TWO CLASSES OF PERFECT GRAPHS

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to Mom and Dad

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Abstract

In this work two classes of graphs are introduced. A graph is weakly triangulated if neither the graph nor its complement contain a chordless cycle with five or more vertices as an induced subgraph. A graph is murky if neither the graph nor its complement contain the chordless cycle with five vertices or the chordless path with six vertices as an induced subgraph. The major results of this thesis are theorems concerning these two classes of graphs. In particular, weakly triangulated graphs and murky graphs are perfect.

Résumé

Dans ce travail on présente deux classes de graphes. Un graphe est appelé faiblement triangulé si ni le graphe ni son complément n'admettent de cycle sans corde de cinq sommets ou plus comme sous-graphe induit. Un graphe est appelé troublé si ni le graphe ni son complément n'admettent de cycle sans corde de cinq sommets ou de chemin sans corde de six sommets comme sous-graphe induit. Les résultats les plus importants dans cette thèse sont des théorèmes qui concernent ces deux classes de graphes. En particulier, les graphes faiblement triangulés et les graphes troublés sont des graphes parfaits.

Preface

The thesis consists of four chapters.

Chapter 1 is an overview of the results of the thesis. A perspective of perfect graph theory is presented which motivates the study of weakly triangulated graphs and murky graphs.

Chapter 2 is a brief description of the background of the thesis, namely perfect graph theory. The first section of the chapter is a description of basic definitions and notations of general graph theory. The second section is a brief outline of selected results in perfect graph theory.

Chapter 3 is a collection of results on weakly triangulated graphs. Included are an examination of the relationship between weakly triangulated graphs and star cutsets, and a proof that weakly triangulated graphs are perfect. The chapter also includes algorithms which solve certain optimization problems for weakly triangulated graphs.

Chapter 4 is a collection of results on murky graphs. The highlight of this chapter is a proof that murky graphs are perfect. The proof involves an examination of properties of unbreakable murky graphs; the chapter concludes with a characterization of such graphs.

Unless otherwise stated, the titled theorems in this thesis are the work of the author.

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Chapter 1

Overview

A *clique* is a set of pairwise adjacent vertices in a graph. The *clique number* of a graph is the number of vertices in a largest clique. The *chromatic number* of a graph is the least number of colours needed to colour the vertices, so that adjacent vertices receive different colours. Note that the chromatic number of a graph must be at least as large as the clique number. Claude Berge defined a graph G to be *perfect* if, for each induced subgraph H of G, the chromatic number of H is *equal* to the clique number of H.

A graph is *minimal imperfect* if it is not perfect and yet every proper induced subgraph is perfect. It is an easy exercise to check that odd chordless cycles with at least five vertices are minimal imperfect; it is only a little more difficult to show that the complements of such chordless cycles are also minimal imperfect. Are there any other minimal imperfect graphs? The celebrated *Strong Perfect Graph Conjecture*, posed by Berge in 1960, asserts that the answer to this question is "no":

The SPGC. *A graph is perfect if and only if neither the graph nor its complement contains an odd chordless cycle with five or more vertices.*

As early attempts to resolve the *SPGC* were unsuccessful, Berge posed a second conjecture (which, since it is implied by the first, was originally known as the *Weak Perfect Graph Conjecture*):

The WPGC. *A graph is perfect if and only if its complement is perfect.*

The *WPGC* was proved by Lovász (see [1972a] and [1972b]), and is now known as the *Perfect Graph Theorem.* The *SPGC* is still open. The *SPGC* has been the primary motivation behind most of the research in perfect graph theory to this date.

We call a graph *Berge* if neither the graph nor its complement contains an odd chordless cycle with five or more vertices. The *SPGC* asserts that a graph is perfect if and only if it is Berge. This wording of the *SPGC* suggests one approach to investigating the conjecture: consider particular classes of Berge graphs, and check to see whether or not the graphs in these classes are perfect.

One such class is the class of *triangulated* graphs, also known as *chordal* graphs, defined as those graphs in which every cycle with four or more vertices has a chord. Let C_k represent the chordless cycle with k vertices, and P_k the chordless path with k vertices. Let $\overline{G}$ represent the complement of the graph G. To see that triangulated graphs are Berge, note that by definition, triangulated graphs do not contain C_k as an induced subgraph, for $k \geq 4$. Also, C_4 is an induced subgraph of $\overline{P}_5$, and $\overline{P}_5$ is an induced subgraph of $\overline{C}_j$, for $j \geq 6$; thus triangulated graphs do not contain $\overline{C}_j$ as an induced subgraph, for $j \geq 6$. Finally, since C_5 is self-complementary, triangulated graphs do not contain $\overline{C}_5$ as an induced subgraph. To summarize, triangulated graphs do not contain C_k, for $k \geq 4$, nor $\overline{C}_j$, for $j \geq 5$, as an induced subgraph. Thus triangulated graphs are Berge.

In 1960 Berge showed that triangulated graphs are perfect; thus triangulated graphs have been known to be perfect almost since the beginning of the history of perfect graph theory. Indeed, Berge's realization that both triangulated graphs and complements of triangulated graphs are perfect (see Hajnal and Surányi [1958]) was part of the motivation that led him to pose the *SPGC* and the *WPGC*.

Another example of a class of Berge graphs is the class of P_4-*free* graphs, defined as those graphs that do not contain P_4, the chordless path with four vertices, as an induced subgraph. Since every C_j contains P_{j-1} as an induced subgraph, P_4-free graphs do not contain C_k, for $k \geq 5$, as an induced subgraph. Also, since P_4 is self-complementary, P_4-free graphs do not contain $\overline{C}_k$, for $k \geq 5$, as an induced subgraph. Thus P_4-free graphs are Berge. Seinsche [1974] proved that P_4-free graphs are perfect.

The main contribution of this thesis is the introduction of two new classes of Berge graphs, together with proofs that such graphs are perfect. In light of the *SPGC*, it is natural to consider classes of Berge graphs defined in terms of forbidden induced subgraphs, and in terms of chordless cycles and complements of chordless cycles. In light of the *Perfect Graph Theorem* (formerly the *WPGC*), it is natural to consider "self-complementary" classes of Berge graphs, i.e. classes of Berge graphs that are closed under complementation. (For example, C_4 is not triangulated, whereas $\overline{C}_4$ is; thus the class of triangulated graphs is not self-complementary. On the other hand, if a graph is P_4-free, then so is its complement; thus the class of P_4-free graphs is self-complementary.) Recall that a graph is triangulated (if and) only if it does not contain C_k, for $k \geq 4$, nor $\overline{C}_j$, for $j \geq 5$ as an induced subgraph. The two aforementioned criteria for selecting a "natural" class of Berge graphs suggest the following generalization of triangulated graphs: define a graph to be *weakly triangulated* if the graph does not contain C_k or $\overline{C}_k$, for $k \geq 5$, as an induced subgraph. Note that the class of weakly triangulated graphs contains all triangulated graphs, all complements of triangulated graphs, and all P_4-free graphs.

The second class of Berge graphs introduced in this thesis also contains all P_4-free graphs (but not all triangulated graphs). Call a graph *murky* if it contains no C_5, P_6 or $\overline{P}_6$ as an induced subgraph. Interest in the class of murky graphs was partly motivated by Hoàng's study of the class of graphs that contain no C_5, P_5 or $\overline{P}_5$ as an induced subgraph (see Hoàng [1983], and Chvátal, Hoàng, Mahadev and De Werra [to appear]).

How can one prove that all graphs in a given class are perfect? One method is to look for some structural property exhibited by all graphs in the class, and then show that no graph with the property can be minimal imperfect. Of particular interest are structural attributes that lead to a decomposition of the graph. For example, suppose that a graph G with vertex set V has a *clique cutset*, that is, a set of vertices C such that C is a clique, and removal of C leaves a disconnected graph. Let A be any set of vertices that induces a component of $G - C$, and let B be the rest of the vertices of G (i.e. $B = V - A - C$). Then it is a simple exercise to show that G is perfect if the subgraphs induced by $A \bigcup C$ and $B \bigcup C$ are perfect: take any two respective minimum colourings of these two graphs, and identify the colours along the clique C. Thus a graph with a clique cutset may be decomposed into two smaller graphs, each an induced subgraph of the original graph, in such a way that the original graph is perfect if the two smaller graphs are perfect. This implies that a graph with a clique cutset cannot be minimal imperfect. Dirac [1961] proved that every triangulated graph is either a complete graph or else has a clique cutset. Thus triangulated graphs are perfect.

Another structural property of a graph that leads to a decomposition is a *homogeneous set*, defined as a subset H of at least two and not all of the vertices of the graph, such that every vertex not in H is adjacent either to all or to none of the vertices of H. From a result due to Lovász (see [1972a]) it follows that if a graph G has a homogeneous set H, and if H and the graph obtained from G by deleting all but one vertex of H are both perfect, then G is perfect. (Note that both of the smaller graphs are induced subgraphs of the original graph.) Thus a graph with a homogeneous set cannot be minimal imperfect.

Seinsche [1974] proved that every P_4-free graph with at least two vertices either is disconnected, or else its complement is disconnected. From this it follows that every P_4-free graph with at least three vertices has a homogeneous set. Thus P_4-free graphs are perfect. (Although P_4-free graphs and homogeneous sets are intimately related, the conclusion that P_4-free graphs are perfect can be reached without using homogeneous sets. It is easy to prove that if a graph or its complement is disconnected, then the graph is not minimal imperfect.)

An attribute of a graph that generalizes both a clique cutset and a homogeneous set is a *star cutset*, defined as a set C of vertices of a graph G, such that some vertex in C is adjacent to all remaining vertices in C, and such that $G - C$ is disconnected. The notion of a star cutset was introduced by Chvátal, with the aim of unifying several structural properties associated with decompositions. Let C be a star cutset of a graph G, with vertex v in C adjacent to all vertices of $C - v$, and let A be a component of $G - C$, and B the vertices of $G - C - A$. Chvátal proved that G is perfect if $G - v$ and the subgraphs induced by $A \bigcup C$ and $B \bigcup C$ are perfect; he also proved the analogous decomposition result for the case in which the complement of a graph has a star cutset. It follows that neither a minimal imperfect graph nor its complement can have a star cutset.

As clique cutsets are associated with triangulated graphs, and homogeneous sets with P_4-free graphs, one might ask whether there is a class of graphs associated with star cutsets. Since a star cutset is a generalization of both a clique cutset and a homogeneous set (see Chvátal [1985a]), such a class of graphs would include triangulated graphs and P_4-free graphs. In fact, there is such a class of graphs, namely weakly triangulated graphs. In Chapter 3 we prove that if a graph is weakly triangulated and has at least three vertices, then either the graph or its complement has a star cutset. Thus weakly triangulated graphs are perfect. Also, if a graph is not weakly triangulated, then the graph has some induced (not necessarily proper) subgraph (namely, C_k or $\overline{C}_k$ with $k \geq 5$) such that neither the induced subgraph nor its complement has a star cutset; thus star cutsets and weakly triangulated graphs are intimately related.

The star cutset decomposition can be used as the starting point in attempting to prove that other classes of Berge graphs, besides weakly triangulated graphs, are perfect. A graph is called *unbreakable* if neither the graph nor its complement has a star cutset. Minimal imperfect graphs are unbreakable; thus, in order to show that the graphs of a particular class of Berge graphs are perfect, it suffices to show that the unbreakable graphs of the class are perfect. What do unbreakable Berge graphs look like? What properties do they have? How do chordless cycles (of even length) and complements of such cycles intersect in unbreakable Berge graphs? These questions motivate our proof that murky graphs are perfect; this is the main result of Chapter 4. As a postscript, we include a characterization of unbreakable murky graphs.

One reason perfect graphs are interesting is that there are certain optimization problems which are NP-complete for arbitrary graphs, but for which there exist algorithms which run in polynomial time if the input graph is perfect. A *stable set* of a graph is a set of pairwise non-adjacent vertices of a graph; the *stability number* is the number of vertices in a largest stable set. The *clique cover number* of a graph is the least number of cliques needed to cover the vertices. Note that the stability number of a graph G is equal to the clique number of $\overline{G}$; the clique cover number of G is equal to the chromatic number of $\overline{G}$. Grötschel, Lovász, and Schrijver [1984] described algorithms that solve the problems of determining the clique number, stability number, chromatic number and clique cover number (and even the weighted versions of these problems) in polynomial time for perfect graphs. Their powerful algorithms are based on the ellipsoid method of linear programming, and on previous work of Lovász [1979] concerning Shannon's capacity of a graph. Given the non-transparent nature of these results, it is of interest to look for simpler algorithms, especially when considering particular classes of perfect graphs. One contribution of this thesis is the presentation of simple combinatorial algorithms which exploit the structure of weakly triangulated graphs to solve the four aforementioned optimization problems (and also the weighted versions of these problems) for the class of weakly triangulated graphs. We have been unable to find analogous algorithms which solve these problems for the class of murky graphs.

Chapter 2
Background

The first section of this chapter is an introduction to the terminology used in the thesis; other definitions will be introduced later as needed. The second section is a brief outline of selected results in perfect graph theory.

2.1 Definitions and Notation

A *graph* consists of a finite non-empty set of *vertices*, together with a finite set of *edges*, or unordered pairs of distinct vertices. If two vertices are in some edge of a graph, then the vertices are said to be *adjacent*, otherwise they are *non-adjacent*. We use the terms "sees" and "misses" as synonyms for adjacency and non-adjacency respectively; thus "a sees b and misses c" is equivalent to "a is adjacent to b, but not to c".

A vertex is called a *neighbour* of another vertex if the two vertices are adjacent. The *neighbourhood* of a vertex x in a graph G, denoted $N(x)$, is the set of all neighbours of x in G; the *non-neighbourhood* of x, denoted $M(x)$, is the set of all non-neighbours of x in $G - x$.

If S is a subset of the vertices of a graph G, then *the subgraph of G induced by S*, denoted G_S, is the graph with vertex set S, whose edges are precisely those edges of G that consist of two vertices of S. An *induced subgraph* of G is a subgraph induced by some S.

A *path* is a sequence of (pairwise distinct) vertices $v_1 v_2 \cdots v_k$, such that every two consecutive vertices v_j, v_{j+1}, are adjacent, for $1 \leq j \leq k-1$; if also v_1 sees v_k, then $v_1 v_2 \cdots v_k$ is called a *cycle*. A *chordless path* is a path $v_1 v_2 \cdots v_k$ such that the *only* edges of the path are (v_j, v_{j+1}), for $1 \leq j \leq k-1$; a *chordless cycle* is a cycle

$v_1 v_2 \cdots v_k$ such that the *only* edges of the cycle are (v_j, v_{j+1}), for $1 \leq j \leq k-1$, and the edge (v_1, v_k). P_k denotes the chordless path with k vertices; C_k denotes the chordless cycle with k vertices.

A graph is *connected* if for every two vertices x and y there is some path $x \ldots y$. A *component* of a graph is a maximal connected subgraph. (Throughout the thesis, the terms "maximal" and "minimal" are used with respect to set inclusion; for example, a maximal connected subgraph is a connected subgraph that is not a proper subgraph of any other connected subgraph of the graph). A *singleton* of a graph is a component with only one vertex; a *big component* is a component with more than one vertex. A *cutset* is a set of vertices of a graph, such that the subgraph induced by the remaining vertices is disconnected. Note that in a disconnected graph, any proper subset of the vertices of any component is a subset.

The *complement* of a graph is the graph obtained by replacing all edges with non-edges, and vice versa. $\overline{G}$ denotes the complement of the graph G. Thus, $\overline{P}_k$ and $\overline{C}_k$ are the respective complements of P_k and C_k.

A *clique* (respectively *stable set*) of a graph is a set of pairwise adjacent (respectively non-adjacent) vertices. The *clique number* (respectively *stablility number*) of a graph is the number of vertices in a largest clique (respectively *stable set*). The *chromatic number* (respectively *clique covering number*) is the minimum number of stable sets (respectively cliques) needed to partition the vertices of a graph. Denote the stablility number, clique number, chromatic number and clique covering number of a graph G by $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$ respectively. A graph is *perfect* if, for each induced subgraph H of G, $\chi(H) = \omega(H)$.

2.2. Some Results in Perfect Graph Theory

In the more than twenty-five years that have passed since Berge posed the *SPGC*, much research has been directed to the study of perfect graphs. Whereas originally most research was directed towards resolving the conjecture, there are aspects of perfect graph theory which are now considered interesting in their own right, independent of whether or not the *SPGC* is true (or even if it is resolved). In particular, the emergence in the past two decades of issues related to computational complexity has inspired much interest in perfect graphs: the question of whether or not perfect graphs are in *NP* is currently the focus of much research.

In this chapter, we sketch a background of perfect graph theory. A more complete history can be found in any of a number of recently published graph theory texts; for instance, see Berge [1985]. Two books devoted entirely to perfect graph theory are Golumbic [1980] and Berge and Chvátal [1984].

2.2.1 The PGT and the SSPGT

When initial attempts to resolve the *SPGC* were unsuccessful, Berge posed a second conjecture, which (since it was implied by the *SPGC*) became known as the *Weak Perfect Graph Conjecture*. This conjecture was proved by Lovász and is now known as the *Perfect Graph Theorem*.

PGT (Lovász [1972a]). *A graph is perfect if and only if its complement is perfect.*

In light of the *PGT*, it is natural to look for properties of perfect graphs that are invariant under complementation. Speculation about such properties led Chvátal [1984a] to define the ***P_4-structure*** of a graph G as the collection of those sets of four vertices that induce a P_4 in G. Since the complement of a P_4 is a P_4, the P_4-structure of a graph is the same as the P_4-structure of its complement. Chvátal conjectured that the perfection of a graph depends only on its P_4-structure. This conjecture, implied by the *SPGC* and implying the *PGT(WPGC)*, was known as the *Semi-Strong Perfect Graph Conjecture* or *SSPGC*. The conjecture was proved by Reed in 1984, and is now known as the *Semi-Strong Perfect Graph Theorem.*

SSPGT (Reed [1985]). *Every graph with the P_4-structure of a perfect graph is perfect.*

The *SSPGC* has inspired several results that consider decompositions of perfect graphs defined in terms of P_4-structure. For example, vertices x and y are called *siblings* if there is a set S of three vertices such that both $S \bigcup \{x\}$ and $S \bigcup \{y\}$ are P_4's. Chvátal proved the following result.

Theorem (Chvátal [1985b]). *Let the vertices of a graph G be coloured with two colours such that every two siblings have the same colour. Then G is perfect if and only if each of the subgraphs induced by the set of all vertices of the same colour is perfect.*

This theorem generalizes two earlier results: Chvátal and Hoàng [1985] showed that if the vertices of a graph can be coloured with two colours such that every P_4 has an even number of vertices of each colour, then the graph is perfect if and only if each of the two mono-chromatic induced subgraphs is perfect; Hoàng [1985b] showed that if the vertices of a graph can be coloured with two colours in such a way that every P_4 has an odd number of vertices of each colour, then the graph is perfect.

Another result concerning P_4-structure is that in a minimal imperfect graph every vertex is in at least four P_4's; this follows from a theorem of Olariu [1986]. (Actually, Olariu's theorem is a much stronger statement; however, it is not primarily related to P_4-structure.)

2.2.2 Some Classes of Perfect Graphs

From the time that Berge first proposed the *SPGC,* much of the energy devoted to the study of perfect graphs has focused on finding new classes of perfect graphs. As has been mentioned, both triangulated graphs and complements of triangulated graphs were known to be perfect by 1960. Other classes of graphs long known to be perfect include line graphs of bipartite graphs (this follows from a theorem due to König [1936] concerning the edge-chromatic number of a bipartite graph) and comparability graphs. A graph is a *comparability graph* if the edges can be directed so that for every three vertices a, b, c, if (a, b) and (b, c) are directed edges, then so is (a, c). It is an exercise to show that comparability graphs are perfect; that complements of comparability graphs are perfect follows from Dilworth's theorem [1950]: the size of a largest anti-chain is equal to the minimum number of chains needed to cover a partially ordered set.

Since the early 1960's many classes of perfect graphs have been discovered. In the rest of this section we briefly discuss two ways of obtaining classes of perfect graphs.

Let P be some forbidden property of minimal imperfect graphs. If every induced subgraph of a certain graph satifies P, then the graph is perfect. Thus the "subgraph property" paradigm can be used to define classes of perfect graphs. For example, Berge and Duchet [1984] defined a graph to be *strongly perfect* if every induced subgraph has a stable set which intersects all maximal cliques. Another class of graphs which fits this paradigm was defined by Meyniel. Call a set $\{x, y\}$ of vertices of a graph an *even pair* if every chordless path between x and y has an even number of edges. Meyniel [1986] defined a graph G to be *quasi-parity* if, for every induced subgraph H of G with at least two vertices, either H or $\overline{H}$ has an even pair. We will say more about quasi-parity graphs in Chapter 3.

Another way to obtain (candidates for) classes of perfect graphs is to forbid certain induced subgraphs from Berge graphs. For instance, Tucker [1977] showed that K_4-free Berge graphs are perfect; Parthasarathy and Ravindra showed that $K_{1,3}$-free Berge graphs [1976] and (K_4-e)-free Berge graphs [1979] are perfect. Chvátal and Sbihi refer to the connected graph with five vertices that consists of a triangle and two pendant edges as a *bull*; they showed that bull-free Berge graphs are perfect [1986].

As was mentioned in Chapter 1, the major contribution of this thesis is the introduction of weakly triangulated graphs and murky graphs. These two classes of graphs clearly fall into the "forbidden subgraph" paradigm: weakly triangulated graphs are Berge graphs with no C_k or $\overline{C}_k$, for k even and $k \geq 6$; murky graphs are Berge graphs with no P_6 or $\overline{P}_6$. In fact, weakly triangulated graphs also fall into the "subgraph property" paradigm; exactly how this is so is discussed in Section 3.2.3.

2.2.3 Properties of Minimal Imperfect Graphs

If the *SPGC* is true, then the only minimal imperfect graphs are chordless odd cycles with at least five vertices, and the complements of such cycles. One approach to the *SPGC* has been to look for properties of minimal imperfect graphs. For instance, (as was noted in the previous chapter), a minimal imperfect graph does not have a clique cutset, nor a homogeneous set, nor a star cutset. (Actually, the fact that a graph does not have a star cutset implies that is has neither a clique cutset nor a homogeneous set; see Chvátal [1985a].) A major result in this area is due to Lovász.

Theorem (Lovász [1972b]). *Every minimal imperfect graph G satisfies* $\alpha(G)\omega(G) = |G| - 1$.

(Recall that $\alpha(G)$ is the stability number of G, and $\omega(G)$ the clique number.)

Padberg [1974] extended Lovász's result by showing that in a minimal imperfect graph G

– there are $|G|$ largest cliques and $|G|$ largest stable sets.

– every vertex is in exactly $\alpha(G)$ largest cliques and $\omega(G)$ largest cliques, and

– every largest stable set intersects all but one largest clique, and vice versa.

