundary with carrier $\{y_1, y_2\}$, m dominates each cell of this carrier [17]; this is illustrated in the last diagram in Figure 10. Further examples of dead cell

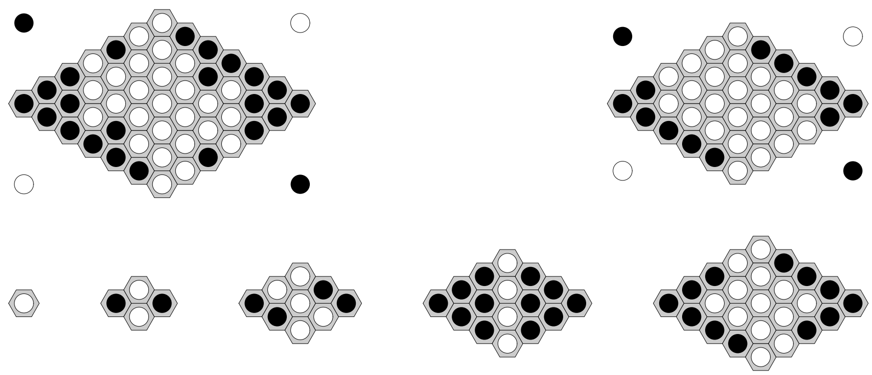

Fig. 11. Winner of the game with perfect play if White has the first move.

analysis appear in the Appendix.

8 Computation

To our mind, the results which come closest to answering Berge's request in [6] to solve Hex problems "using nontrivial theorems about combinatorial properties of sets" involve the mathematics of virtual-connections and dead cells. High-speed computers have recently facilitated the use of such results.

As we have mentioned, Anshelevich presented combining rules for finding virtual-connections of a fixed board-state B; however, these static-state rules can not find all win-links [2]. On the other hand, the following dynamic-state virtual-connection combining rules, based on an algorithm described by Van Rijswijck [35], will find all win-links.

Theorem 8 *Consider a board-state B and players P, Q. If B has a P-winning chain then $(B, P{:}{-}\phi{-})$ is a win-link. If B has no P-winning or Q-winning chain then P wins (B, P, Q) if and only if P wins $(B + m, Q, P)$ for some unoccupied cell m of B; in particular*

- *if P wins $(B + m, Q, P)$ with a win-link $(B + m, P{:}{-}S{-})$ then $(B, P{:}{-}\{S + m\}{-})$ is a win-semilink,*
- *if, for each unoccupied cell m_j, Q wins $(B + m_j, Q)$ with some win-link $(B + m_j, Q{:}{-}S_j{-})$, then the intersection of the sets S_j is empty,*
- *if there is a set M of unoccupied cells such that, for each m_j in M, Q wins $(B + m_j, Q)$ with some win-link $(B + m_j, Q{:}{-}S_j{-})$ and the intersection of the sets S_j is empty, $(B, Q{:}{-}S{-})$ is a win-link, where S is the union of the sets S_j.*

Using an algorithm which combines the fixed-state virtual-connection rules of Anshelevich [2], the preceding dynamic-state virtual-connection rules of Van Rijswijck [35], and some dead cell analysis due to Hayward [17], Hayward et. al. recently solved all 7×7 single-move openings [19], as shown in Figure 11. In the figures, we assume that White moves first; the colour of the disc in each cell indicates the winner with perfect play if White opens in that cell. As most of these openings were previously unsolved, it appears as though "combinatorial properties of sets" are being used to solve new Hex problems, as Berge had hoped.

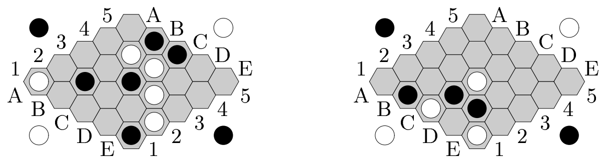

Fig. 12. Puzzles 1 and 2 from [5]. Puzzle 2 (right) was composed by Piet Hein [20]. White to play and win.

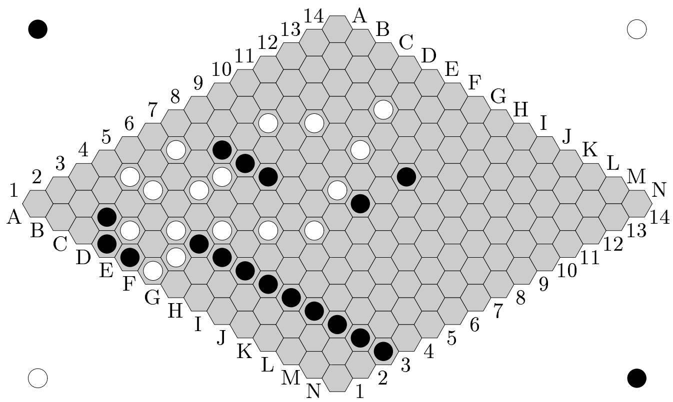

Fig. 13. Berge's Puzzle 3: Black to play and win.