Define the graph C_n^t as follows: $v_1, \ldots, v_n$ are the vertices, with v_i and v_j adjacent if $|i - j| \leq t$, for every pair of vertices v_i, v_j. Observe that C_{2k+1} is C_{2k+1}^1, and $\overline{C}_{2k+1}$ is C_{2k+1}^{k-1}. In fact, the graph $C_{\alpha\omega+1}^{\omega-1}$ satisfies the conditions of Lovász and Padberg. Chvátal [1984c] showed the *SPGC* is equivalent to stating that every minimal imperfect graph has a spanning subgraph isomorphic to $C_{\alpha\omega+1}^{\omega-1}$. However, Chvátal, Graham, Perold and Whitesides [1979] found infinitely many graphs which do not contain $C_{\alpha\omega+1}^{\omega-1}$ as a spanning subgraph, and yet which satisfy the conditions of Lovász and Padberg; Bland, Huang and Trotter [1979] independently discovered two of these graphs. Thus the list of properties of minimal imperfect graphs described so far is insufficient to imply the *SPGC*.

2.2.4 Complexity and a Changing Perspective

Since the time that the *SPGC* was first posed, ideas have emerged in the theory of computer science that have significantly altered the way problems are approached by computer scientists. One such idea is the notion of a *good algorithm*, suggested by Edmonds [1965] as an algorithm which computes the answer to a problem in such a way that the number of operations required by the algorithm is bounded above by some polynomial in the size of the problem. This immediately raises the question "for which problems do there exist good algorithms?".

From this point of view, one of the most important open problems in perfect graph theory is "does there exist a polynomial time algorithm to recognize perfect graphs?" A related question is whether or not there exists a certificate of perfection that could be verified in polynomial time (i.e. whether or not perfect graphs are in *NP*). Whitesides has suggested (see Berge and Chvátal [1984], page xii) that perhaps perfect graphs can be created from certain "primitive classes" of perfect graphs using perfection preserving operations. If the graphs in the primitive classes are in *NP*, and if the perfection preserving operations can be performed in polynomial time, then it would follow that perfect graphs are in *NP*. For example, *clique identification* is the process of combining two graphs by identifying a clique of one with a clique of the other. It follows from Dirac's theorem that triangulated graphs can be created from cliques using the perfection preserving operation of clique identification. Whitesides [1984] has shown how to reverse this process, so that every triangulated graph can be decomposed into cliques in polynomial time. (There are faster ways to recognize triangulated graphs; for instance, see Rose, Tarjan and Leuker [1976]. However, the example presented here suffices to illustrate our paradigm.) Although this approach has been successful in showing that certain classes of perfect graphs are in *NP* (or even in *P*), the question of whether or not perfect graphs are in *NP* is still open. On the other hand, *imperfect* graphs are in *NP*. We close the chapter with this result.

Bland, Huang, and Trotter [1979] call a graph G *partitionable* if there are integers $r \geq 2$ and $s \geq 2$ such that for each vertex v of G, the vertices of $G - v$ partition into r cliques of size s and s stable sets of size r. They noted that Lovász's theorem (see the previous section) implies that a graph is minimal imperfect if and only if it contains a partitionable induced subgraph. As Cameron and Edmonds remarked (see Cameron [1982]), this implies that imperfect graphs are in *NP*.

Chapter 3

Weakly Triangulated Graphs

3.1 Introduction

Recall that a graph is *weakly triangulated* if it contains no C_k, and no $\overline{C}_k$, for $k \geq 5$. In this chapter we decribe some properties of weakly triangulated graphs, and show that weakly triangulated graphs are perfect. In particular, we describe a relationship between weakly triangulated graphs and star cutsets. Finally, we describe polynomial time algorithms which solve the maximum clique, maximum independent set, minimum colouring and minimum clique cover problems for weakly triangulated graphs.

An attractive feature of weakly triangulated graphs is that they can be recognized in polynomial time. One such recognition algorithm is as follows: for each vertex in a graph, determine if the vertex is contained in a chordless cycle with five or more vertices; repeat the process for the complement of the graph. Whether or not a vertex v is contained in a chordless cycle with five or more vertices can be checked as follows: for each pair of non-adjacent vertices x and y which are both adjacent to v, remove all vertices of the graph adjacent to both x and y, as well as all vertices adjacent to v (except x and y), and then check whether or not there is a path from x to y in the resulting graph. The vertex v is contained in a chordless cycle with at least five vertices if and only if there exists such a path from x to y. For a graph with n vertices and e edges, determining whether or not there is a path between a specified pair of vertices can be done in time $O(e)$. Since the total number of edges in a graph and its complement is $O(n^2)$, the above algorithm recognizes weakly triangulated graphs in time $O(n^5)$.

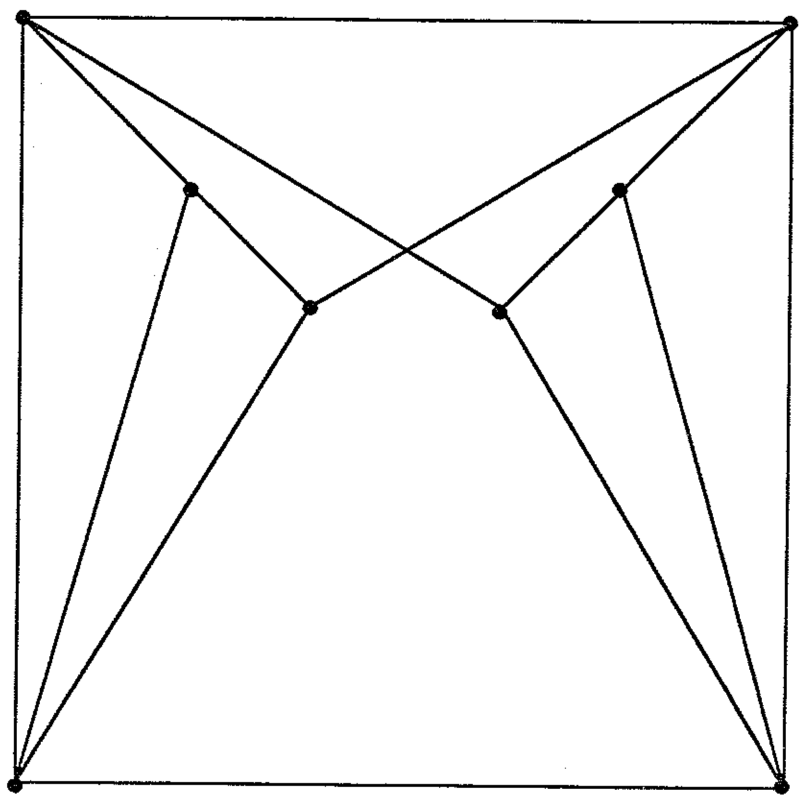


Figure 3.1

3.2 Weakly Triangulated Graphs, Star Cutsets, and Perfection

3.2.1 Why Star Cutsets?

In attempting to analyze the structure of weakly triangulated graphs, we begin by examining two special cases: triangulated graphs and P_4-free graphs.

Dirac [1961] proved that every minimal cutset in a triangulated graph is a clique. A theorem due to Seinsche [1974] implies that every P_4-free graph with at least three vertices has a homogeneous set. However, there are weakly triangulated graphs with no clique cutset, no clique cutset in the complement, and no homogeneous set. The smallest such graph appears in *Figure 3.1.*

In attempting to unify certain structural properties associated with decompositions of perfect graphs, Chvátal [1985a] conceived the following notion: a *star cutset* is a set C of vertices of a graph G such that some vertex in C is adjacent to all other vertices in C, and such that $G - C$ is disconnected. (In particular, if a graph has a clique cutset, then it has a star cutset; if a graph has a homogeneous set, then either the graph or its complement has a star cutset.) Let G be a graph with star cutset C, with vertex v in C adjacent to all vertices of $C - v$, and let A be a component of $G - C$, and B the vertices of $G - C - A$. Chvátal proved that G is perfect if the three subgraphs $G_{A \bigcup C}$, $G_{B \bigcup C}$, and $G - v$ respectively are perfect; he also proved the analogous decomposition result for the case in which the complement of a graph has a star cutset. The following is a consequence of these two results:

The Star Cutset Lemma (Chvátal [1985a]). *If a graph is minimal imperfect, then neither the graph nor its complement has a star cutset.*

Chvátal conjectured that every weakly triangulated graph with at least three vertices either has a star cutset, or else its complement has a star cutset. This conjecture will be proved as the *WT Star Cutset Theorem.*

3.2.2 Perfection

The *WT Star Cutset Theorem* follows easily from the following theorem.

The WT Min Cut Theorem. *Let N be a minimal cutset of a weakly triangulated graph G, and let N induce a connected subgraph of $\overline{G}$. Then each connected component of $G - N$ includes at least one vertex adjacent to all the vertices of N.*

Proof of the WT Min Cut Theorem. We first show that

every two non-adjacent vertices in N
have a common neighbour in each component of $G - N$. (1)

For this purpose, consider arbitrary non-adjacent vertices x and y in N, and an arbitrary component A of $G - N$. Since the cutset N is minimal, each vertex in N has at least one neighbour in A; now connectedness of A implies the existence of a path from x to y with all interior vertices in A; the shortest such path P is chordless. The same argument, applied to another component B of $G - N$, shows the existence of a chordless path Q from x to y with all interior vertices in B. The two paths P and Q combine into a chordless cycle in G; since G contains no chordless cycle with five or more vertices, each of the two paths must have only one interior vertex. In particular, the interior vertex of P is a common neighbour of x and y in A, and (1) is proved.

Next, let us show that

the theorem holds whenever no two vertices in N are adjacent. (2)

To prove (2), we use induction on $|N|$. When $|N| = 1$, the conclusion follows from the fact that the cutset N is minimal. When $|N| = 2$, the conclusion is guaranteed by (1). When $|N| \geq 3$, choose distinct vertices x, y, z in N and consider an arbitrary component A of $G - N$. Note that $N - x$ is a minimal cutset of $G - x$, and that $(G-x) - (N-x) = G - N$. Hence the induction hypothesis guarantees the existence of a vertex u in A that is adjacent to all vertices in $N - x$. By the same argument, some vertex v in A is adjacent to all vertices in $N - y$, and some vertex w in A is adjacent to all vertices in $N - z$. We will show that at least one of the vertices u, v, w is adjacent to all the vertices in N. Assuming the contrary, note that u, v, w must be distinct. Now u cannot be adjacent to v (else y, u, v, x, and any common neighbour of x and y in $G - N - A$, whose existence is guaranteed by (1), would induce a chordless cycle in G); by the same argument, u cannot be adjacent to w, nor v to w. But then x, w, y, u, z, v induce a chordless cycle in G. This contradiction completes the proof of (2).

To prove the theorem in its full generality, we again use induction on $|N|$. When $|N| \leq 2$, the conclusion follows from (2). When $|N| \geq 3$, we may assume that at least two vertices in N are adjacent (else the conclusion is guaranteed by (2) again). Now we claim that N includes distinct vertices x and y such that

(i) x and y are adjacent in G, and

(ii) both $N - x$ and $N - y$ induce connected subgraphs of $\overline{G}$.

(To justify this claim, we only need choose x and y so that, in the subgraph of $\overline{G}$ induced by N, the shortest path from x to y is as long as possible.) Consider an arbitrary component A of $G - N$. By the induction hypothesis, A includes vertices u and v such that u is adjacent to all the vertices in $N - x$ and v is adjacent to all the vertices in $N - y$. We will show that at least one of the vertices u and v is adjacent to all the vertices in N. Assuming the contrary, note that u and v must be distinct. By (i), the shortest path P from x to y in the subgraph of $\overline{G}$ induced by N has at least one interior vertex. Now u and v must be adjacent: else u,v and P would induce a chordless cycle in $\overline{G}$. Next, the argument showing the existence of v in A shows also the existence of a vertex r in $G-N-A$ such that r is adjacent to all the vertices in $N-y$. If r is not adjacent to y then u,r and P induce a chordless cycle in $\overline{G}$; else u,r,v and P induce a chordless cycle in $\overline{G}$. This contradiction completes the proof. ■

The WT Star Cutset Theorem. *If G is a weakly triangulated graph with at least three vertices then G or $\overline{G}$ has a star cutset.*

Proof of the WT Star Cutset Theorem. The star cutset may be found as follows. Choose an arbitrary vertex w in G. For each vertex x other than w, put x in the set N if x is adjacent to w; else put x in the set M. If N is empty then stop: $\{u\}$ is a star cutset in G for every vertex u in M. If M is empty then stop: $\{v\}$ is a star cutset in $\overline{G}$ for every vertex v in N.

Now, both M and N are non-empty. If M induces a disconnected subgraph of G then stop: $\{w\} \bigcup N$ is a star cutset in G. If N induces a disconnected subgraph of $\overline{G}$ then stop: $\{w\} \bigcup$ M is a star cutset in $\overline{G}$.

Now, M induces a nonempty connected subgraph of G and N induces a nonempty connected subgraph of $\overline{G}$. If some vertex v in N is adjacent to no vertex in M then stop: $\{w\} \bigcup (N-v)$ is a star cutset in G. In the other case, each vertex in N is adjacent to at least one vertex in M; note that N is a minimal cutset in G. Now, the *WT Min Cut Theorem* guarantees that some vertex u in M is adjacent to all the vertices in N. Stop: $\{w\} \bigcup (M-u)$ is a star cutset in $\overline{G}$. ■

Corollary. *All weakly triangulated graphs are perfect.*

Proof. Argue by contradiction; let G be an imperfect weakly triangulated graph. Then there is some induced subgraph H of G such that H is minimal imperfect; H is also weakly triangulated. Graphs with one or two vertices are perfect; thus H has at least three vertices. But now the *WT Star Cutset Theorem* says that either H or $\overline{H}$ has a star cutset, contradicting Chvátal's *Star Cutset Lemma.* ∎

3.2.3 Star Cutsets and Generating Classes of Perfect Graphs

Chvátal has pointed out that a forbidden property of minimal imperfect graphs may be used to generate large classes of perfect graphs from smaller ones. For example, the star cutset may be used in such a way. Specifically, given any class C of graphs, denote by C^* the class of graphs defined recursively by the following two rules:

(i) if $G \in C$ then $G \in C^*$,

(ii) if G or $\overline{G}$ has a star cutset, and if $G - v \in C^*$ for all $v \in G$, then $G \in C^*$.

Chvátal's *Star Cutset Lemma* implies that C^* is a class of perfect graphs whenever C is. For example, let *Triv* denote the class of all graphs with at most two vertices. What can we say about the class of graphs $Triv^*$? By the *WT Star Cutset Theorem* it follows that $Triv^*$ contains the class of weakly triangulated graphs. On the other hand, neither chordless cycles with five or more vertices nor the complements of such cycles have star cutsets; thus a graph in $Triv^*$ cannot contain C_k or $\overline{C}_k$, for $k \geq 5$, as an induced subgraph. It follows that $Triv^*$ is exactly the class of weakly triangulated graphs. Thus weakly triangulated graphs are *the* class of graphs associated with the property "either a graph or its complement has a star cutset".

Another class of graphs associated with star cutsets is the class Bip^*, where Bip denotes the class of bipartite graphs. Although Bip^* contains $Triv^*$, as well as many other classes of perfect graphs, it is not known whether or not graphs in Bip^* can be recognized in polynomial time.

3.2.4 Which Weakly Triangulated Graphs Have Star Cutsets?

Note that the *WT Star Cutset Theorem* states only that a weakly triangulated graph with at least three vertices, *or its complement*, has a star cutset. We now answer the question "exactly which weakly triangulated graphs have star cutsets?". The following theorem is a strictly stronger statement than the *WT Star Cutset Theorem.* However, we have included both theorems because the proof of the *WT Star Cutset Theorem* is much simpler than the proof of the following theorem, and because the *WT Star Cutset Theorem* suffices to prove that weakly triangulated graphs are perfect. In fact, it is the *WT Star Cutset Theorem* that appears in Hayward [1985].

The Second WT Star Cutset Theorem. *Let G be a weakly triangulated graph. Then exactly one of the following is true:*

(i) *G is a clique,*

(ii) *every component of $\overline{G}$ consists of a single edge,*

(iii) *G has a star cutset.*

Before proving the theorem we present a lemma; before presenting the lemma, we introduce some definitions. A vertex x is said to be *dominated* by a vertex y if every vertex (different from x and y) that is adjacent to x is also adjacent to y. We call a graph with no dominated vertex *domination-free*. Recall that $N(x)$ and $M(x)$ are respectively the neighbourhood and non-neighbourhood of a vertex x.

The WT Domination-Free Lemma. *If G is a domination-free weakly triangulated graph with at least two vertices, then $\overline{G}$ has a star cutset.*

Proof of Lemma. First, we propose to find a vertex v and a component J of $M(v)$ such that

every vertex in $N(v)$ has a neighbour in J. (1)

For this purpose, we borrow a trick from Ravindra [1982]: find a vertex t and a component F of $M(t)$ such that the number of vertices in F is minimized (over all choices of t and F). We claim that (1) holds whenever $v \in F$ and J is the component of $M(v)$ that contains t. To justify this claim, consider an arbitrary x in $N(v)$. We may assume that $x \notin N(t)$, for otherwise t is the neighbour of x in J; hence $x \in F$. Since x is not dominated by v, it has a neighbour y in $M(v)$; trivially, $y \in F \bigcup N(t)$. Now we only need verify that $F \bigcap M(v) \subseteq J$ and $N(t) \bigcap M(v) \subseteq J$. The second of these inclusions is obvious; to verify the first, we only need verify that every y in $F \bigcap M(v)$ has a neighbour in $N(t) \bigcap M(v)$. If the last assertion were false then the component of $M(v)$ that contains y would be contained in $F - v$, contradicting our choice of t and F. Hence (1) holds.

Now consider the subgraph H of G induced by $\{v\} \bigcup N(v) \bigcup J$. It follows from (1) that $N(v)$ is a minimal cutset in H. Next, the *WT Min Cut Theorem* guarantees that the complement of the subgraph induced by $N(v)$ must be disconnected (otherwise v would be dominated by some vertex of J in H, and therefore also in G). But then $\{v\} \bigcup M(v)$ is a star cutset in $\overline{G}$. ■

Proof of the Second WT Star Cutset Theorem. We shall argue by induction; the cases where G has at most four vertices can be checked by inspection. Now suppose that G has at least five vertices. If G is domination-free then $\overline{G}$ is domination-free, and G has a star cutset by the *WT Domination-Free Lemma*. Suppose then that G is not domination-free: in this case there are vertices u and v in G, such that v dominates u, i.e. $N(u) - v$ is a subset of $N(v)$.

Case 1: suppose that u is not adjacent to all the vertices of $G - v$. In this case $\{v\} \bigcup N(u)$ is a star cutset.

Case 2: suppose that u is adjacent to all the vertices of $G - v$. Then, since v dominates u, v is adjacent to all of $G - u$. There are two subcases to consider.

Case 2.1: suppose that u is adjacent to v (thus $N(v) = G - v$ and $N(u) = G - u$). Then either G is a clique, or else there are non-adjacent vertices x and y in G, in which case $G - x - y$ is a star cutset (v is adjacent to all of $G - v$).

Case 2.2: suppose that u is not adjacent to v (thus $N(v) = N(u) = G - u - v$). We now use the inductive hypothesis on $G - u - v$. If $G - u - v$ is a clique, then $G - u - v$ is a star cutset in G. If $G - u - v$ has a star cutset C, then $C \bigcup \{u, v\}$ is a star cutset in G. Finally, note that the complement of G consists of the complement of $G - u - v$ together with a component consisting of the edge induced by $\{u, v\}$. Thus, if every component of $\overline{G - u - v}$ is a single edge, then every component of $\overline{G}$ is a single edge. This completes the proof of the *Second WT Star Cutset Theorem.* ■

Vertices x and y of a graph G are called *twins* if every vertex of $G - x - y$ is adjacent either to both x and y or to neither x nor y. A corollary of the *Second WT Star Cutset Theorem* is that every twin-free weakly triangulated graph with at least three vertices has a star cutset. This is a stronger statement than the *WT Domination-Free Lemma.*

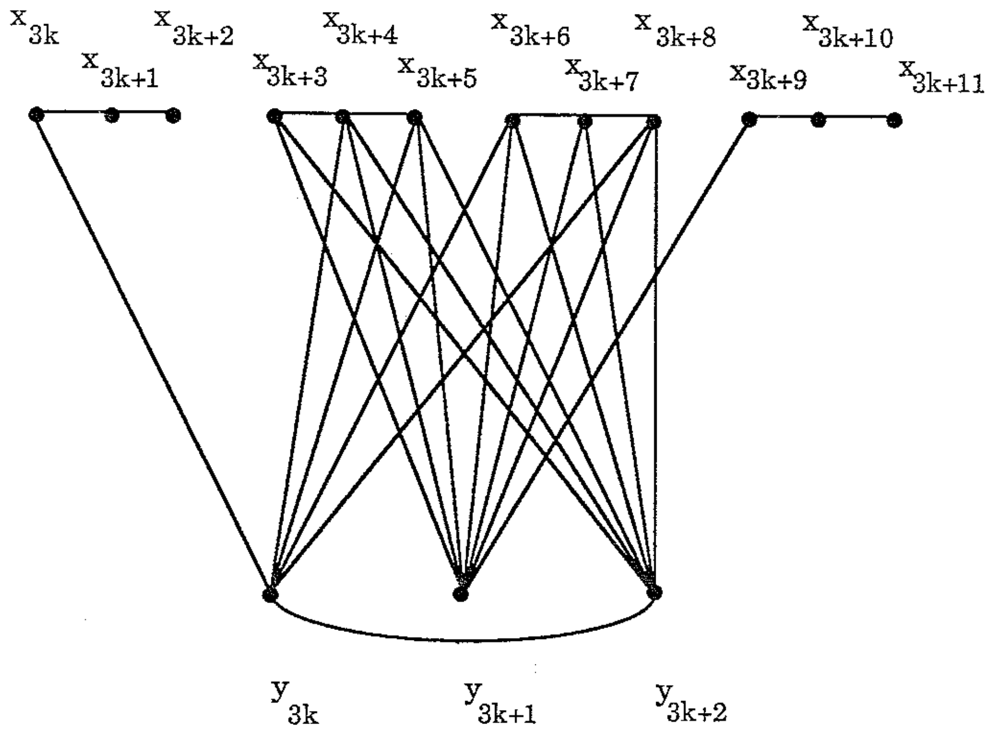

x3k
x3k+1
x3k+2
x3k+3
x3k+4
x3k+5
x3k+6
x3k+7
x3k+8
x3k+9
x3k+10
x3k+11
y3k
y3k+1
y3k+2

Figure 3.2

3.2.5 A Domination-Free Weakly Triangulated Graph

Domination-free weakly triangulated graphs are mentioned in the proof of the *Second WT Star Cutset Theorem.* In this section we describe such a graph W. Our search for a domination-free weakly triangulated graph was motivated by Mahadev [1984].

The set of vertices of W is the union of the set $X = \{ x_0, x_1, x_2, ..., x_{11} \}$ and the set $Y = \{ y_0, y_1, y_2, ..., y_{11} \}$. The only edges of W with both endpoints in X are (x_{3k}, x_{3k+1}) and (x_{3k+1}, x_{3k+2}), for $k = 0,1,2,3$. The only edges of $\overline{W}$ with both endpoints in Y are (y_{3k}, y_{3k+1}) and (y_{3k+1}, y_{3k+2}), for $k = 0,1,2,3$. Finally, for $k = 0,1,2,3$, (all indices are modulo 12)

the only edge of W between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k}, x_{3k+1}, x_{3k+2}\}$ is the edge (y_{3k}, x_{3k}),

the only edge of $\overline{W}$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+3}, x_{3k+4}, x_{3k+5}\}$ is the edge (y_{3k}, x_{3k+3}),

the only edge of $\overline{W}$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+6}, x_{3k+7}, x_{3k+8}\}$ is the edge (y_{3k}, x_{3k+7}),

the only edge of W between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+9}, x_{3k+10}, x_{3k+11}\}$ is the edge (y_{3k+1}, x_{3k+9}).

Table I lists that part of the adjacency matrix of W representing edges of the form (x_i, y_j). *Figure 3.2* is a drawing of the subgraph of W induced by $X \bigcup \{y_{3k}, y_{3k+1}, y_{3k+2}\}$, and *Figure 3.3* is a drawing of the whole of W. Note that W is self-complementary: the permutation P defined by $P(x_i) = y_i$ and $P(y_i) = x_{i+3}$ for $i = 0,1,...,11$ sends edges of W onto edges of $\overline{W}$ and vice versa.

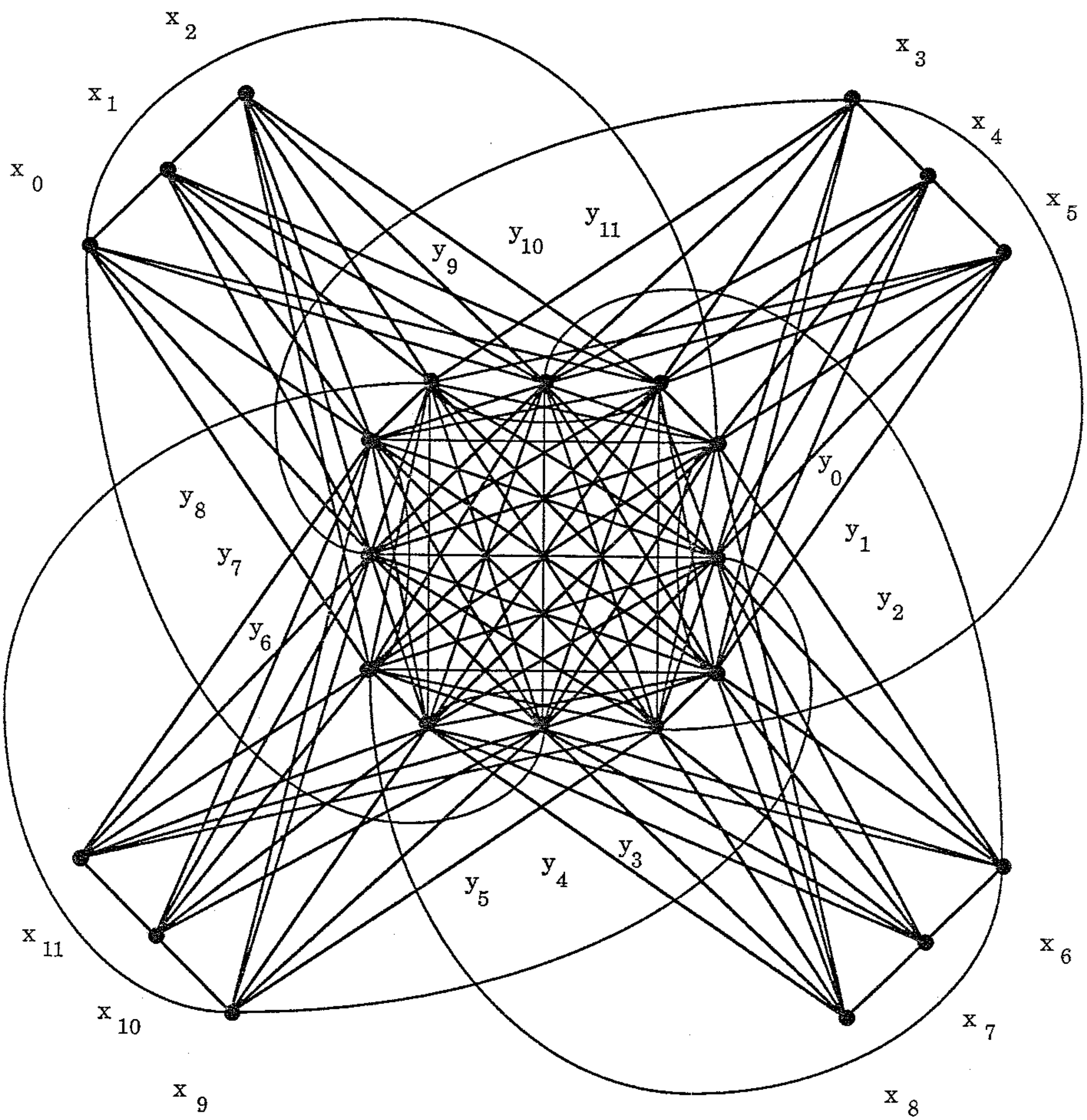


Figure 3.3. The domination-free graph W

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0	1	1	1	1	1
2	0	0	0	0	0	0	1	1	1	1	1	1
3	0	1	1	1	0	0	0	1	0	1	1	1
4	1	1	1	0	0	0	0	0	0	0	1	1
5	1	1	1	0	0	0	0	0	0	1	1	1
6	1	1	1	0	1	1	1	0	0	0	1	0
7	0	1	1	1	1	1	0	0	0	0	0	0
8	1	1	1	1	1	1	0	0	0	0	0	0
9	0	1	0	1	1	1	0	1	1	1	0	0
10	0	0	0	0	1	1	1	1	1	0	0	0
11	0	0	0	1	1	1	1	1	1	0	0	0

Table I. $a_{ij} = 1$ **if and only if** x_i **is adjacent to** y_j **in W**

Since W is self-complementary, in order to prove that W is weakly triangulated it is sufficient to show that W has no chordless cycle C with at least 5 vertices. Argue by contradiction: suppose that W contains such a C. Recall that

(i) the subgraph of W induced by X consists of four disjoint P_3's,

(ii) the subgraph of $\overline{W}$ induced by Y consists of four disjoint P_3's.

It is a routine matter to verify the following three claims:

(iii) W contains no chordless path (p_1, p_2, p_3, p_4) whose intersection with X is the set $\{p_2, p_3\}$,

(iv) W contains no chordless path $(p_1, p_2, p_3, p_4, p_5)$ whose intersection with X is $\{p_2, p_3, p_4\}$,

(v) W contains no chordless cycle $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with X is $\{c_2, c_3, c_4\}$.

From (v) and the fact that both W and C_5 are self-complementary, it follows that

(vi) W contains no chordless $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with X is the set $\{c_1, c_3\}$.

Because of (i), C cannot be properly contained in X. Because of (ii), C cannot be properly contained in Y. Hence, let C_X be the subgraph of W induced by those vertices of C in X and C_Y be the subgraph of W induced by those vertices of C in Y. Both C_X and C_Y must consist of disjoint chordless paths. Because of (i), C_X contains no P_k with $k > 3$. Because of (iv) and (v), C_X contains no P_3. Because of (iii), C_X contains no P_2. Thus C_X consists of pairwise non-adjacent vertices. C_X cannot consist of a single vertex, because then C_Y would contain a P_k, with $k \geq 4$, contradicting (ii). Thus C_X consists of at least two non-adjacent vertices; hence C_Y consists of (at least two) disjoint chordless paths. But C_Y cannot contain three or more disjoint chordless paths, because then $\overline{C}_Y$ would contain a triangle, contradicting (ii). Thus C_Y consists of exactly two disjoint paths; now (ii) implies that one of these paths is an isolated vertex, and the other has two vertices (each subgraph of W induced by at least four vertices in Y is connected). But then the cycle would have to consist of exactly five vertices $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with Y is $\{c_2, c_4, c_5\}$, contradicting (vi). Thus, W is weakly triangulated.

To verify that W is domination-free, assume the contrary: some vertex u is dominated by a vertex v. First, consider the case when u is in X. By symmetry, we may assume that $u = x_i$ with $0 \leq i \leq 2$. To see that v cannot be in Y, consult *Table II.*

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
x_0	x_1	x_1	x_1	x_1	x_1	x_1	y_7	y_8	y_7	y_{10}	y_{11}	y_{10}
x_1	x_2	x_2	x_2	x_2	x_2	x_2	y_7	y_8	y_7	y_{10}	y_{11}	y_{10}
x_2	x_1	x_1	x_1	x_1	x_1	x_1	y_7	y_8	y_7	y_{10}	y_{11}	y_{10}

Table II. Neighbours of x_i non-adjacent to y_j in W.

Thus we must have $v = x_j$ for some j; considering the subgraph of W induced by X, we conclude easily that $0 \leq j \leq 2$. But now we only need observe that

y_0 sees x_0 and misses x_1, x_2,

y_9 sees x_1, x_2 and misses x_0,

y_6 sees x_2 and misses x_1,

x_0 sees x_1 and misses x_2.

Thus u cannot be in X.

Next, consider the case when u is in Y. By symmetry, we may assume that $u = y_i$ with $0 \leq i \leq 2$. To see that v cannot be in X, observe that u is adjacent to both x_2 and x_8, at least one of which is non-adjacent to v. The only remaining subcase, with u and v both in Y, is reduced to a previous subcase by considering the permutation P that sends W onto its complement: clearly, $P(v)$ is dominated by $P(u)$, and both $P(u)$ and $P(v)$ are in X. Thus W is domination-free.

Incidentally, W has neither a clique cutset nor a homogeneous set. Furthermore, W is not strongly perfect. (Recall from Chapter 2 that a graph is strongly perfect if in every induced subgraph there is a stable set that meets all maximal cliques.) In the subgraph of W induced by $Z = \{ x_0, x_1, x_2, x_6, x_7, x_8, y_0, y_1, y_6, y_7 \}$ no stable set meets all maximal cliques. To see this, note that the maximal cliques of this graph are

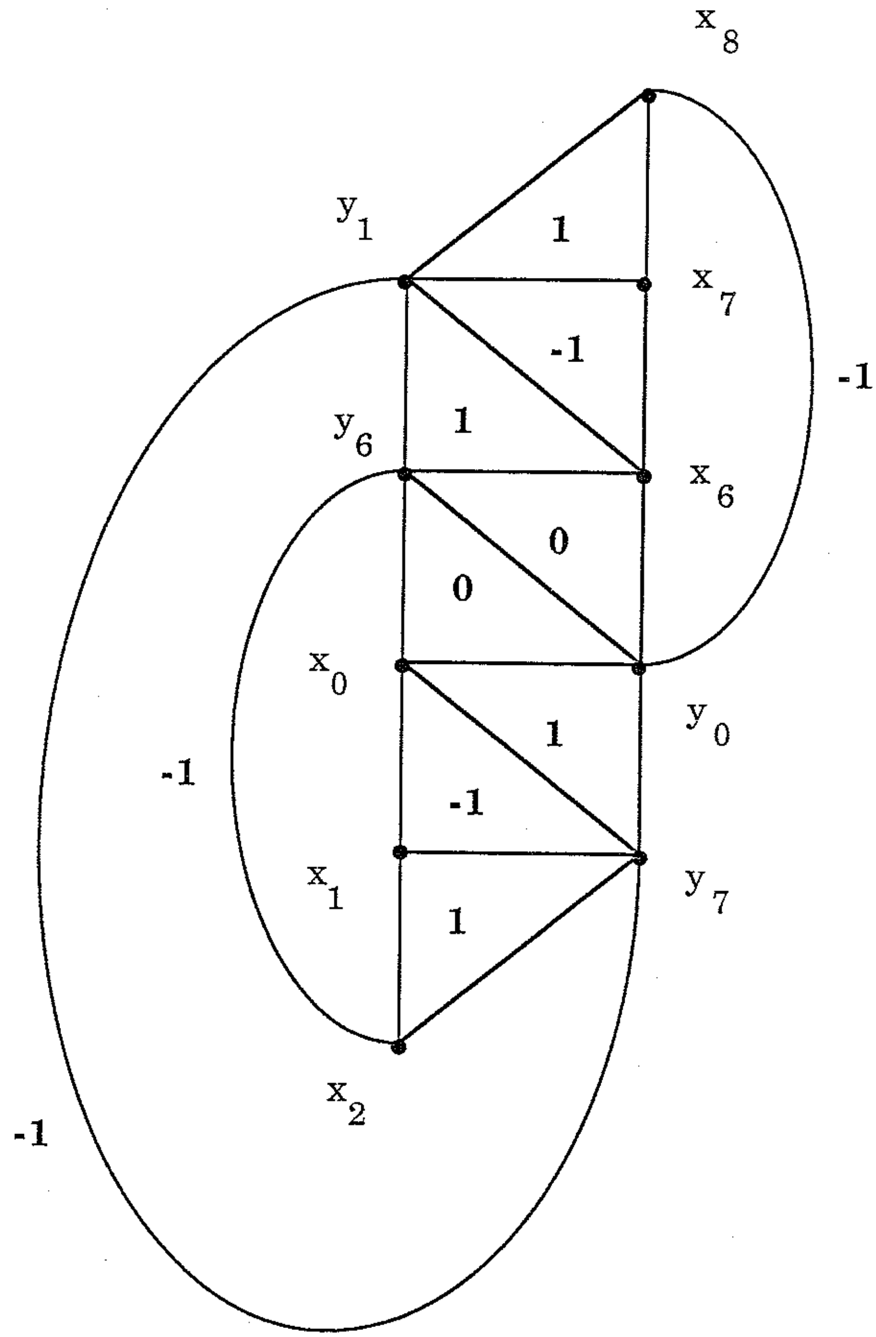


Figure 3.4. The subgraph W_Z, with weights on maximal cliques

$$\{ x_0, x_1, y_7 \}, \{ x_6, x_7, y_1 \}, \{ x_0, y_0, y_6 \}, \{ x_6, y_6, y_0 \},$$

$$\{ x_0, y_0, y_7 \}, \{ x_6, y_6, y_1 \}, \{ x_1, x_2, y_7 \}, \{ x_7, x_8, y_1 \},$$

$$\{ x_2, y_6 \}, \{ x_8, y_0 \}, \text{and } \{ y_1, y_7 \}.$$

Assign to these cliques the integers -1,-1,0,0,1,1,1,1,-1,-1,-1 respectively. The sum of the integers is -1, and yet for each vertex v, the sum of the integers of the cliques that contain v is 0. On the other hand, let S be a stable set that meets every maximal clique of a graph G. Since a stable set meets a clique in at most one vertex, each maximal clique of G meets precisely one vertex of S. Thus, if integers are assigned to the maximal cliques of G such that for each vertex v, the sum of the integers of the cliques that contain v is 0, then the sum of the integers must also be 0. Thus W_Z is not strongly perfect, and so neither is W. A drawing of W_Z, with the maximal cliques labelled as described above, is shown in *Figure 3.4*.

3.3 Weakly Triangulated Graphs and Two-Pairs

An *even pair* is a pair of (non-adjacent) vertices in a graph, such that every chordless path between the two vertices has an even number of edges. Meyniel defined a graph G to be *strict quasi-parity* if every induced subgraph H of G which is not a clique has an even pair. A graph G is *quasi-parity* if every induced subgraph H of G, or its complement $\overline{H}$, is either a clique or has an even pair. Meyniel proved that strict quasi-parity graphs and quasi-parity graphs are perfect. Recently Hoàng and Maffray [1986] proved that weakly triangulated graphs are strict quasi-parity. It is not known whether or not strict quasi-parity graphs, or quasi-parity graphs, can be recognized in polynomial time.

Hoàng and Maffray showed that weakly triangulated graphs are strict quasi-parity by proving that every weakly triangulated graph which is not a clique has an even pair. In fact, a slightly stronger statement is true. We call a pair of vertices a *two-pair* if every chordless path which joins the vertices has exactly two edges. The original theorem of Hoàng and Maffray was easily modified to yield the following theorem.

The WT Two-Pair Theorem. *Every weakly triangulated graph which is not a clique has a two-pair.*

Proof. We shall prove a stronger assertion, namely, that all weakly triangulated graphs G other than cliques have the following two properties:

(1) if G has no clique cutset then each cutset of G contains a two-pair,

(2) G contains a two-pair.

Arguing by induction on the number of vertices, we may assume that both (1) and (2) hold for all weakly triangulated graphs with fewer vertices than G. To prove (1) for G, consider any minimal cutset C of G. By assumption, C is not a clique. We shall distinguish between two cases.

Case 1. Suppose that $\overline{G}_C$ is disconnected. Let D be the set of vertices of some component of $\overline{G}_C$ with at least two vertices (since C is not a clique, there must be such a set D). Note that every vertex of $C - D$ is adjacent to every vertex of D, and that D is a minimal cutset, not a clique, of $G - (C - D)$. Thus by inductive assumption, D contains a two-pair of $G - (C - D)$; obviously, this two-pair is a two-pair of G.

Case 2. Suppose that $\overline{G}_C$ is connected. Let $B_1, \ldots, B_t$ be the vertex sets of the components of $G - C$. Now use the *WT Min Cut Theorem*: in each component B_j, there is some vertex that is adjacent to all of C.

Case 2.1. Suppose that $|B_j| = 1$ for all j. Then, by inductive assumption the graph G_C contains some two-pair $\{x, y\}$. Clearly $\{x, y\}$ is a two-pair of G.

Case 2.2. Suppose that $|B_j| \geq 2$ for some j. Let z be any vertex of B_j that is adjacent to all of C; let D be the set of vertices of C that are adjacent to some vertex of $B_j - z$. Now D is a minimal cutset of $G - z$. Note that D is not empty, and not a clique (otherwise $D \bigcup \{z\}$ is a clique cutset of G, contradiction). Thus, by inductive assumption D contains a two-pair of $G - z$ which is clearly a two-pair of G.

To prove (2) for G, we may assume that G has a clique cutset C (otherwise the desired conclusion follows from (1)). Let $B_1, B_2, \ldots, B_t$ be the vertex sets of the components of $G-C$. If some $G-B_j$ is not a clique then by the induction hypothesis $G-B_j$ contains a two-pair; since every chordless path in G with both endpoints in $G-B_j$ is fully contained in $G-B_j$, this two-pair is also a two-pair in G. Hence we may assume that each $G-B_j$ is a clique. This implies that $t = 2$ and that $\{x, y\}$ is a two-pair whenever $x \in B_1$, $y \in B_2$. ■

A noteworthy distinction between an even pair and a two-pair is that it is easy to check in polynomial time whether or not a pair of vertices is a two-pair: remove the common neighbours, and check whether the original two vertices are in different components of the resulting graph. (We know of no polynomial time algorithm to determine if a pair of vertices is an even pair.) In the next section we build upon this property and derive polynomial time algorithms for solving certain optimization problems for weakly triangulated graphs.

3.4 Optimizing Weakly Triangulated Graphs

3.4.1 Introduction

In this section algorithms are presented which solve the following problems for weakly triangulated graphs in polynomial time.

The Maximum Clique Problem. *Find a largest clique in a graph.*

The Maximum Stable Set Problem. *Find a largest stable set in a graph.*

The Minimum Colouring Problem. *Find a partition of the vertices into the smallest number of stable sets.*

The Minimum Clique Covering Problem. *Find a partition of the vertices into the smallest number of cliques.*

Algorithms are also presented which solve the weighted versions of these problems. In each of the following problems, assume that a graph G with vertices $v_1,..,v_n$ and positive integers $w(v_1),...,w(v_n)$ are given. These integers are referred to as *weights.*

The Maximum Weighted Clique Problem. *Find a clique K of G, such that the sum of the weights of the vertices of K is maximum, over all cliques of G.*

The Maximum Weighted Stable Set Problem. *Find a stable set S of G, such that the sum of the weights of the vertices of S is maximum, over all stable sets of G.*

The Minimum Weighted Colouring Problem. *Find stable sets $S_1,...,S_t$ and integers $X(S_1),...,X(S_t)$, such that*

(1) for every vertex v_j, the sum of the integers $X(S_i)$ of all sets S_i such that $v_j \in S_i$ is at least $w(v_j)$, and such that

(2) the sum of all integers $X(S_1) + ... + X(S_t)$ is minimum, over all sets of integers that satisfy (1).

The Minimum Weighted Clique Covering Problem. *Find cliques $K_1,\dots,K_t$ and integers $X(K_1),\dots,X(K_t)$, such that*

(1) for every vertex v_j, the sum of the integers $X(K_i)$ of all sets K_i such that $v_j \in K_i$ is at least $w(v_j)$, and such that

(2) the sum of all integers $X(K_1) + \dots + X(K_t)$ is minimum, over all sets of integers that satisfy (1).

An algorithm which solves any of the weighted problems can be used to solve the unweighted version of the problem by assigning the weight "1" to all vertices. However, since our algorithms for the unweighted problems are more transparent and more efficient (in the sense of worst time complexity) than the algorithms for the weighted problems, we include both sets of algorithms.

Actually, we present only two algorithms. *Algorithm OPT* solves the maximum clique and minimum colouring problem for weakly triangulated graphs; *Algorithm W-OPT* solves the weighted versions of these problems. Since the complement of a weakly triangulated graph is weakly triangulated, *Algorithms OPT* and *W-OPT* can also be used to solve the unweighted and weighted versions respectively of the maximum stable set and minimum clique covering problems: to find a largest stable set of a graph G, find a largest clique of $\overline{G}$; to find a minimum clique covering of a graph G, find a minimum colouring of $\overline{G}$.

Our algorithms rely on the fact that every weakly triangulated graph is either a clique or else has a two-pair (see the previous section). The aforementioned optimization problems are easily solved for graphs which are cliques. Given a weakly triangulated graph other than a clique, our algorithms repeatedly find a two-pair, each time transforming the graph in question into a smaller weakly triangulated graph by "identifying" the two-pair. (We will define this term shortly.) Eventually the original graph is transformed into a clique; the optimization problem is solved for the clique, and the two-pair identification process is reversed, transforming the solution of the optimization problem for the clique to the solution of the optimization problem for the original graph.

3.4.2 The Unweighted Case

Let $G(xy \rightarrow z)$ be the graph obtained by replacing vertices x and y of G with a vertex z, such that z sees exactly those vertices of $G - \{x, y\}$ that see at least one of $\{x, y\}$. The *identification* of x and y and G is the process of replacing G with $G(xy \rightarrow z)$.

In the following algorithm, we specify a colouring by a function f_G that assigns some integer from 1 to t to each vertex, such that adjacent vertices are assigned different integers. Assume that $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices of G.

Algorithm OPT(G).

Input: a weakly triangulated graph G.

Output: a largest clique K_G and a minimum colouring f_G.

Step 1. Look for a two-pair $\{x, y\}$ of G.

If G has no two-pair, then

(a) $K_G \leftarrow V(G)$,

(b) for $i = 1$ to n do $f_G(v_i) \leftarrow i$, and

(c) STOP.

Step 2. $J \leftarrow G(xy \rightarrow z)$.

Step 3. $K_J, f_J \leftarrow$ OPT(J).

Step 4a. If $z \notin K_J$ then $K_J \leftarrow K_G$, else ($z \in K_J$ and...)

if x sees all of $K_J - \{z\}$ then $K_G \leftarrow K_J - \{z\} + \{x\}$,

else $K_G \leftarrow K_J - \{z\} + \{y\}$.