9 Puzzles

We recall Sam Loyd, the greatest chess puzzle composer of all time, who claimed that his main goal was to compose a problem where the first move is the opposite of what 999 players out of 1000 would propose. This criterion also applies in Hex: in many puzzles the solution seems paradoxical.

With these words Berge introduced in [5] a series of five puzzles, which we reproduce here. Solutions to the puzzles are given in the appendix. The Hex boards are labelled with coordinates to denote moves analogously to algebraic chess notation. Berge starts with two warm-up puzzles on a 5×5 board.

Puzzles 1 and 2: Figure 12. Both puzzles have a unique winning move. Puzzle 2 is from Hein's original newspaper articles [20] and also appears in [15].

Puzzle 3: Figure 13. Black to play and win. It looks as though White has an impenetrable "wall" from G1 to the northeast. But Black can break through. How?

Puzzle 4: Figure 14. Black to play and win. The two black chains seem completely cut off, one by a white wall from A11 to N14, the other by a white wall from B8 to N1. The gaps in the former wall are disjoint from the gaps in the latter. It thus seems as though Black's two chains are completely useless. And yet ...

Puzzle 5: Figure 15. White to play and win. This is a study more than a puzzle. One question is whether White can run from C12 to the southwest; while this looks straightforward, Black can complicate matters considerably by well-timed moves to E4 and E3. A more interesting question is whether White can connect G11 to the northeast; Black ends up trying to connect G12 to the southeast, and there are many variations to consider.

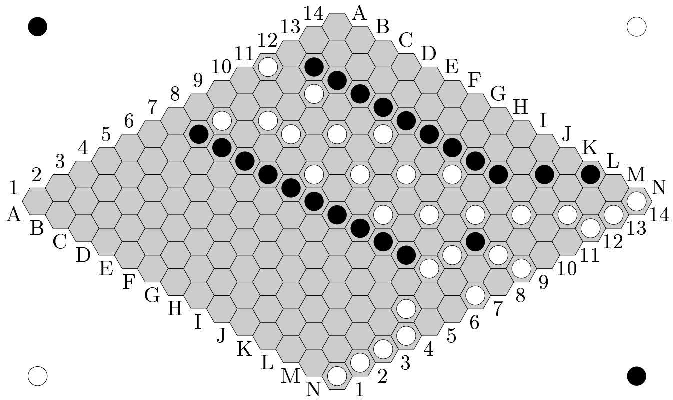

Fig. 14. Berge's Puzzle 4: Black to play and win.

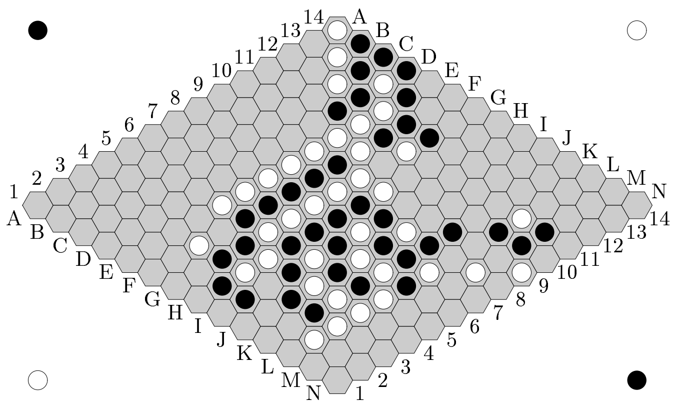

Fig. 15. Berge's Puzzle 5: White to play and win.

The puzzle which is the subject of [6] was intended to be only a minor variation of Puzzle 4, but inadvertently appeared in [6] with the black stone on K7 missing. Berge's puzzle was constructed with great care, and the omission of this one stone was enough to spoil the solution: without it, there is no longer a winning move for Black. The analysis in [6] is thus invalid for the accompanying diagram; this was pointed out by Browne in [8], who was unaware of the aforementioned omission. Of course, Berge's analysis is correct for the diagram as it was supposed to appear in the article, as Browne himself was quick to point out as soon as the matter was brought to his attention [9]. A revised version of Berge's analysis from [5] of these puzzles, including the notorious Puzzle 4, appears in the Appendix.