Step 4b. $f_G(x) \leftarrow f_G(y) \leftarrow f_J(z)$;

for each $v_i \in J - \{x, y\}$ do

$f_G(v_i) \leftarrow f_J(v_i)$. ∎

To prove the correctness of *Algorithm OPT*, we need to establish several properties concerning the identification of a two-pair in a weakly triangulated graph. One such property is described in the following lemma.

The Identification Lemma. *Let G be a weakly triangulated graph with a two-pair* $\{x,y\}$. *Then* $G(xy \rightarrow z)$ *is weakly triangulated.*

Proof. Let $H = G(xy \rightarrow z)$. We prove that if H is not weakly triangulated, then neither is G. Assume that H is not weakly triangulated. Then there is some subset C of the vertices of H, such that the subgraph H_C of H induced by C is either C_k or $\overline{C}_k$, with $k \geq 5$. If $z \notin C$, then clearly G is not weakly triangulated. Thus we may assume that $z \in C$.

Case 1. H_C is a chordless cycle $c_1 \ldots c_k$ with $k \geq 5$.

Assume without loss of generality that $z = c_1$. Then $c_2 \ldots c_k$ is a chordless path in G. Since z sees c_2, c_k, and none of $c_3, \ldots, c_{k-1}$, at least one of $\{x, y\}$ sees c_2, and similarly c_k, and neither x nor y sees any of $\{c_3, \ldots, c_{k-1}\}$. Now observe that at least one of $\{x, y\}$ must see both of $\{c_2, c_k\}$. (Suppose not; assume w.l.o.g. that x sees c_2 but not c_k and that y sees c_k but not c_2. Then $(x, c_2, \ldots, c_k, y)$ is a chordless path with at least six vertices, contradicting the assumption that $\{x, y\}$ is a two-pair.) Thus assume w.l.o.g. that x sees both of $\{c_2, c_k\}$. Then $\{x, c_2, \ldots, c_k\}$ induces a C_k in G, G is not weakly triangulated, and the theorem holds in this case.

Case 2. $\overline{H}_C$ is a chordless cycle $c_1 \ldots c_k$ with $k \geq 5$.

Assume without loss of generality that $z = c_1$. Thus $c_2 \ldots c_k$ is a $\overline{P}_{k-1}$ in G, and

(i) c_2 sees neither x nor y and c_k sees neither x nor y, and

(ii) every vertex in $\{c_3, \ldots, c_k\}$ sees at least one of $\{x, y\}$.

Now observe that

(iii) x or y sees both c_3 and c_4.

(Assume the contrary. By (ii) either x or y sees c_3; assume w.l.o.g. that x sees c_3. Since (iii) does not hold, x does not see c_4; thus by (ii) y sees c_4, and since (iii) does not hold, y does not see c_3. But then (x, c_3, c_k, c_4, y) is a P_5, contradicting the fact that $\{x, y\}$ is a two-pair in G.)

Assume w.l.o.g. that x sees both c_3 and c_4; let m be the smallest index greater than four such that x does not see c_m. Then $x\, c_2 \ldots c_m$ is a $\overline{C}_k$, with $k \geq 5$, G is not weakly triangulated, and the theorem holds in this case. ■

Another result that will be used in proving the correctness of *Algorithm OPT* is that two-pair identification does not change the clique size. This follows from a lemma due to Meyniel.

The Clique Size Lemma (Meyniel [1986]). *If vertices x and y of a graph G are not joined by any chordless path with three edges, then $\omega(G(xy \rightarrow z)) = \omega(G)$.*

The Clique Size Corollary. *If $\{x,y\}$ is a two-pair of the weakly triangulated graph G, then $\omega(G(xy \rightarrow z)) = \omega(G)$.*

The Correctness Theorem. *Algorithm OPT finds a largest clique and a minimum colouring of G.*

Proof. Throughout the proof we let $|f_G|$ and $|f_J|$ denote the number of colours of f_G and f_J respectively. Since the clique size of a graph is never greater than the chromatic number, to prove the theorem it suffices to show that K_G is a clique, that f_G is a colouring, and that $|K_G| = |f_G|$. The proof is by induction on the number of calls of OPT. (Since identification decreases the number of vertices by one, OPT is called at most n times; thus the algorithm terminates.) If OPT is called only once, then the algorithm terminates at Step 1. By the *WT Two-Pair Theorem*, $K_G = V(G)$ is a clique, f_G is a colouring with $n = |K_G|$ colours, and the theorem holds.

Suppose then that OPT is called more than once; thus the algorithm terminates with Step 4b. Since (by the *Identification Lemma*) J is weakly triangulated, by the inductive hypothesis we may assume that K_J and f_J are a respectively a clique and a colouring of J, such that $|K_J| = |f_J|$. If $z \notin K_J$, then $K_G = K_J$, and $|K_G| = |K_J|$. If $z \in K_J$, then either x or y must see all vertices of $K_J - z$. (Suppose not. Then x misses some $v_i \in K_J$; however, y sees v_i, else z would miss v_i. Similarly, y misses some $v_j \in K_J$ that sees x. But then $xv_j v_i y$ is a chordless path, contradicting the assumption that $\{x, y\}$ is a two-pair.) Thus $|K_G| \geq |K_J|$. Since K_J is a largest clique of J, the *Identification Lemma* implies that $|K_G| = |K_J|$.

Since no pair of adjacent vertices a, b of J satisfy $f_J(a) = f_J(b)$, no pair of adjacent vertices a, b of $G - \{x, y\}$ satisfy $f_G(a) = f_G(b)$. Finally, let c be a vertex of G that sees at least one of $\{x, y\}$; then c sees z in J, and so

$$f_G(c) = f_J(c) \neq f_J(z) = f_G(x) = f_G(y).$$

Thus no pair of adjacent vertices u, v of G satisfy $f_G(u) = f_G(v)$, and f_G is a colouring. Note that $|f_G| = |f_J|$. Thus $|K_G| = |K_J| = |f_J| = |f_G|$, and the theorem is proved. ∎

A corollary of the *Correctness Theorem* is that $\omega(G) = \chi(G)$ if G is weakly triangulated. Thus (since every induced subgraph of a weakly triangulated graph is weakly triangulated) the *Correctness Theorem* yields another proof that weakly triangulated graphs are perfect.

We now analyze the complexity of *Algorithm OPT(G)*. Let e be the number of edges of G, and n the number of vertices. Note that a pair of non-adjacent vertices x and y in a graph G is a two-pair if and only if there is no path from x to y in $G - N$, where N is the set of all vertices of G that see both x and y. Determining whether or not two vertices are in the same component of a graph can be done in time $O(n+e)$. Thus determining whether or not a pair of vertices is a two-pair can be done in time $O(n+e)$, and Step 1 can be done in time $O((n+e)n^2)$. Step 2 can be done in time $O(n)$, as can Steps 4a and 4b. Since Step 3 is executed at most $n-1$ times, the worst-case complexity of *Algorithm OPT* is $O((n+e)n^3)$.

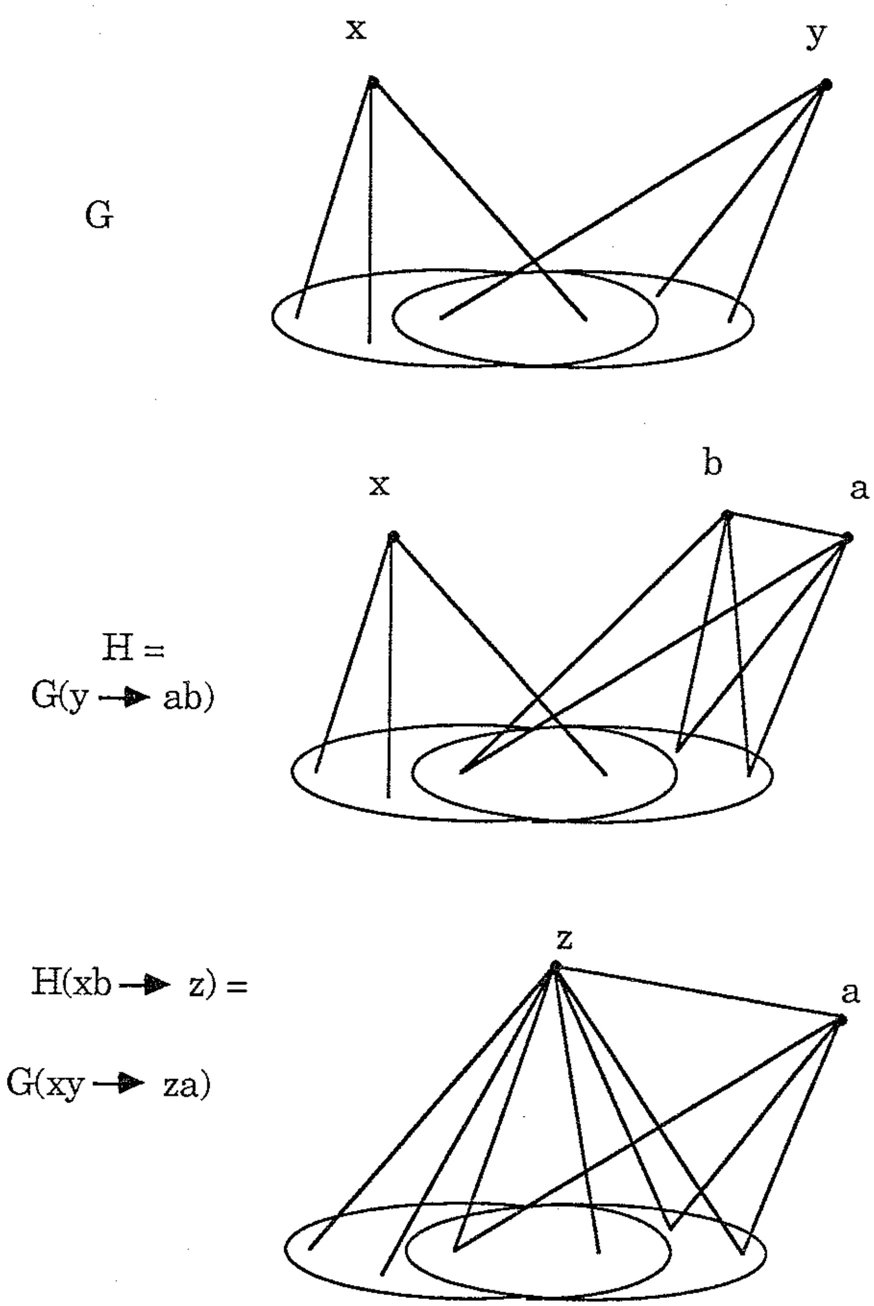


Figure 3.5. Quasi-identification

3.4.3 The Weighted Case

In this section we present polynomial time algorithms that solve the weighted versions of the maximum clique, maximum stable set, minimum colouring and minimum clique covering problems for weakly triangulated graphs.

One way to solve the weighted clique problem for a graph G is to replace every vertex v of G with a clique of size $w(v)$, and then solve the unweighted clique problem on the resulting graph. However, this transformation is inefficient if the weights are large. Our solution is more direct.

Define $G(u \rightarrow vw)$ to be the graph obtained from the graph G by replacing the vertex u with vertices v and w, such that v sees w, and such that u,v,w see exactly the same vertices of $G - u$. This process is referred to as *duplication.*

We now define an operation that combines identification and duplication. Define $G(xy \rightarrow za)$ to be the graph $H(xb \rightarrow z)$, where $H = G(y \rightarrow ab)$. We refer to the process of replacing G with $G(xy \rightarrow za)$ as *quasi-identification.*

Quasi-identification is represented in *Figure 3.5.* Note that $G(xy \rightarrow za)$ is the graph obtained from G by replacing x,y with z,a respectively, such that z sees a, z sees exactly those vertices of $G - \{x,y\}$ that see at least one of $\{x,y\}$, and a sees exactly those vertices of $G - \{x,y\}$ that see y.

In the following algorithm, the weighted colouring f_G consists of stable sets S_{G_1}, S_{G_2}, ..., S_{G_r} and associated positive integers $X(S_{G_1})$, $X(S_{G_2})$, ..., $X(S_{G_r})$.

Algorithm W-OPT(G).

Input: a weakly triangulated graph G.

Output: a max. weighted clique K_G and a min. weighted colouring f_G.

Step 1. Look for a two-pair $\{x, y\}$ of G.

If G has no two-pair then

(a) $K_G \leftarrow V(G)$,

(b) for $i \leftarrow 1$ to n do

$$S_{G_i} \leftarrow \{v_i\},$$

$$X(S_{G_i}) \leftarrow w(v_i)$$

(c) **STOP.**

Step 2. Assume that $w(x) \leq w(y)$.

If $w(x) = w(y)$ then

$$J \leftarrow G(xy \rightarrow z),$$

$$w(z) \leftarrow w(x);$$

else { ... thus $w(x) < w(y)$ ... }

$$J \leftarrow G(xy \rightarrow za),$$

$$w(z) \leftarrow w(x),$$

$$w(a) \leftarrow w(y) - w(x).$$

Step 3. $K_J, f_J \leftarrow$ **W-OPT**(J).

Step 4a. If $z \notin K_J$ then $K_G \leftarrow K_J$, else ($z \in K_J$ and ...)

if y sees all of $K_J - \{a, z\}$ then $K_G \leftarrow K_J - \{a, z\} + y$

else (...x sees all of $K_J - \{a, z\}$...) $K_G \leftarrow K_J - \{a, z\} + x$.

Step 4b. For each set S_{J_i} of f_J do

(i) if $z \in S_{J_i}$ then $S_{G_i} \leftarrow S_{J_i} - z + \{x, y\}$, else

if $a \in S_{J_i}$ then $S_{G_i} \leftarrow S_{J_i} - a + y$, else

$S_{G_i} \leftarrow S_{J_i}$,

(ii) $X(S_{G_i}) \leftarrow X(S_{J_i})$. ■

The proof of correctness of *Algorithm W-OPT* parallels the proof of correctness of *Algorithm OPT*. We first show that quasi-identification of a two-pair of a weakly triangulated graph yields a weakly triangulated graph.

The Quasi-Identification Lemma. *Let G be a weakly triangulated graph with a two-pair $\{x, y\}$. Then $G(xy \rightarrow za)$ is weakly triangulated.*

Proof. $G(xy \rightarrow za) = H(xb \rightarrow z)$, where $H = G(y \rightarrow ab)$. It is easy to check that H is weakly triangulated and that $\{x, b\}$ is a two-pair of H. Now the result follows from the *Identification Lemma.* ∎

Next we prove that the process of quasi-identification, together with the reweighting of the new vertices as described in *Algorithm W-OPT*, does not change the weighted clique number of G. Let $\Omega(G)$ represent the weighted clique number of G (i.e. the weight of a maximum weighted clique of G).

The Weighted Clique Number Lemma. *Let G be a weighted weakly triangulated graph with a two-pair $\{x,y\}$ such that $w(x) \leq w(y)$. Let $F = G(xy \rightarrow za)$, and let $w(z) = w(x)$ and $w(a) = w(y) - w(x)$. Then $\Omega(G) = \Omega(F)$.*

Proof. $F = G(xy \rightarrow za) = H(xb \rightarrow z)$, where $H = G(y \rightarrow ab)$. Let $w(b) = w(x)$; clearly $\Omega(H) = \Omega(G)$. To prove the lemma we need only show that $\Omega(F) = \Omega(H)$.

Let K_H be a clique of H of maximum weight. Since x, b are non-adjacent, K_H contains at most one of these two vertices. If K_H contains neither x nor b, then K_H is a clique of F. If K_H contains x, then $K_H - x + z$ is a clique of F with the same weight as K_H; if K_H contains b, then $K_H - b + z$ is a clique of F with the same weight as K_H. Thus $\Omega(F) \geq \Omega(H)$.

Now let K_F be a clique of F of maximum weight. If $z \notin K_F$ then K_F is a clique of H; if $z \in K_F$ then either $K_F - z + x$ or $K_F - z + b$ is a clique of H, and both have the same weight as K_F. Thus $\Omega(H) \geq \Omega(F)$. ∎

The Weighted Correctness Theorem. *Algorithm W-OPT solves the Maximum Weighted Clique Problem and the Minimum Weighted Colouring Problem for a weakly triangulated graph G.*

Proof. Let K_G and f_G be as described in *Algorithm W-OPT*. It is easy to check that K_G is a clique, and that S_{G_i} is a stable set, for all i. Let $|K_G| = \sum_{v \in K_G} w(v)$ and let $|f_G| = \sum_i X(G_i)$. We wish to show that f_G satisfies property (1) of the definition of the *Minimum Weight Colouring Problem,* and that $|K_G| = |f_G|$. Note that if K is any clique of a weighted graph, and if f is any colouring that satisfies (1), then $|K| \leq |f|$; thus the equality $|K_G| = |f_G|$ implies that both K_G and f_G are optimal.

We first show that (1) holds for f_G. Argue by induction on the number of times Step 1 is executed in *W-OPT(G)*. If Step 1 is executed only once, then $X(S_{G_i}) = w(v_i)$ for all $i = 1,..,n$, and (1) holds.

Suppose then that Step 1 is executed at least twice. Thus the algorithm terminates with Step 4. Assume by induction that (1) holds for the colouring f_J of J. Recall that in Step 4b,

the vertex z is replaced (in every set S_{J_i} of f_J that contains z) with the pair of vertices x, y, and, if $w(x) < w(y)$,

the vertex a is replaced (in every set S_{J_i} of f_J that contains a) with the vertex y.

In the case where $w(x) = w(y)$, we have $w(z) = w(x) = w(y)$, and so

$$w(x) = w(z) = \sum_{S_{J_i} \supseteq z} X(S_{J_i}) = \sum_{S_{G_i} \supseteq x} X(S_{G_i}),$$

$$w(y) = w(z) = \sum_{S_{J_i} \supseteq z} X(S_{J_i}) = \sum_{S_{G_i} \supseteq y} X(S_{G_i}).$$

In the case where $w(x) < w(y)$, we have $w(x) = w(z)$ and $w(y) = w(a) + w(z)$, and so

$$w(x) = w(z) = \sum_{S_{J_i} \supseteq z} X(S_{J_i}) = \sum_{S_{G_i} \supseteq x} X(S_{G_i}),$$

$$w(y) = w(z) + w(a) = \sum_{S_{J_i} \supseteq z} X(S_{J_i}) + \sum_{S_{J_i} \supseteq a} X(S_{J_i}) = \sum_{S_{G_i} \supseteq y} X(S_{G_i}).$$

Thus property (1) holds for f_G.

Now we wish to show that $|K_G| = |f_G|$. Argue by induction on the number of executions of Step 1; the result clearly holds if Step 1 is executed exactly once. Assume then that Step 1 is executed more than once; thus the algorithm terminates with Step 4. By the induction hypothesis, $|K_J| = |f_J|$.

Now an argument similar to that used in the *Correctness Theorem* establishes that $|K_G| = |K_J|$; thus to finish the proof, we need only show that $|f_G| = |f_J|$. But this is obviously the case, because there is a one-to-one correspondence between the stable sets of f_G and f_J, namely S_{G_i} corresponds to S_{J_i}, and $X(S_{G_i}) = X(S_{J_i})$ for all i. ∎

We now analyze the complexity of *Algorithm W-OPT(G)*. Let e be the number of edges of G, and n the number of vertices. As in *Algorithm OPT(G)*, Step 1 can be done in time $O((n+e)n^2)$, and Steps 2, 4a and 4b can be done in time $O(n)$. Now consider Step 3. The graph J is either $G(xy \rightarrow z)$ or $G(xy \rightarrow za)$. In the former case J has one vertex fewer than G; in the latter case, J has at least one edge more than G (z

sees every vertex of $G - \{x, y\}$ that x sees, a sees every vertex of $G - \{x, y\}$ that y sees, and z sees a whereas x misses y). Thus Step 3 is executed at most $n - 1 + \binom{n}{2} - e$ times, and the worst-case complexity of *Algorithm W-OPT* is $O((n+e)n^4)$.

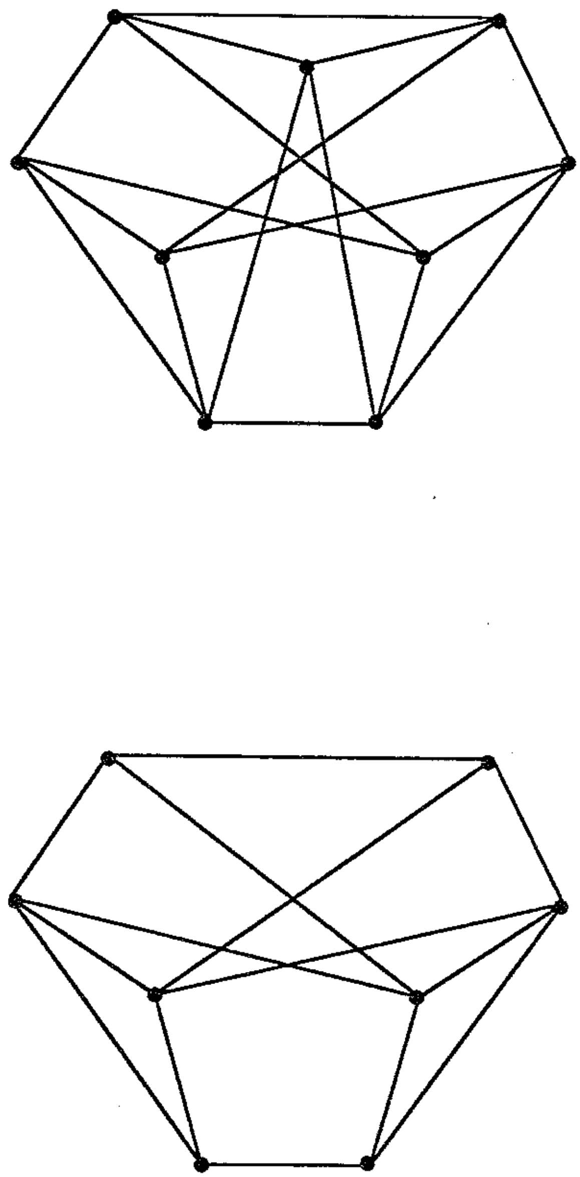


Figure 4.1. L_8 and L_9 (bottom and top)

Chapter 4

Murky Graphs

4.1 The Main Result

In this chapter we introduce a new class of Berge graphs, namely murky graphs, and prove that murky graphs are perfect. A graph is *murky* if it contains neither C_5, P_6, nor $\overline{P}_6$ as an induced subgraph.

Recall (see Chapter 1) that a graph is unbreakable if neither the graph nor its complement has a star cutset. A class H of graphs is called *hereditary* if every induced subgraph of a graph in H is in H. Since minimal imperfect graphs are unbreakable, to prove that the graphs in some hereditary class C are perfect, we only need prove that the unbreakable graphs in C are perfect. Clearly murky graphs are hereditary; thus to prove that murky graphs are perfect we need only prove that unbreakable murky graphs are perfect.

The *line graph L(G)* of a graph G is the graph whose vertices correspond to the edges of G, such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share a vertex. $K_{3,3}$ is the graph with six vertices whose complement consists of two disjoint triangles. $K_{3,3}$-e is the graph obtained by removing any edge from $K_{3,3}$. We let L_8 and L_9 denote the line graphs of $K_{3,3}$-e and $K_{3,3}$ respectively. Drawings of L_8 and L_9 are shown in *Figure 4.1.*

There are two kinds of unbreakable murky graphs: those that contain L_8 as an induced subgraph, and those that do not. Let U be an unbreakable murky graph. If U contains L_8 as an induced subgraph, then U is either L_8 or L_9. If U does not contain L_8 as an induced subgraph, then U can be constructed by taking two copies of a P_4-free graph, and adding a specified set of edges between the two copies. The following is a

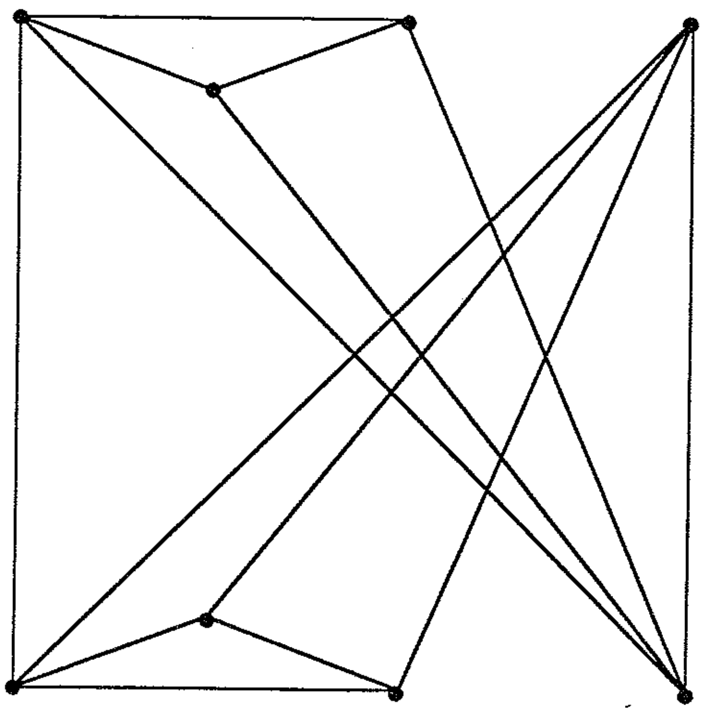


Figure 4.2. A mirror graph

formal definition of such graphs, which we call "mirror graphs".

Define a *mirror partition* $[R, S]$ of a graph G to be a partition of the vertices into sets $R = \{r_1, ..., r_t\}$ and $S = \{s_1, ..., s_t\}$ such that

(1) G_R *and* G_S *are* P_4*-free, and*

(2) r_i *sees* r_j *if and only if* s_i *sees* s_j *if and only if*

r_i *misses* s_j *if and only if* s_i *misses* r_j, *for* $1 \leq i < j \leq t$.

(Note that one consequence of (2) is that G_R and G_S are isomorphic.)

Any graph that has a mirror partition is called a *mirror graph.* With respect to a mirror partition $[R, S]$ of a mirror graph, a pair of corresponding vertices $\{r_j, s_j\}$ is a *couple,* and r_j is the *mate* of s_j (and vice versa). Note that in a mirror graph the vertices of a couple may or may not be adjacent. A mirror graph is shown in *Figure 4.2.*

Recall that vertices x and y are *twins* in a graph G if every vertex in $G - \{x, y\}$ sees both or neither of $\{x, y\}$. Lovász [1972a] showed that a minimal imperfect graph does not have twins. Olariu calls vertices u and v in a graph G *anti-twins* if every vertex in $G - \{u, v\}$ sees exactly one of $\{u, v\}$; he proved that a minimal imperfect graph does not have anti-twins [1986]. (His proof of this result appears in the appendix.)

Burlet and Uhry (see *Lemma 5* in [1984]) observed that every P_4-free graph with at least two vertices has twins. (We use this fact in the proof of the following proposition, and frequently throughout the chapter.) We prove a similar result for mirror graphs.

The Mirror Proposition. *Let F be an induced subgraph of a mirror graph G. If F has at least two vertices then F contains twins or anti-twins.*

Proof. Let $[R, S]$ be a mirror partition of G. Define

$$A = \{i : r_i \in F\}, \quad B = \{j : s_j \in F\}.$$

If some k belongs to $A \cap B$ then r_k, s_k are anti-twins in F. Hence we may assume that $A \cap B = \phi$. Now let F^* be the graph induced by all r_k with $k \in A \cup B$; let r_i, r_j be twins in F^*. If $i \in A$, $j \in A$ (or $i \in B$, $j \in B$) then r_i, r_j (or s_i, s_j) are twins in F; if $i \in A$, $j \in B$ (or $i \in B$, $j \in A$) then r_i, s_j (or s_i, r_j) are anti-twins in F. ∎

The main results of this chapter are summarized by the following two theorems. The proof of *Theorem 4.1* takes up most of the rest of the chapter. The proof of *Theorem 4.2* follows almost immediately from *Theorem 4.1,* and is presented below.

Theorem 4.1. *If G is an unbreakable murky graph, then G is L_8, L_9, or a mirror graph.*

Theorem 4.2. *Murky graphs are perfect.*

Proof of Theorem 4.2. By the *Star Cutset Lemma* and the fact that murky graphs satisfy the hereditary property, we need only prove that unbreakable murky graphs are perfect; by *Theorem 4.1* we need only prove that L_8, L_9 and mirror graphs are perfect. It is a routine exercise to check that L_8 and L_9 are perfect (actually, all line graphs of bipartite graphs are perfect: this follows from a theorem due to König [1936] concerning the edge-chromatic number of a bipartite graph). That mirror graphs are perfect follows from the *Mirror Proposition,* and the fact that a minimal imperfect graph contains neither twins nor anti-twins. ∎

The proof of *Theorem 4.1,* which appears at the end of Section 4.3, is preceded by several intermediate results: Sections 4.2 and 4.3 contain lemmas concerning properties of unbreakable mirror graphs. As a postscript, in Section 4.4 we present a theorem which extends *Theorem 4.1* to a characterization of unbreakable murky graphs.

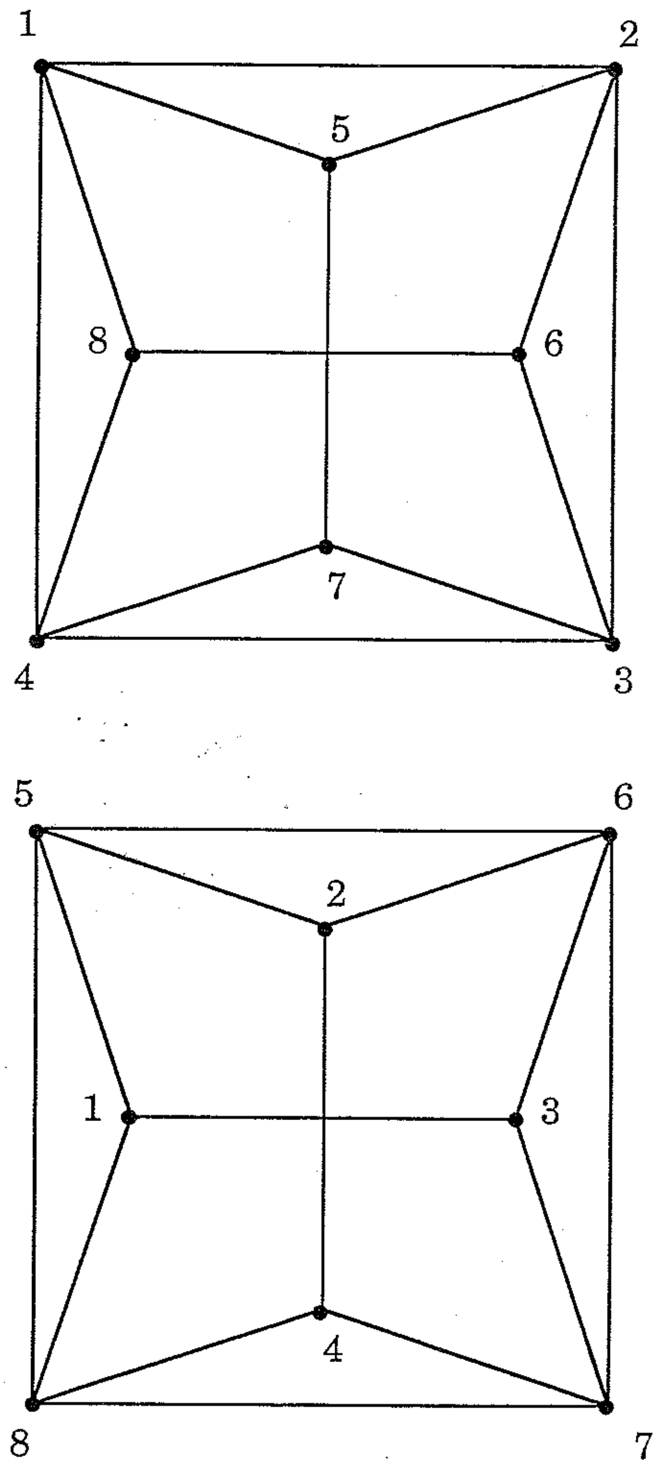


Figure 4.3. L_8 (top) and its complement (bottom)

4.2 Local Properties of Unbreakable Mirror Graphs

In this section we prove several lemmas concerning unbreakable murky graphs. As almost every result in this section is concerned with graphs which contain or do not contain other graphs as induced subgraphs, the following abbreviation will be adopted: we shall say that a graph *contains* some other graph if the latter is an induced subgraph of the former. Similarly, a graph *properly* contains some other graph if the latter is a *proper* induced subgraph of the former.

The definition of "twins" is extended as follows: given vertices x and y and a subset H of the vertices of G, the vertices x and y are called *twins with respect to* H if x and y see exactly the same set of vertices of $H \cap (G - \{x, y\})$. Given a vertex v and a subset X of the vertices of a graph, we say that v is (respectively) *null, partial,* or *universal* on X if v sees (respectively) none, some but not all, or all, of the vertices of X.

The L_8 Lemma. *If an unbreakable murky graph contains L_8, then it is either L_8 or L_9.*

Before proving the lemma, we present two claims. The first states how a vertex can attach to L_8 in a murky graph; the second is a similar statement, but with the added hypothesis that the graph is unbreakable.

Claim Attach. *Let X be a subset of the vertices of a murky graph G such that X induces L_8, and such that some vertex v of $G - X$ is partial on X. Then either there is some vertex u in X such that u and v are twins with respect to X, or else $X + v$ induces L_9.*

Proof of Claim. Label the vertices of X as in *Figure 4.3*. Let v be an arbitrary vertex outside X. Consider the following four cases.

Case 1: v misses all of 1,2,3,4.

Since v sees at least one vertex in X, assume w.l.o.g. that v sees 5. Now v sees 6 (to avoid a P_6 on $v\,51436$); by rotational symmetry, v seeing 6 forces v to see 7, and v seeing 7 forces v to see 8. But then $X + v$ induces L_9.

Case 2: v misses all of 5,6,7,8.

Since v sees at least one vertex in X, assume w.l.o.g. that v sees 1. If v sees 3 then $v\,1573$ is a C_5; if v misses 3 then $v\,15736$ is a P_6. Hence this case cannot occur.

Case 3: v sees 1 but misses 2 and 3.

Now v misses 7 (to avoid a C_5 on $v\,1237$).

Subcase 3.1: v sees 6.

Now v misses 5 (to avoid a C_5 on $v\,6375$) and v sees 4 (to avoid a C_5 on $v\,6341$). But then v and 8 are twins with respect to X.

Subcase 3.2: v misses 6.

Now v sees 8 (to avoid a P_6 on $v\,18637$) and v sees 5 (to avoid a P_6 on $v\,15736$). But then $v\,5268$ is a C_5. Hence this subcase cannot occur.

Case 4: v sees 6 but misses 5 and 7.

Now v misses at least one of 1,3 (to avoid a C_5 on $v\,1573$) and v misses at least one of 2,4 (to avoid a C_5 on $v\,2574$). But then this case reduces to Case 1 or (possibly rotated) Case 3.

We now show that the proof reduces to one of the previous cases. If v misses all of 1,2,3,4 then it satisfies the hypothesis of Case 1; if v sees all of 1,2,3,4 then it satisfies the hypothesis of Case 2 on $\overline{G}_X$. Hence we may assume that v is partial on $\{1,2,3,4\}$; next, rotational symmetry allows us to assume that v sees 1 and misses 2. If v misses 3 then it satisfies the hypothesis of Case 3; if v sees 3 then it satisfies the hypothesis of Case 4 on $\overline{G}_X$. This concludes the proof of *Claim Attach.* ∎

Claim No-Twins. *Let X be a subset of the vertices of an unbreakable murky graph G such that X induces L_8. Then there is no vertex v in $G - X$ such that v is a twin with respect to X of some vertex of X.*

Proof of Claim. Assume the contrary: there is a vertex u in X such that the set S of all twins of u with respect to X (including u itself) has size at least two. Without loss of generality, we may assume that $u = 1$ (all other cases reduce to this one by rotation and complementation). Note that S includes no vertices of X except 1. Since G is unbreakable, S is not a homogeneous set in G. Hence some vertex v outside S sees some a in S and misses some b in S; trivially, $v \notin X$. Let A and B denote the subgraphs of G induced by $X + a - 1$ and $X + a - b$ respectively. Note that v must be partial on X (else v would have precisely one neighbour in A or precisely seven neighbours in B, contradicting *Claim Attach*) and that $X + v$ does not induce L_9 (else v would contradict *Claim Attach* with A in place of X). By *Claim Attach,* v must be a twin with respect to X of some w in X; since $v \notin S$, we have $w \neq 1$; now symmetry (swapping 5 with 8, 2 with 4, and 6 with 7) allows us to assume that w is one of 2,3,5,6. If $w = 3$ or $w = 6$ then v contradicts *Claim Attach* with A in place of X; if $w = 2$ or $w = 5$, then v contradicts *Claim Attach* with B in place of X. This completes the proof of *Claim No-Twins.* ∎

Proof of the $\mathbf{L_8}$ Lemma. Let X be a proper subset of the vertices of an unbreakable murky graph G such that X induces L_8. Since G is unbreakable, X is not a homogeneous set of G, and therefore some vertex u of $G - X$ is partial on X.

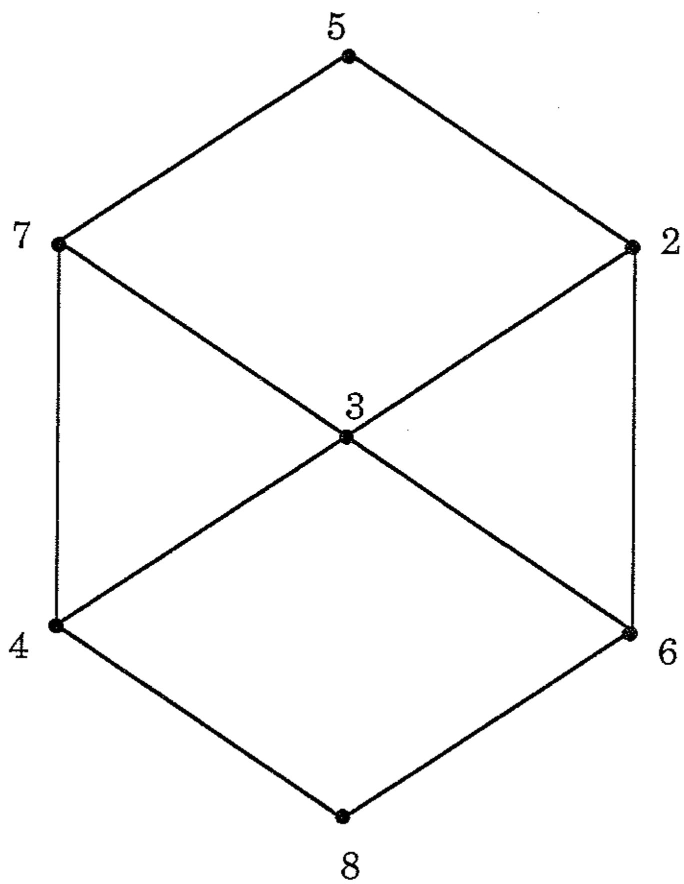


Figure 4.4. L_7

Let $Y = X + \{u\}$. *Claim Attach* together with *Claim No-Twins* imply that Y induces L_9. Now we need only show that there are no vertices in $G - Y$. Assume the contrary; then there is some vertex w in $G - Y$ that is partial on Y. But then it is possible to delete some vertex v of Y so that w sees either at most three or at least five vertices of $Y - \{v\}$. But $Y - v$ induces L_8, and since w does not see exactly four vertices of $Y - \{v\}$, $Y + w - v$ does not induce L_9. Now either *Claim Attach* or *Claim No-Twins* is contradicted. ■

Let L be the class of murky unbreakable graphs that contain L_8 as an induced subgraph and M the class of all other unbreakable murky graphs. From the L_8 *Lemma* it follows that L contains *at most* two graphs, namely L_8 and L_9. (We have not yet determined whether L_8 and L_9 are in L. In fact, they are. However, since it is not necessary to establish this in order to prove *Theorem 4.1*, we postpone this task until Section 4.4.)

We now turn our attention to M. By definition, no graph in M contains L_8 as an induced subgraph. The following lemma shows that in fact the class M is even more restricted. We define L_7 to be the graph obtained by removing any vertex of degree four from L_8.

The $\mathbf{L_7}$ Lemma. *No graph in M contains L_7.*

Proof. Let G be a graph in M. Argue by contradiction; suppose that X is a set of vertices such that G_X is L_7, labelled as in *Figure 4.4*. (The graph in *Figure 4.4* can be obtained from the graph in *Figure 4.3* by removing vertex 1.) Since G is unbreakable, there must be some path from 5 to 8, none of whose vertices is 3 or sees 3. Consider any shortest such path P. Since G is murky, P contains at most three interior vertices.

Claim 1: P does not contain exactly one interior vertex.

Suppose it did; label the interior vertex 1, so that $P = 518$. Note that 1 misses at least one of 2,4,6,7 (to avoid a $\overline{P}_6$ on 137245); assume without loss of generality that 1 misses 7. Now 1 sees 4 (to avoid a C_5 on 15748), and 1 misses 6 (to avoid a C_5 on 16375), and so 1 sees 2 (to avoid a C_5 on 18625). But then $\{1,\ldots,8\}$ induces L_8, contradiction.

Claim 2: P does not contain exactly two interior vertices.

Suppose it did; label the vertices 0 and 1 so that $P = 5018$. Then 0 sees 7 (suppose not: then (if 0 sees 6) 05736 is a C_5 or (if 0 misses 6) 057368 is a P_6). By symmetry, 0 sees 2, 1 sees 4, and 1 sees 6. Now, 0 misses 4 (to avoid a $\overline{P}_6$ on 035427). By symmetry, 0 misses 6, 1 misses 2, and 1 misses 7. But then 02341 is a C_5, contradiction.

Claim 3: P does not contain exactly three interior vertices.

Suppose it did; label the vertices 9,0,1 so that $P = 59018$. Arguing as in Claim 2, vertex 9 sees 7 and 2 but misses 4 and 6, vertex 1 sees 4 and 6 but misses 7 and 2. Now the graph induced by $\{9,7,4,1,6,2,3\}$ is isomorphic to that induced by $\{2,\ldots,8\}$; furthermore, 0 sees 9 and 1 but misses 3. Therefore, by Claim 1, $\{9,7,4,1,6,2,3,0\}$ induces L_8, contradiction. ■

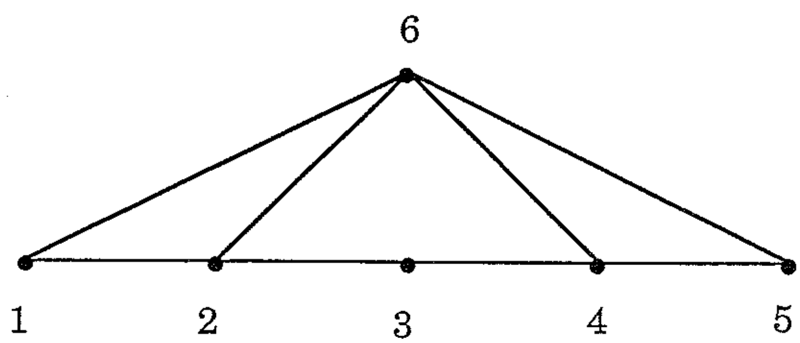


Figure 4.5.1

The next lemma is of the following form: if a graph G in M properly contains a certain subgraph S, then a certain subgraph T of G properly contains S. In this case, $S = P_5$ and $T = C_6$. Later, we present another lemma of this form.

The $\mathbf{P_5}$ Lemma. *Let G be a graph in M. Then every P_5 in G is contained in a C_6.*

Proof. We will call a P_5 *bad* if it is not contained in a C_6. We begin with a simple observation.

If abcde is a bad P_5 in G, and some vertex f sees a but not c, then f sees b. (*)

(Otherwise, *fabcd* is a C_5 or *fabcde* is a P_6.)

Define a *bypass* of a P_5 *abcde* to be a chordless path P from a to e, such that every interior vertex of P misses c. Note that in an unbreakable graph, every P_5 *abcde* has a bypass (otherwise, c is in some star cutset that separates a and e); we will use this fact repeatedly in the proof. Define the *index* of a P_5 (in an unbreakable graph) to be the number of interior vertices in a shortest bypass. Note that in a murky graph, the index of a P_5 is at most three.

Let G be a graph in M. To prove the lemma, we will show that there is no bad P_5 in G; we do this by showing that there is no bad P_5 with index one, two, or three.

Claim 1: No bad P_5 has index one.

Assume the contrary; let 12345 be a bad P_5, with bypass $P = 165$. By (*), 6 sees 2 and 4. The graph induced by $\{1,\ldots,6\}$ is shown in *Figure 4.5.1.* Now, 63142 is a P_5 in $\overline{G}$; furthermore, it is a bad P_5 of $\overline{G}$. (Assume the contrary; then there is a vertex 7 that sees 3,1,4 but misses 2,6 in G. If 7 sees 5 then 73265 is a C_5, else $\{1,\ldots,7\}$ induces $\overline{L}_7$; contradiction.) Now 63142 must have a bypass in $\overline{G}$.

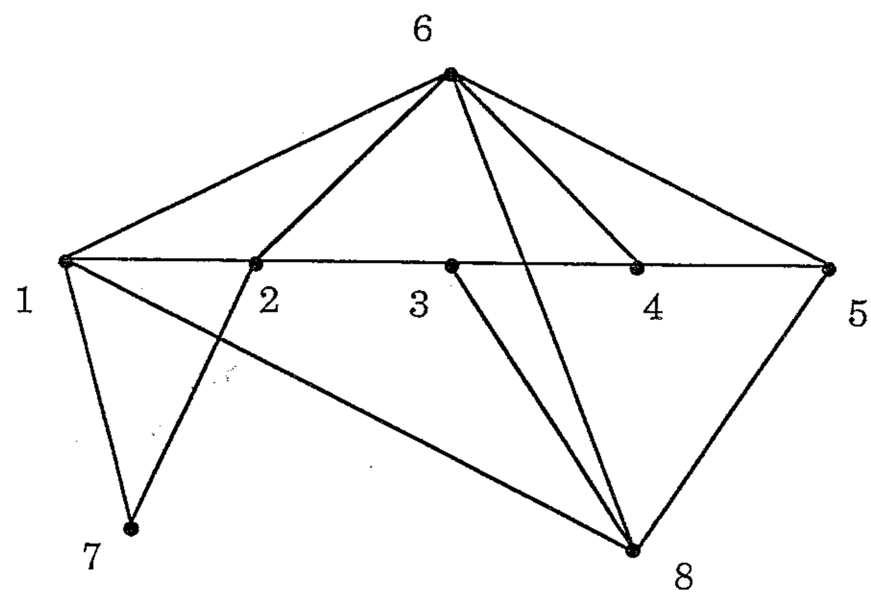


Figure 4.5.2

Claim 1.1: 63142 does not have index one.