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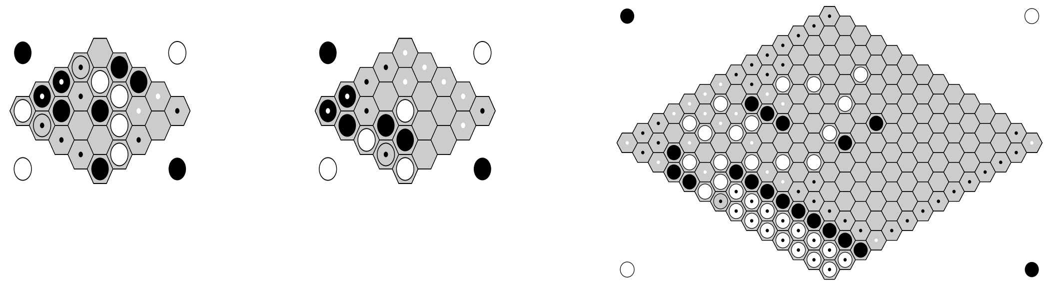

Fig. A.1. Dead cell analysis of Berge Puzzles 1-3.

A Puzzle solutions and dead cell analysis

The figures here show the dead cell analysis of Berge's puzzles. Dead cells are grey; captured cells are dotted. Dotted empty cells represent dominated nodes which can be ignored by the player to move; the colour of the dot indicates which player dominates the node.

Puzzle 1. The sets $\{A2, A3, B3, C2, D2\}$ and $\{A2, A3, C2, D1, D2\}$ are Black win-semilinks, as are $\{A2, A4, B3, C1, C2, D1, D2\}$ and $\{A3, A4, B3, C1, C2, D1, D2\}$. The resulting White must-play is the intersection of these four sets, namely $\{C2\}$, and it is easy to confirm that 1. C2! is a winning move. The preceding win-semilinks yield Black winning responses to any other White move, for example [1. A2 (or A3) B3!]; [1. D1 D2!]; and [1. D2 D1!].

Puzzle 2. The winning move is [1. B3!], and Black is powerless. The most difficult Black reply is [1. – C4], to which the only correct reply is [2. D3!]. All other White opening moves lose. For example, [1. E2? D3!] [2. B3 C4!]; [1. D3? E2!]; and [1. B2? B3!].

Puzzle 3. Black plays [1. E4] threatening [2. F5]. The intermediate move B7 serves only to delay matters, and eventually White must reply with one of the moves {E5, E6, F5, F6, G4}. Black continues with D4 and then connects at either D3 or C5.

Puzzle 4. Black starts with [1. M6 L6], [2. M11 L12]. These two moves are interchangeable. Black then continues: [3. L10 K11], [4. K9 K8], [5. J10 J11], [6. I10 H11], [7. I9 J8], and so on. Eventually Black reaches B9 and connects to the northwest via A9 or C8.

Puzzle 5. The key is to maintain as long as possible the double threat of joining the stone at L11 with either the stone at G11 or the stone at M8, since the latter stone can be easily be joined to the southwest.

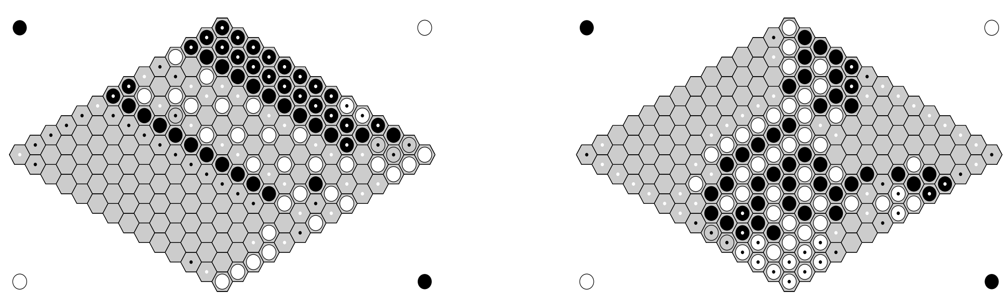

Fig. A.2. Dead cell analysis of Berge Puzzles 4-5.

For example, one winning move is [1. H11], with a typical continuation being [1. – H12] [2. I11 I12] [3. K11!]. The white stones at K11,L11 are now virtually joined to the southwest: White can respond to [4. – J11] with [5. K10!], threatening both J10 and L9. They are also joined to the northeast; White can reply at J13 to any Black move to a location not in {J12, K12, J13, I14, J14}, and at L12 to any Black move to a location not in {L12, M12, K13, L13, M13, J14, K14, L14, M14}. The intersection of these two must-play regions is J14, and if Black moves there then [5. – J14] [6. K13 K14] [7. M13!] wins.

The move [1. K10?] is also tempting, since if Black decides to eliminate the White double threat with [1. K10 L9?] then White can reply with [2. H11!], leading eventually to a White win. But Black has a better reply, namely [1. – I11!], and now White loses.