Assume the contrary; let $Q = 672$ be a bypass of 63142 in $\overline{G}$. Thus, using (*) with 63142, in G, 7 sees 1 but misses 6,3,4,2. But 7 seeing 1 and missing 2,3 contradicts (*) with 12345. This concludes Claim 1.1.

Claim 1.2: 63142 does not have index two.

Assume the contrary; let $Q = 6782$ be a bypass of 63142 in $\overline{G}$. Thus, in G, vertex 7 sees 2,1, but misses 6,8; vertex 8 sees 6,1 but misses 2,7. Using (*) with 63142, 7 misses 3, and 8 misses 4; using (*) with 12345, 8 sees 3. Now it follows that

7 misses 4	(to avoid a C_5 on 74381),
7 misses 5	(to avoid a C_5 on 72345),
8 sees 5	(to avoid a P_6 on 718345).

The subgraph of G induced by $\{1,...,8\}$ is now the graph in *Figure 4.5.2*. Now note that 71643 is a bad P_5. (Assume the contrary: let 716439 be a C_6. Then 9 sees 7,3 but misses 1,4,6. Thus 9 misses 5 (to avoid a C_5 on 97165) and 9 sees 8 (to avoid a C_5 on 97183); finally, if 9 misses 2 then 97268 is a C_5, if 9 sees 2 then 913782 is a $\overline{P}_6$.)

Claim 1.2.1: 71643 does not have index one.

Assume the contrary; let $R = 793$ be a bypass of 71643. Thus, using (*) with 71643, vertex 9 sees 7,1,4,3 but misses 6. But if 9 misses 2 then 97264 is a C_5, if 9 sees 2 then 963142 is a $\overline{P}_6$. This concludes Claim 1.2.1.

Claim 1.2.2: 71643 does not have index two.

Assume the contrary; let $R = 7903$ be a bypass of 71643. By (*) with 71643, vertex 9 sees 1,7,0 but misses 3,6; vertex 0 sees 3,4,9 but misses 7,6. Now

0 misses 1	(if 0 sees 1 then 0 misses 2 (to avoid a $\overline{P}_6$ on 063142), and so 05623 is a C_5 or $\{1,2,3,4,5,6,0\}$ induces $\overline{L}_7$);

9 sees 4 (to avoid a C_5 on 91640),

9 sees 8 (to avoid a C_5 on 91834),

0 misses 8 (to avoid a $\overline{P}_6$ on 960148),

9 sees 2 (to avoid a C_5 on 91234),

0 misses 2 (to avoid a $\overline{P}_6$ on 960142), and finally

if 0 sees 5, then 05623 is a C_5, else 045812 is a P_6. This concludes Claim 1.2.2.

Claim 1.2.3: 71643 does not have index three.

Assume the contrary; let $R = 79x03$ be a bypass of 71643. By (*) with 71643, vertex 9 sees 1,7,x but misses 3,6,0; vertex x sees 9,0 but misses 3,6,7; vertex 0 sees 3,4,x but misses 6,7,9. Now

0 misses 1 (if 0 sees 1 then 0 misses 2 (to avoid a $\overline{P}_6$ on 063142), and so 05623 is a C_5 or $\{1,2,3,4,5,6,0\}$ induces $\overline{L}_7$);

0 sees 8 (if 0 misses 8 then 0 sees 5 (to avoid a P_6 on 045817), but then either 02185 or 05623 is a C_5),

0 misses 5 (if 0 sees 5 then 05623 is a C_5 or 063524 is a $\overline{P}_6$),

x sees 4 (if x misses 4 then x0461 is a C_5 or x04617 is a P_6),

x misses 1 (if x sees 1 then x1834 is a C_5 or x60148 is a $\overline{P}_6$),

9 sees 4 (to avoid a C_5 on 9164x),

9 sees 2 (to avoid a C_5 on 91234),

9 sees 8 (to avoid a C_5 on 94381),

9 sees 5 (to avoid a $\overline{P}_6$ on 695148),

x misses 2 (to avoid a $\overline{P}_6$ on 96x142),

x misses 8 (to avoid a $\overline{P}_6$ on 96x148),

x sees 5 (to avoid a P_6 on x45812),

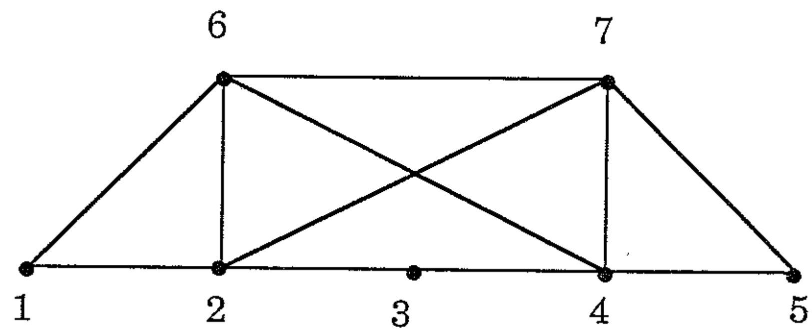


Figure 4.5.3

0 misses 2 (to avoid a C_5 on $x\,5620$).

But now $\{x,0,3,2,6,5,8\}$ induces L_7. This contradiction justifies Claim 1.2.3, and therefore Claim 1.2.

Claim 1.3: 63142 does not have index three.

Assume the contrary; let $Q = 67892$ be a bypass of 63142 in $\overline{G}$. Thus, by (*), in G vertex 7 sees 1,2,9 but misses 3,6,8; vertex 8 sees 1,2,6 but misses 7,9; vertex 9 sees 1,6,7 but misses 2,4,8. By (*) with 12345, 9 sees 3. Now

8 misses 3 (to avoid a $\overline{P}_6$ on 137892),

8 misses 4 (to avoid a C_5 on 84391),

8 misses 5 (to avoid a C_5 on 82345),

9 sees 5 (to avoid a P_6 on 543918),

7 sees 4 (if 7 misses 4 then 72345 is a C_5 or 827954 is a P_6),

7 sees 5 (to avoid a $\overline{P}_6$ on 675149).

But then removing vertex 6 and relabelling vertices 7,8,9 as 6,7,8 respectively gives the graph in *Figure 4.5.2*, and we are done by Claim 1.2. This concludes Claim 1.3, which (finally) concludes Claim 1.

Claim 2: No bad P_5 has index two.

Assume the contrary; let 12345 be a bad P_5 with bypass $P = 1675$. By (*) with 12345, 6 sees 1,2,7 but misses 3,5; 7 sees 5,4,6 but misses 1,3. Now 7 must see 2; suppose not. By Claim 1 (with 7 in place of 5), 12347 must extend into a C_6, say 123478. But then (*) is contradicted by 12345 and 8. Thus 7 sees 2; by symmetry, 6 sees 4. The graph induced by $\{1,...,7\}$ is shown in *Figure 4.5.3*.

Now note that in $\overline{G}$ 63142 is a bad P_5. (Assume the contrary; let 863142 be a $\overline{C}_6$ in G. Then either 84721 is a C_5 or 682417 is a $\overline{P}_6$.)

Claim 2.2: 63142 does not have index two.

Assume the contrary; let $S = 6892$ be a bypass of 63142 in $\overline{G}$. Arguing as in the beginning of Claim 2, in $\overline{G}$ both 8 and 9 see 3 and 4. But then, in G, 9 sees 1 but misses 2,3, which contradicts (*) with 12345. This concludes Claim 2.2.

Claim 2.3: 63142 does not have index three.

Assume the contrary; let $S = 68092$ be a bypass of 63142 in $\overline{G}$. Thus, using (*) with 63142, in G, 8 sees 1,2,9 but misses 3,6,0; 9 sees 1,6,8 but misses 2,4,0; 0 sees 1,2,6 but misses 8,9. Now

9 sees 3	(if 9 misses 3 then (*) with 12345 is contradicted),
0 misses 3	(to avoid a $\overline{P}_6$ on 138092),
0 misses 4	(to avoid a C_5 on 01934),
8 misses 5	(if 8 sees 5 then 12345 is bad P_5 with index one),
0 misses 5	(if 0 sees 5 then 12345 is a bad P_5 with index one),
9 sees 5	(to avoid a P_6 on 019345), and
8 sees 4	(to avoid a P_6 on 028954).

Now 83149 extends to a C_6 in $\overline{G}$, say $83149x$. (Suppose not; in $\overline{G}$, 0 sees 8,9 but misses 1, and so 83149 is a bad P_5 with index one, contradicting Claim 1.) Then, in G, x misses 2 (to avoid a $\overline{P}_6$ on $x83142$). But in $\overline{G}$, x sees 2 but misses 1,4, which contradicts (*) with 63142. This concludes Claim 2.3, and also Claim 2.

Claim 3: No bad P_5 has index three.

Assume the contrary; let 12345 be a bad P_5 with bypass $P = 16785$. Thus 16785 is a chordless path such that 3 misses 6,7,8. Since 12345 has index three, 1 misses 7 and 8, 5 misses 6 and 7; by (*), 6 sees 2 and 8 sees 4.

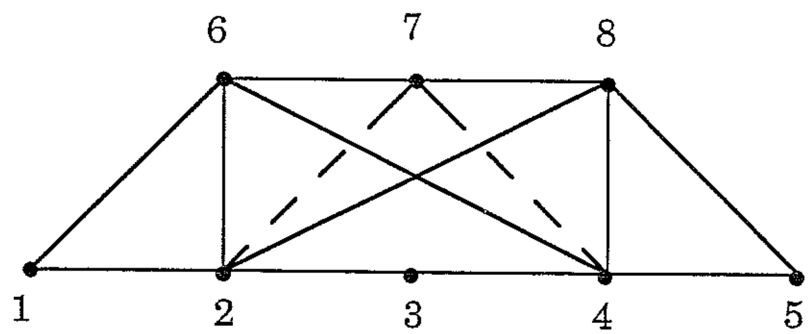


Figure 4.5.4

If 6 misses 4 then 62345 is a P_5 of index at most two (consider 6785) and hence not a bad P_5, by Claims 1 and 2; thus there is a C_6 of the form 623459, contradicting the assumption that 12345 has index three (consider 1695). Hence 6 sees 4; by symmetry, 8 sees 2. The subgraph of G induced by $\{1,...,8\}$ is shown in *Figure 4.5.4* (the vertex 7 may or may not see 2, and may or may see 4).

Now suppose that $\overline{G}$ contains a C_6 of the form 631420. Then

0 sees 8 (to avoid a C_5 on 01284),

0 sees 7 (to avoid a C_5 on 01678),

2 misses 7 (to avoid a $\overline{P}_6$ on 718602),

and finally 76230 is a C_5, a contradiction.

Hence we may assume that 63142 is a bad P_5 in $\overline{G}$; by Claims 1 and 2, its index is three. But then we obtain the desired contradiction by forgetting all about 7 and 8 and following the proof of Claim 2.3 (which does not refer at all to vertex 7 of *Figure 4.5.3*). This concludes the proof of Claim 3, and the P_5 *Lemma*. ■

The next lemma is a stronger statement than the L_7 *Lemma,* in that it implies that two particular six-vertex induced subgraphs of L_7 (and their complements) are forbidden induced subgraphs of graphs in M. This lemma will be used in the proof of the C_6 *Lemma.*

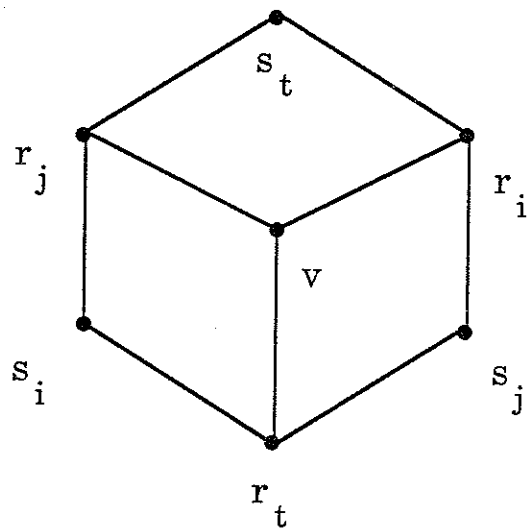
The Stronger Lemma. *If G is a graph in M, then G does not contain either*

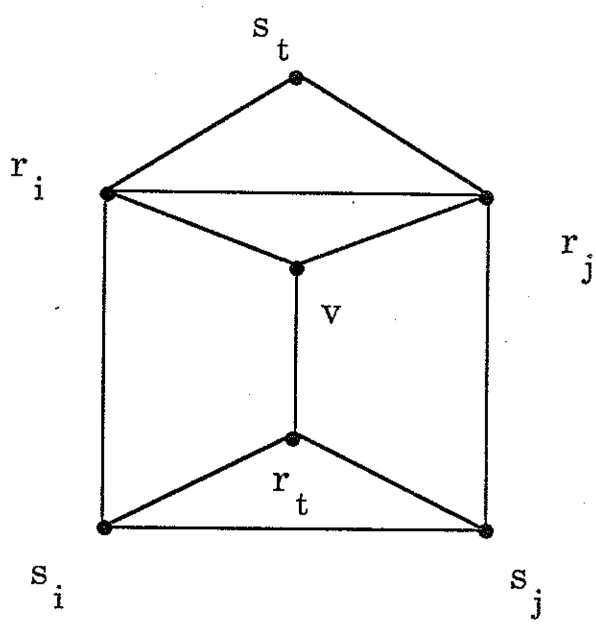
(*) *a P_5 12345 and a vertex 6 that sees 1,2,4,5 but misses 3, or*

(**) *a P_5 12345 and a vertex 6 that sees 2,3 but misses 1,4,5.*

Proof. To prove (*), note that by the P_5 *Lemma* the $\overline{P}_5$ 24136 must extend to a $\overline{C}_6$. Thus there is a vertex 7 that sees 1,3,4 but misses 2,6 in G. But this is impossible, since if 7 sees 5 then 23756 is a C_5, whereas if 7 misses 5 then $\{1,...,7\}$ induces $\overline{L}_7$.

To prove (**), note that by the P_5 *Lemma* the P_5 12345 must extend to a C_6. Thus there is a vertex 7 that sees 1,5 but misses 2,3,4 in G. But this is impossible, since if 7 sees 6 then 34576 is a C_5, whereas if 7 misses 6 then 634571 is a P_6. ∎





Figures 4.6.1 and 4.6.2 (top and bottom)

The following lemma describes restrictions on the ways in which vertices of a graph G in M can attach to one of two particular seven-vertex subgraphs of G. This result will be used in the C_6 *Lemma*, and also in the *Second Extension Lemma.*

The Little Local Lemma. *Let* $X = \{r_i, r_j, r_t, s_i, s_j, s_t, v\}$ *be a subset of the vertices of a graph G in M.*

(1A) *If* G_X *is the graph in Figure 4.6.1 and there are vertices* $w_i, w_j \in G - X$ *such that* w_i *sees* s_j, s_t *but misses* r_j, r_t, v, *and* w_j *sees* s_i, s_t *but misses* r_i, r_t, v, *then either* w_i *or* w_j *sees* s_i, s_j, s_t *but misses* r_i, r_j, r_t, v.

(1B) *If* G_X *is the graph in Figure 4.6.1 and there is a vertex* $w \in G - X$ *such that* w *sees* s_i, s_t *but misses* r_j, r_t, v, *then* w *sees* s_i, s_j, s_t *but misses* r_i, r_j, r_t, v.

(2A) *If* G_X *is the graph in Figure 4.6.2 and there is a vertex* $w \in G - X$ *such that* w *sees* s_j, s_t *but misses* r_j, r_t, v, *then* w *sees* s_i, s_j, s_t *but misses* r_i, r_j, r_t, v.

(2B) *If* G_X *is the graph in Figure 4.6.2 and there is a vertex* w *in* $G - X$ *such that* w *sees* s_i, s_t *but misses* r_j, r_t, v, *then* w *sees* s_i, s_j, s_t *but misses* r_i, r_j, r_t, v.

Proof. To prove (1A), assume the contrary. Now w_i must see r_i (if not, then w_i must see s_i to avoid a P_6 on $w_i\, s_t\, r_i\, v\, r_t\, s_i$, and we are done) and therefore miss s_i (to avoid a C_5 on $w_i\, r_i\, v\, r_t\, s_i$). By symmetry, w_j must see r_j and miss s_j. But then $w_i\, w_j\, s_i\, r_t\, s_j$ is a C_5 or $w_i\, s_j\, r_t\, v\, r_j\, w_j$ is a P_6, a contradiction.

To prove (1B), note that w misses r_i (to avoid a C_5 on $w\, r_i\, v\, r_t\, s_i$) and sees s_j (to avoid a P_6 on $w\, s_t\, r_j\, v\, r_t\, s_j$).

To prove (2A), note that w misses r_i (to avoid a C_5 on $w\, r_i\, v\, r_t\, s_j$) and sees s_i (to avoid a P_6 on $w\, s_t\, r_j\, v\, r_t\, s_i$).

To prove (2B), note that w sees s_j (to avoid a C_5 on $w\, s_i\, s_j\, r_j\, s_t$) and thus misses r_i (to avoid a C_5 on $w\, s_j\, r_t\, v\, r_i$). ■

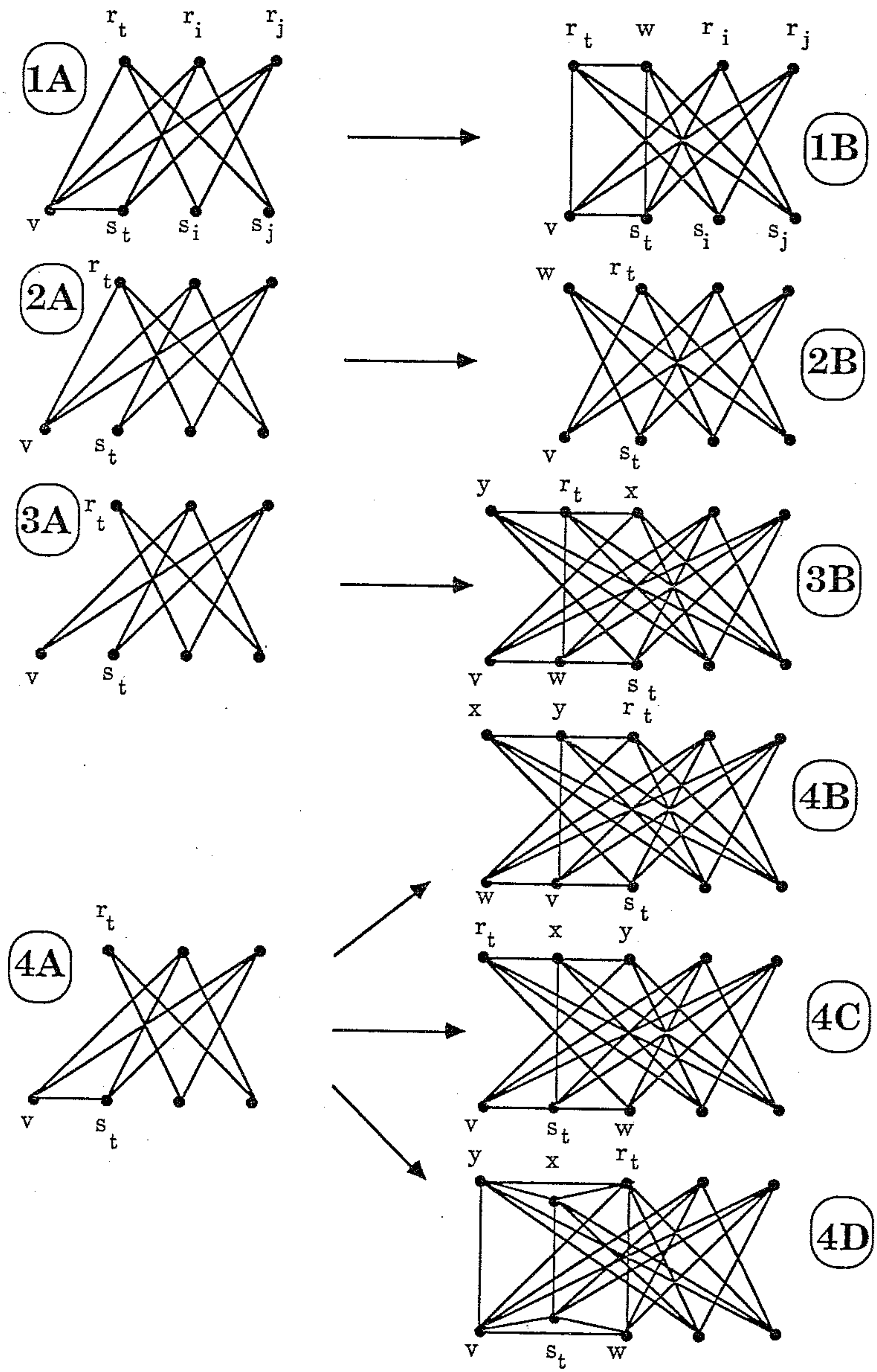


Figure 4.7

The following lemma describes how certain seven-vertex induced subgraphs (of graphs in M) that contain C_6 extend to other induced subgraphs. This lemma will be used as the basis case in the proof of *Theorem 4.1.*

The C_6 Lemma. *Let* $X = \{r_i, r_j, r_t, s_i, s_j, s_t, v\}$ *be a subset of vertices of a graph G in M.*

(1) *If* G_X *is the graph in Figure 4.7.1A, then there is a vertex w in G, such that* $G_{X \bigcup \{w\}}$ *is the graph in Figure 4.7.1B.*

(2) *If* G_X *is the graph in Figure 4.7.2A, then there is a vertex w in G, such that* $G_{X \bigcup \{w\}}$ *is the graph in Figure 4.7.2B.*

(3) *If* G_X *is the graph in Figure 4.7.3A, then there are vertices w,x,y in G, such that* $G_{X \bigcup \{w,x,y\}}$ *is the graph in Figure 4.7.3B.*

(4) *If* G_X *is the graph in Figure 4.7.4A, then there are vertices w,x,y in G, such that* $G_{X \bigcup \{w,x,y\}}$ *is the graph in Figure 4.7.4B, Figure 4.7.4C, or 4.7.4D.*

Before proving the lemma, we present a claim which will be used in two of the four cases of the proof. Throughout the claim, (*) and (**) refer back to the *Stronger Lemma.*

Claim. *Let 123456 be a* C_6 *in a graph G in M, and let 7 be a vertex of G that sees 2,6 but not 3,4,5 (7 may or may not see 1). Then there is a vertex 8 in G that sees 1,3,5 but not 2,6,7 (8 may or may not see 4).*

Proof of Claim. Since G is unbreakable, there must be a path from 1 to 3, none of whose vertices sees 7. Let P be any shortest such path. Note that P is chordless.

Case 1: P has exactly one interior vertex.

Let P = 183. If 8 sees 2 then

8 sees 6 (if 8 misses 6 then 8 misses 5 (to avoid a C_5 on 82765), but now 81654 is a C_5 or 827654 is a P_6),

8 misses 5 (if 8 sees 5 then 8 with 32765 contradicts (*)), and then

if 8 sees 4 then 8 with 34561 contradicts (*), else 83456 is a C_5; contradiction.

So 8 misses 2. Now 8 misses 6 (to avoid a C_5 on x3276), and finally 8 sees 5 (to avoid a P_6 on 832765). Thus 8 is the desired vertex.

Case 2: P has exactly two interior vertices.

Let P = $1xy3$. If x sees 5, then we are in Case 1: switch 2 with 6 and 2 with 5. Hence we may assume that x misses 5. Then x must see 6 (if not, x1654 is a C_5 or x16543 is a P_6). Thus x must see 2 (else x with 56123 contradicts (**)).

If x sees 4 then, by (1) with x in place of 7, some vertex 8 sees 1,3,4,5 and misses 2,6,x; note that 8 misses 7 (to avoid a C_5 on 872x4).

Hence we may assume that x misses 4. Applying the argument of Case 1 with x in place of 1 and with y in place of 8, we conclude that y sees x,3,5 and misses 2,6,7. Now by (1) with y in place of 4 and with x in place of 7, some vertex 8 sees 1,3,y,5 and does not see 2,6,x; note that 8 misses 7 (to avoid a C_5 on 876xy).

Case 3: P has exactly three interior vertices.

Let P = $1xyz3$. As in Case 2, we may assume that x misses 5, sees 6, sees 2 and misses 4. By Case 2, there is a vertex w that sees x,3,5 and not 2,6,7. If w sees 1, then we may set 8 = w with x in place of 1; hence we may assume that w misses 1. By (1) with w in place of 4 and with x in place of 7, some vertex 8 sees 1,3,w,5 and misses 2,6,x; note that 8 misses 7 (to avoid a C_5 on 876xw). This concludes the proof of the Claim. ∎

Proof of the C_6 Lemma. To prove (1), by the P_5 *Lemma* the $\overline{P}_5$ $r_i\, r_t\, s_t\, s_j\, v$ must extend to a $\overline{C}_6$; suppose that some vertex w sees s_t, s_j, r_t but misses r_i, v. Then w does not see r_j (else $w\, r_j\, v\, r_i\, s_j$ is a C_5) and w sees s_i (else $w\, s_t\, r_j\, s_i\, r_t$ is a C_5). Thus (1) is proved.

To prove (2), by the P_5 *Lemma* the P_5 $s_t\, r_i\, v\, r_t\, s_i$ must extend to a C_6; thus there is a vertex w_j that sees s_t, s_i but misses r_i, r_t, v. Similarly, the P_5 $s_t\, r_j\, v\, r_t\, s_j$ must extend to a C_6; thus there is a vertex w_i that sees s_t, s_j but misses r_t, r_j, v. Now, by (1A) of the *Little Local Lemma,* it follows that either w_i or w_j is the desired vertex w.

To prove (3), by the Claim (with vertex v and the C_6 $s_t\, r_i\, s_j\, r_t\, s_i\, r_j$ in place of 7 and the C_6 123456 respectively) there is a vertex x that sees s_t, s_j, s_i but misses r_i, r_j, v. Similarly, (by the Claim with vertex s_t and the C_6 $v\, r_i\, s_j\, r_t\, s_i\, r_j$) there is a vertex y that sees v, s_j, s_i but misses r_i, r_j, s_t. Next, (by the Claim with vertex x and the C_6 $r_t\, s_i\, r_j\, v\, r_i\, s_j$) there is a vertex w that sees r_i, r_t, r_j and misses x, s_j, s_i.

Now

x misses y	(to avoid a C_5 on $x\, s_t\, r_i\, v\, y$),
x sees r_t	(to avoid a P_6 on $r_t\, s_j\, x\, s_t\, r_j\, v$),
y sees r_t	(to avoid a P_6 on $r_t\, s_j\, y\, v\, r_j\, s_t$),
w sees s_t	(to avoid a C_5 on $w\, r_t\, x\, s_t\, r_i$),
w misses y	(to avoid a C_5 on $s_t\, w\, y\, s_i\, x$),
w sees v	(to avoid a C_5 on $w\, r_t\, y\, v\, r_i$), and (3) is proved.

To prove (4), argue as in the beginning of the proof of (3): there are vertices x, y such that vertex x sees s_t, s_j, s_i but misses r_i, r_j, v, and vertex y sees v, s_j, s_i but misses r_i, r_j, s_t. Note that x sees y (to avoid a C_5 on $s_t\, v\, y\, s_i\, x$). There are three cases to consider.

Case 1: r_t misses x .

Applying the Claim to the C_6 on $r_t\, s_i\, r_j\, v\, r_i\, s_j$ and to vertex x , we find a vertex w that sees r_i ,r_t ,r_j but misses x ,s_j ,s_i . Now

w misses s_t	(to avoid a C_5 on $s_t\, w\, r_t\, s_i\, x$),
w misses y	(to avoid a C_5 on $w\, y\, x\, s_t\, r_i$),
y sees r_t	(to avoid a P_6 on $y\, s_i\, r_t\, w\, r_i\, s_t$),
w sees v	(to avoid a P_6 on $v\, r_j\, w\, r_t\, s_j\, x$),

and the graph induced by $\{s_t ,r_i ,s_j ,r_t ,s_i ,r_j ,v ,x ,y ,w \}$ is that shown in *Figure 4.7.4B.*

Case 2: r_t misses see y .

Applying the Claim to the C_6 on $r_t\, s_i\, r_j\, s_t\, r_i\, s_j$ and to vertex y, we find a vertex w that sees r_i ,r_t ,r_j but misses s_j ,s_i ,y . Now

w misses v	(to avoid a C_5 on $v\, w\, r_t\, s_i\, y$),
w misses x	(to avoid a C_5 on $w\, x\, v\, r_i$),
x sees r_t	(to avoid a P_6 on $x\, s_i\, r_t\, w\, r_i\, v$),
w sees s_t	(to avoid a P_6 on $s_t\, r_j\, w\, r_t\, s_j\, y$),

and the graph induced by $\{s_t ,r_i ,s_j ,r_t ,s_i ,r_j ,v ,x ,y ,w \}$ is that shown in *Figure 4.7.4C.*

Case 3: r_t sees x and y .

Applying (2) to the $\overline{C}_6$ on $s_j\, v\, x\, r_i\, y\, s_t$ and to vertex r_t we find a vertex w that sees (in G) s_t ,r_i ,r_t ,v and does not see s_j ,x ,y . Now

w misses s_i	(to avoid a C_5 on $w\, s_i\, x\, s_j\, r_i$),
w sees r_j	(to avoid a P_6 on $w\, r_i\, s_j\, x\, s_i\, r_j$),

and the graph induced by $\{s_t ,r_i ,s_j ,r_t ,s_i ,r_j ,v ,x ,y ,w \}$ is that shown in *Figure 4.7.4D.* This concludes the proof of the C_6 *Lemma.* ■

4.3 Strong Mirror Graphs

It is easier to prove *Theorem 4.1* by dealing only with a certain subclass of mirror graphs that includes all unbreakable mirror graphs, rather than by dealing with all mirror graphs. This subclass is the class of "strong mirror graphs"; we present a formal definition shortly. It turns out that a mirror graph is unbreakable if and only if it is a strong mirror graph. As we did with L_8 and L_9, we will postpone the proof of unbreakability, i.e. the "if" part of the previous statement, until Section 4.4.

We shall say that a P_4-free graph G is *strong* unless (and only unless) G or $\overline{G}$ has precisely two components and one of these components is a singleton. The following lemma is a useful tool for working with strong P_4-free graphs. The graph $2K_2$ referred to in the lemma is the graph with two components, each of which is a single edge.

The Rip-Off Lemma. *Let G be a strong P_4-free graph with at least four vertices such that neither G nor $\overline{G}$ is $2K_2$. Then G contains twins x,y such that $G - x$ and $G - y$ are strong P_4-free graphs. Furthermore, if G has an isolated vertex z, then we can choose x,y both distinct from z.*

Proof. First, let us prove only that G contains twins c, d such that both $G - c$ and $G - d$ are strong P_4-free graphs. Let a, b be twins in G. Since $G - a$ and $G - b$ are isomorphic, we may assume that $G - a$ is not strong (otherwise we are done by setting $c = a$, $d = b$). Replacing G by $\overline{G}$ if necessary, we may assume that $G - a$ has precisely two components and that one of them is a singleton. Note that the singleton is b (else G would not be strong); call the other component Q; observe that Q is a component of G. Now let c, d be any twins in Q.

To complete the proof, assume that one of c, d is isolated in G (otherwise we can set $x = c$ and $y = d$). Then both c and d are isolated in G. If G has no edges at

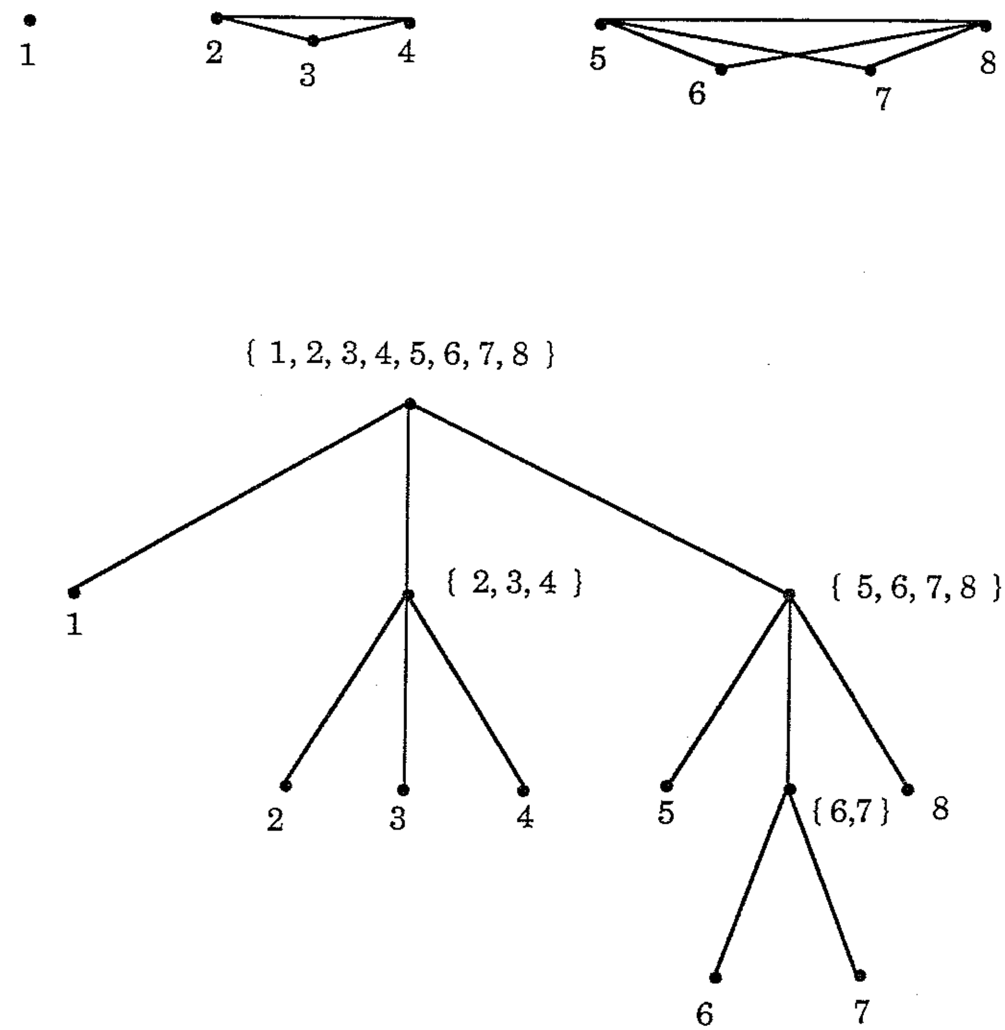


Figure 4.8. A P_4- free graph and its decomposition tree

all then any two vertices x, y distinct from c and d will do; else G has a big component Q and any twins x, y in Q will do. ∎

Strong mirror graphs are defined as follows: start with the definition of a mirror graph, insist that the P_4-free graph G_R be strong, and specify exactly which couples of the partition induce edges of the graph (that is, for which couples $\{r_j, s_j\}$ the vertices r_j and s_j are adjacent). This specification is in the form of a certain 0-1 function f; this function is defined in terms of a decomposition of P_4-free graphs that follows from repeatedly applying Seinsche's theorem. (Recall Seinsche's theorem from Chapter 1: if a P_4-free graph has at least two vertices, then either the graph or its complement is disconnected.)

We now present a recursive definition of a graph $DT(G)$ whose vertices correspond to subsets of vertices of another graph G. In order to avoid ambiguity, we will refer to the vertices of $DT(G)$ as *nodes*. The *decomposition tree* $DT(G)$ of a P_4-free graph G is the rooted tree such that:

> *(1) if G has only one vertex v, then the root of $DT(G)$ is the vertex v, and there are no other nodes in $DT(G)$, and*
>
> *(2) if G has more than one vertex, then the root of $DT(G)$ is the set of all vertices of G, and the nodes adjacent to the root are $DT(G_1)$, ..., $DT(G_k)$, where G_1, ..., G_k are the induced subgraphs of G that correspond to the components of whichever of G or $\overline{G}$ is disconnected.*

A P_4-free graph and its decomposition tree are shown in *Figure 4.8.* Note that every vertex of G is a leaf of $DT(G)$. Also, every leaf of $DT(G)$ is a vertex of G, and every node of $DT(G)$ that is not a leaf is a subset of at least two of the vertices of G. Note also that $DT(G)$ is identical to $DT(\overline{G})$.

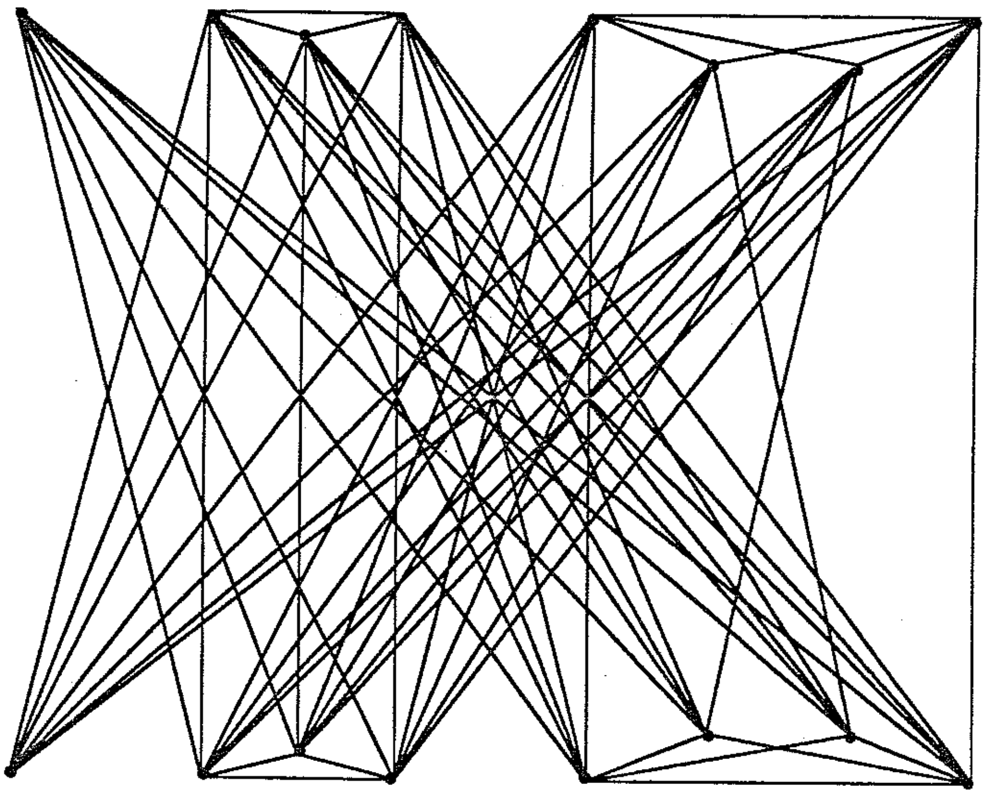


Figure 4.9. A strong mirror graph

We need one more definition before we can define the 0-1 function f. Let G be a P_4-free graph with at least two vertices. For every vertex v of a P_4-free graph G with at least two vertices, define the *parent* $P(G,v)$ to be the parent of v in $DT(G)$, (i.e. the node of $DT(G)$ adjacent to the leaf v). For example, with respect to the P_4-free graph G shown in *Figure 4.8*, the parent of 1 is the root of $DT(G)$ (namely, the set of all vertices of G), the parent of 2,3, and 4 is the node $\{2,3,4\}$, the parent of 5 and 8 is the node $\{5,6,7,8\}$, and the parent of 6 and 7 is the node $\{6,7\}$.

Now define the function $f(G,v)$ so that

$$f(G,v) = 0 \quad \text{if } G_{P(G,v)} \text{ is disconnected,} \qquad \text{and}$$

$$f(G,v) = 1 \quad \text{if } G_{P(G,v)} \text{ is connected.}$$

Note that v is a singleton in whichever of $G_{P(G,v)}$ or $\overline{G}_{P(G,v)}$ is disconnected. For the graph G shown in *Figure 4.8*, $f(G,v) = 0,1,1,1,1,0,0,1$, for $v = 1,2,\ldots,8$ respectively.

Now that $f(G,v)$ is defined, we can formally define strong mirror graphs. A partition $[R,S]$ of the vertices of a graph G is called a *strong mirror partition* if conditions (1) and (2) of the definition of a mirror partition are satisfied, and if

(3) G_R *is a strong* P_4*-free graph, and*

(4) r_j *sees* s_j *if and only if* $f(G_R, r_j) = 1$, *for all* $r_j \in R$.

A graph with a strong mirror partition is a *strong mirror graph*. A strong mirror graph is shown in *Figure 4.9*.

The classes of P_4-free graphs and murky graphs are self-complementary. We now show that the same is true for the classes of mirror graphs and strong mirror graphs.

The Complement Lemma. *Let [R,S] be a (strong) mirror partition of G. Then the partition [R,S], with vertices labelled as in the partition of G, is a (strong) mirror partition of* $\overline{G}$.

Proof. The conditions (1), (2), (3), (4) mentioned in the proof refer to the definitions of mirror partition and strong mirror partition.

Let G be a graph with mirror partition $[R ,S]$. Since the complement of a P_4-free graph is P_4-free, the partition $[R ,S]$ of $\overline{G}$ satisfies condition (1).

Let r_m and r_p be any two vertices of R ; r_m sees r_p in $\overline{G}$ if and only r_m misses r_p in G . From (2) it follows that in G

r_m misses r_p if and only if s_m misses s_p if and only if

r_m sees s_p if and only if s_m sees r_p .

Thus in $\overline{G}$

r_m sees r_p if and only if s_m sees s_p if and only if

r_m misses s_p if and only if s_m misses r_p ,

and (2) holds for the partition $[R ,S]$ of $\overline{G}$. Thus $[R ,S]$ is a mirror partition of $\overline{G}$.

Now assume that $[R ,S]$ is a *strong* mirror partition of G ; we will prove that it is also a strong mirror partition of $\overline{G}$. By the previous argument we need only prove that (3) and (4) hold for $[R ,S]$ with respect to $\overline{G}$. But (3) holds trivially. To see that (4) holds, note that $DT(H) = DT(\overline{H})$ for any P_4-free graph H; thus $H_{P(H,v)}$ is the complement of $\overline{H}_{P(\overline{H},v)}$, and so (if H has at least two vertices) $f(H,v) + f(\overline{H},v) = 1$. Now set $H = G_R$, and use the fact that (4) holds for $[R ,S]$ with respect to G . ■

The graph shown in *Figure 4.9* is a strong mirror graph, since the partition suggested by the drawing is a strong mirror partition. (Partition the vertices into the "upper set" and the "lower set"; the couples are the pairs of vertically aligned vertices. Note that the subgraphs induced by "upper set" and "lower set" respectively are isomorphic to the graph shown in *Figure 4.8*.) On the other hand, the partition suggested by the drawing of the mirror graph in *Figure 4.2* is *not* a strong mirror partition (in fact this graph has no strong mirror partition). In Section 4.4 we will say more about which mirror graphs have strong mirror partitions. First, however, we we wish to prove *Theorem 4.1*. With this goal in mind, we state two results concerning the function f.

The Localization Lemma. *Let G be a P_4-free graph and let H be a homogeneous set in G. Then $f(G,x) = f(G_H,x)$, for all $x \in H$.*

Proof. Consider an arbitrary vertex x in H. The *Complement Lemma* allows us to assume that $f(G,x) = 0$. We may assume that $f(G_H,x) = 1$, for otherwise we are done. Let A be the parent of x in $DT(G)$; since $f(G,x) = 0$, vertex x is isolated in A. Let B be the parent of x in $DT(G_H)$; since $f(G_H,x) = 1$, vertex x sees all the remaining vertices in B. It follows that the intersection of A and B contains only x. Since both A and B have at least two vertices, there is some vertex $a \in A - B$, and some vertex $b \in B - A$.

Note that A is homogeneous in G and that B (being homogeneous in G_H) is also homogeneous in G. Since a misses x, it must miss all of B; in particular, a misses b. Since b sees x, it must see all of B; in particular, b sees a; contradiction. ∎

A special case of the *Localization Lemma* asserts that $f(G,x) = f(G,y) = 1$ whenever x, y are adjacent twins and that $f(G,x) = f(G,y) = 0$ whenever x, y are non-adjacent twins.

The following lemma is also concerned with f and with twins.

The Twin Lemma. *Let G be a P_4-free graph with at least three vertices. If x,y are twins in G then $f(G,z) = f(G-x, z) = f(G-y, z)$, for all z in $G - \{x,y\}$.*

Proof. Argue by induction on $|G|$. Since $f(G,x) + f(\overline{G},x) = 1$ for all P_4-free graphs G, we may assume that G is disconnected: its vertices can be partitioned into non-empty disjoint sets S_1, S_2, so that no edge has one vertex in each S_i. If x and y belong to distinct S_i's, then each vertex distinct from both x and y misses at least one of them, and therefore it misses both; in that case, we can redefine S_1, S_2 by setting $S_1 = \{x,y\}$ and letting S_2 consist of all the remaining vertices.

Hence we may assume that $x, y \in S_1$. To prove the lemma for all z in S_1, distinct from both x and y, we may assume that $|S_1| \geq 3$ (else there is nothing to prove); the induction hypothesis guarantees that

$$f(G_{S_1}, z) = f(G_{S_1}-x, z) = f(G_{S_1}-y, z) \text{ whenever } z \in S_1, z \neq x, y;$$

the *Localization Lemma* guarantees that

$$f(G, z) = f(G_{S_1}, z), \quad f(G-x, z) = f(G_{S_1}-x, z), \quad f(G-y, z) = f(G_{S_1}-y, z).$$

Now combining these two sets of equalities yields the desired conclusion.

To prove the lemma for all $z \in S_2$, we may assume that $|S_2| \geq 2$ (else $f(G,x) = f(G-x,z) = f(G-y,z) = 0$ for the singleton z in S_2, and we are done). Clearly, S_2 is a homogeneous set of G, $G-x$, and $G-y$; now the *Localization Lemma* implies the desired conclusion. ∎

Having built up a repetoire of results concerning f, we are able to present some lemmas concerning strong mirror graphs.

The Reduction Lemma. *Let G be a strong mirror graph with at least eight vertices such that neither G_R nor $\overline{G}_R$ is $2K_2$. Then there are twins r_i, r_j in G_R such that either*

(a) *$[R-r_i, S-s_i]$ is a strong mirror partition of $G-\{r_i, s_i\}$,*

$[R-r_j, S-s_j]$ is a strong mirror partition of $G-\{r_j, s_j\}$, and

$$f(G_R-r_i, r_j) = f(G_R-r_j, r_i) = f(G_R, r_i) = f(G_R, r_j), \qquad or$$

(b) *$[R-r_i, S-s_j]$ is a strong mirror partition of $G-\{r_i, s_j\}$,*

$[R-r_j, S-s_i]$ is a strong mirror partition of $G-\{r_j, s_i\}$, and

$$f(G_R-r_i, r_j) = f(G_R-r_j, r_i) \neq f(G_R, r_i) = f(G_R, r_j).$$

In all cases, all sets $\{r_k, s_k\}$ with $k \neq i,j$ are couples of these strong mirror partitions. Furthermore, if G_R has an isolated vertex r_t, then we can choose i,j both distinct from t.

Proof. By the *Rip-Off Lemma*, we find twins r_i, r_j in G_R such that G_R-r_i and G_R-r_j are strong P_4-free graphs, and such that, for any given isolated vertex of r_t, both i, j are distinct from t. By the *Twin Lemma*,

$$f(G_R-r_i, r_k) = f(G_R-r_j, r_k) = f(G_R, r_k) \text{ whenever } k \neq i, j.$$

Note that G_R-r_i, G_R-r_j, G_S-s_i, G_S-s_j are all isomorphic and that

$$f(G_R-r_i, r_j) = f(G_R-r_j, r_i).$$

In addition, note that

$$f(G_R, r_i) = f(G_R, r_j) = 1 \quad \text{if } r_i \text{ sees } r_j,$$

$$f(G_R, r_i) = f(G_R, r_j) = 0 \quad \text{if } r_i \text{ misses } r_j$$

(use the *Localization Lemma* with $G = G_R$, $H = \{r_i, r_j\}$). Hence (a) holds if $f(G_R-r_i, r_j) = f(G_R, r_j)$, and (b) holds in the other case. ∎

One difficulty that must be overcome in proving theorems that concern strong mirror graphs is that a strong mirror graph can have more than one strong mirror partition. For example, the strong mirror graphs shown in *Figures 4.10, 4.11,* and *4.12* are isomorphic and yet have different strong mirror partitions. The following two lemmas show how this non-uniqueness can be exploited. In particular, the first of these lemmas shows that under certain hypotheses it is possible to "repartition" a strong mirror graph, i.e. find some other strong mirror partition of the graph. The second lemma shows that any given strong mirror graph has a strong mirror partition that "isolates" any given vertex of the graph.

The Repartitioning Lemma. *Let G be a strong mirror graph with a strong mirror partition $[R,S]$, and suppose that whichever of G_R or $\overline{G}_R$ is disconnected has some big component. Let R_1 be the set of vertices of such a component and let S_1 be the set of mates of vertices in R_1. Define $R' = R_1 + S - S_1$, and $S' = S_1 + R - R_1$. Then the partition $[R',S']$ in which the couples are the same as the couples of $[R,S]$ is a strong mirror partition of G.*

Proof of Lemma. Label the vertices of R' and S' so that couples of $[R',S']$ are couples of $[R,S]$, i.e. let $r_i' = r_i$ for all r_i in R_1 and let $r_j' = s_j$ for all s_j in $S - S_1$; let $s_i' = s_i$ for all s_i in S_1 and let $s_j' = r_j$ for all s_j in $R - R_1$. To prove the lemma it suffices to confirm that the following four properties hold.

(1) $G_{R'}$ and $G_{S'}$ are P_4-free,

(2) r_i' sees r_j' if and only if s_i' sees s_j' if and only if
r_i' misses s_j' if and only if s_i' misses r_j', for all $i \neq j$,

(3) $G_{R'}$ and $G_{S'}$ are strong P_4-free graphs,

(4) r_j' sees s_j' if and only if $f(G_{R'}, r_j') = 1$, for all r_j' in R'.

Both G_{R_1} and G_{S-S_1} are P_4-free, and there are either no edges or all edges between G_{R_1} and G_{S-S_1}; thus $G_{R'}$ is P_4-free. By symmetry, so is $G_{S'}$, and (1) holds.

To see that (3) holds, consider first the case in which G_R is disconnected; note that $\overline{G}_{R'}$ is disconnected. Since R_1 induces a big component in G_R, $\overline{G}_{R'}$ has at least three components (at least two are induced by R_1, and at least one is induced by $S - S_1$), and so $G_{R'}$ is a strong P_4-free graph. Similarly, in the case where $\overline{G}_R$ is disconnected, $G_{R'}$ is disconnected and has at least three components, and (3) holds.

Note that (2) is equivalent to the following:

for every two distinct couples $X = \{a,b\}$ and $Y = \{c,d\}$,

each vertex of X sees exactly one vertex of Y.

Since this property holds with respect to the partition $[R,S]$, and since the couples of $[R,S]$ are the same as the couples of $[R',S']$, it follows that this property holds with respect to the partition $[R',S']$. Thus (2) holds.

Finally, to show that (4) holds, let r_j' be any vertex in R'. Consider first the case $r_j' \in R_1$. Note that R_1 is a homogeneous set of both G_R and $G_{R'}$; by the *Localization Lemma*, $f(G_{R'}, r_j') = f(G'_{R_1}, r_j')$ and $f(G_R, r_j) = f(G_{R_1}, r_j)$. Since $r_j' = r_j$ and $G'_{R_1} = G_{R_1}$, it follows that $f(G_{R'}, r_j') = f(G_R, r_j)$. This, together with the fact that (by the repartitioning) r_j' sees s_j' if and only if r_j sees s_j, and the fact that (since $[R,S]$ is a strong mirror partition) r_j sees s_j if and only if $f(G_R, r_j) = 1$, imply that r_j' sees s_j' if and only if $f(G_{R'}, r_j') = 1$. Thus (4) holds in this case.

In the other case, $r_j' \in S - S_1$. Since G_R is a strong P_4-free graph, $|S - S_1| \geq 2$, and $S - S_1$ is a homogeneous set of both $G_{R'}$ and G_S. Now a similar argument to that of the previous case implies that (4) holds in this case as well. ■

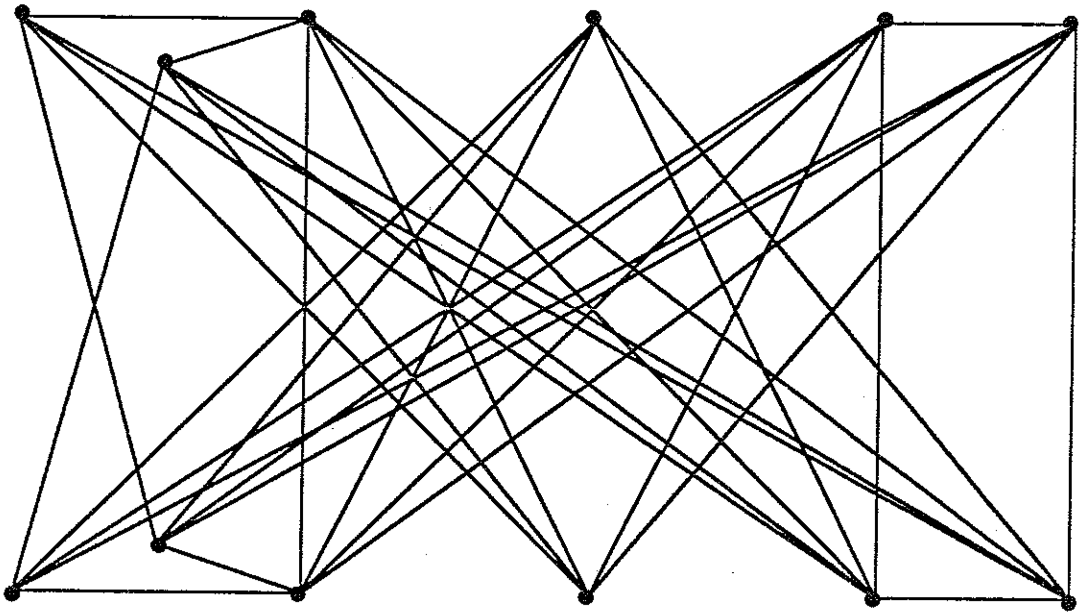


Figure 4.10

Figure 4.10 shows a strong mirror graph; the partition suggested by the drawing (i.e. "upper part" and "lower part") is a strong mirror partition. The graph in *Figure 4.11* can be obtained by repartitioning the graph in *Figure 4.10* as follows: let R_1 be the leftmost component of the "upper part" of the graph in *Figure 4.10,* and repartition as described in the *Repartition Lemma.* Similarly, the graph in *Figure 4.12* can be obtained by repartitioning the graph in *Figure 4.11.*

The Isolation Lemma. *Let G be a graph with a strong mirror partition $[R,S]$, and let v be any vertex of G. Then there is a strong mirror partition $[R',S']$ of G such that the couples of $[R',S']$ are the couples of $[R,S]$ and such that v is a singleton in whichever of $G_{R'}$ or $\overline{G}_{R'}$ is disconnected.*

Proof of Lemma. Assume that G_R is disconnected (the following argument holds if G_R is replaced with $\overline{G}_R$). Let R_1 be the set of vertices of R in the component of G_R that contains v. The proof is by induction on $|R_1|$.

If $|R_1| = 1$, then $[R,S]$ is the desired partition. Suppose then that $|R_1| \geq 2$. Let R' and S' be as defined in the *Repartitioning Lemma.* Consider the strong mirror partition $[R',S']$ of G. Note that $\overline{G}_{R'}$ is disconnected and has at least three components. Let R_2' be the set of vertices of R' induced by the component of $\overline{G}_{R'}$ that contains v. Since R_2' is a proper subset of R_1, $|R_2'| < |R_1|$. The lemma now follows by inductive hypothesis and the *Repartitioning Lemma.* ∎

The *Isolation Lemma* is illustrated by the graphs shown in *Figures 4.10, 4.11* and *4.12.* Let v be the upper leftmost vertex in the graph in *Figure 4.10. Figure 4.11* and *Figure 4.12* show the sequence of two repartitions that isolate v. (The vertex v appears as the upper leftmost vertex in all three drawings.)

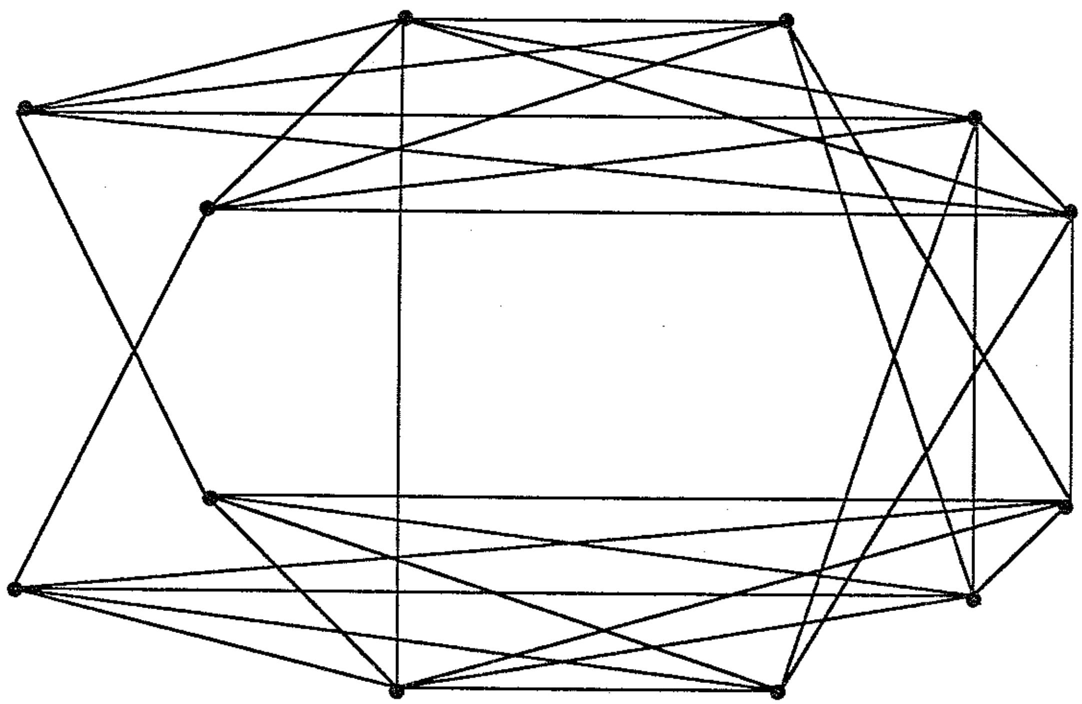


Figure 4.11

The following two lemmas describe restrictions on how vertices in graphs in M can "attach" to strong mirror subgraphs.

The Zero-Two Lemma. *Let H be a graph in M, let G be a strong mirror subgraph of H, and let v be a vertex of $H - G$ that is partial on G. If v is universal or null on some couple $\{r_t, s_t\}$ of a strong mirror partition of G, then v is a twin of one of r_t, s_t with respect to $G - \{r_t, s_t\}$.*

Proof. Argue by induction on the number of vertices in G. By the *Complement Lemma* and the *Isolation Lemma,* we may assume that r_t is isolated in G_R.

If G has precisely six vertices then $r_t\, s_i\, r_j\, s_t\, r_i\, s_j$ is a C_6. There are two cases.

Case 1: v is null on $\{r_t, s_t\}$.

Since v sees at least one vertex of G, by swapping R and S if necessary, and also i and j, we may assume that v sees s_i. Now v must see s_j (if not, either $v\, s_i\, r_t\, s_j\, r_i$ is a C_5 or $v\, s_i\, r_t\, s_j\, r_i\, s_t$ is a P_6). This implies that v misses r_j (if v sees r_j, then either $v\, s_j\, r_i\, s_t\, r_j$ is a C_5 or $\{v, r_t, s_t, r_i, s_i, r_j, s_j\}$ induces L_7); by symmetry, v also misses r_i. Now observe that v is a twin with respect to r_t of $G - \{r_t, s_t\}$.

Case 2: v is universal on $\{r_t, s_t\}$.

Since v misses at least one vertex of G, by swapping R and S if necessary, and also i and j, we may assume that v misses s_i. Now v sees r_j (to avoid a C_5 on $v\, r_t\, s_i\, r_j\, s_t$), misses s_j (else v and the P_5 $s_j\, r_t\, s_i\, r_j\, s_t$ contradict (*) of the *Stronger Lemma*), and finally sees r_i (to avoid a C_5 on $v\, r_t\, s_j\, r_i\, s_t$). Now v is a twin with respect to s_t of $G - \{r_t, s_t\}$.

If G has at least eight vertices then (since r_t is isolated in G_R) neither G_R nor $\overline{G}_R$ is $2K_2$, and so we can apply the *Reduction Lemma.* Let r_i, r_j be as in the

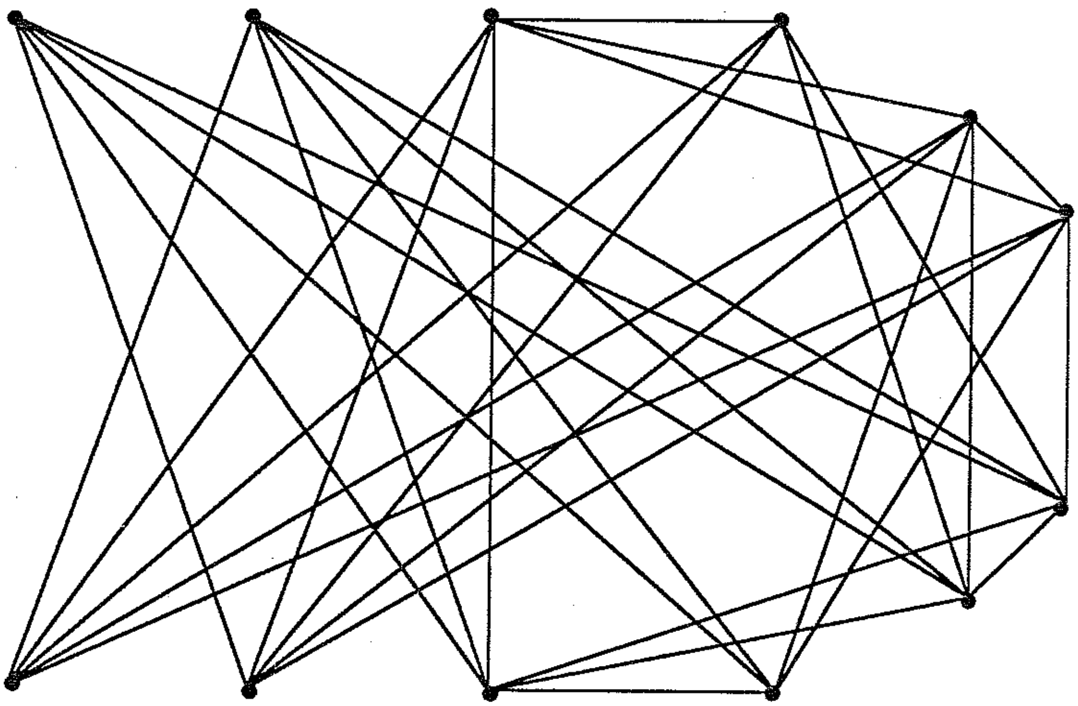


Figure 4.12

conclusion of the *Lemma*; set $G_i = G - \{r_i, s_i\}$, $G_j = G - \{r_j, s_j\}$ in case (a) and $G_i = G - \{r_i, s_j\}$, $G_j = G - \{r_j, s_i\}$ in case (b). By the induction hypothesis, there is a vertex w_i in $\{r_t, s_t\}$ such that v is a twin of w_i with respect to $G_i - \{r_t, s_t\}$ and there is a vertex w_j in $\{r_t, s_t\}$ such that such that v is a twin of w_j with respect to $G_j - \{r_t, s_t\}$. We need only prove that $w_i = w_j$.

Assume the contrary: $w_i \neq w_j$. Now w_i and w_j are anti-twins in G. However, v is a twin of both with respect to the non-empty graph $G - \{r_i, s_i, r_j, s_j, r_t, s_t\}$, a contradiction. This concludes the proof of the *Zero-Two Lemma*. ∎

The Attachment Lemma. *Let H be a graph in M, let G be a strong mirror subgraph of H, and let v be a vertex of $H - G$ that is partial on G. Then either*

(i) *there is a strong mirror partition $[R,S]$ of G such that v is universal on R and null on S, or*

(ii) *in every strong mirror partition $[R,S]$ of G there is a couple $\{r_t, s_t\}$ such that v is a twin of one of r_t, s_t with respect to $G - \{r_t, s_t\}$.*

Proof. We may assume that (ii) does not hold; now the *Zero-Two Lemma* guarantees the existence of a strong mirror partition $[R, S]$ of G such that v is partial on every couple $\{r_t, s_t\}$.

First we claim that

(1) if G_R has at least three components then v is partial on at most two of them.

To justify this claim, assume the contrary: G_R has components R_1, R_2, R_3 (and possibly others) such that v is partial on R_1 and R_2. Let S_1, S_2, S_3 be the corresponding components of G_S. Now there are adjacent vertices a and z in R_1, such that v sees z and misses a; let b denote the mate of z. Now $a \in R_1$, $b \in S_1$, and

a, b, v are pairwise non-adjacent. By symmetry, there are vertices c and d such that $c \in R_2$, $d \in S_2$ and such that c, d, v are pairwise non-adjacent.

Finally, let $\{x, y\}$ be a couple with $x \in R_3$, $y \in S_3$. Swapping R and S if necessary, we may assume that v sees x. Now we wish to find a vertex z in S_3 that misses x. If $R_3 = \{x\}$ then $f(G_R, x) = 0$, and so we may set $z = y$; else let z be the mate of any neighbour of x in R_3. Now observe that $azcbxd$ is a C_6. Since v sees x and misses a, b, c, d, either $vxdaz$ is a C_5 or $vxdazc$ is a P_6, a contradiction.

Next we claim that

(2) G_R has no components R_1, R_2, R_3 such that v is partial on R_1, universal on R_2, and null on R_3.

To justify this claim, assume the contrary. As in the proof of (1), we find a vertex a in R_1 and b in S_1 such that a, b, v are pairwise non-adjacent. Now let c be any vertex in R_2. There is a vertex d in S_2 that misses c: if $R_2 = \{c\}$ then $f(G_R, c) = 0$, and we may let d be the mate of c, else we may let d be the mate of any neighbour of c in R_2. Finally, let e be any vertex in R_3. Note that v is null on S_2; it follows that $vcbeda$ is a P_6, a contradiction.

Finally, replacing H by $\overline{H}$ if necessary, we may assume that G_R is disconnected. Let us distinguish between two cases.

Case 1: v is partial on no component of G_R.

In this case, let R_1 be the set of neighbours of v in R, and let S_1 be the set of non-neighbours of v in S. Note that $|R_1| \geq 2$ (else $R_1 = \{r_t\}$ and v is a twin of r_t with respect to $G - \{r_t, s_t\}$). Note that R_1 and $R - R_1$ are homogeneous in G_R; by the *Localization Lemma*, $[R_1 + S - S_1, S_1 + R - R_1]$ is a strong mirror partition of G. Since v is universal on $R_1 + S - S_1$ and null on $S_1 + R - R_1$, property (i) holds.

Case 2: v is partial on some component of G_R .

By (1), v is partial on precisely one component R_1 of G_R . We shall argue by induction on $|R_1|$. By (2), v is universal or null on $R - R_1$. Note that $|R - R_1| \geq 2$ (because G_R is strong); hence R_1 and $R - R_1$ are homogeneous in G_R . Set $R' = R_1 + S - S_1$, $S' = S_1 + R - R_1$. By the *Localization Lemma*, $[R',S']$ is a strong mirror partition of G . Note that R' induces a disconnected subgraph of $\overline{G}$, and so does R_1. By (2), v is partial on at most one component of $\overline{G}_{R_1}$. If v is partial on precisely one such component then we are done by the induction hypothesis applied to the mirror partition $[R',S']$ of $\overline{G}$; if v is partial on no such component then we are done because the mirror partition $[R',S']$ of G satisfies the hypothesis of Case 1. This concludes the proof of the *Attachment Lemma.* ■

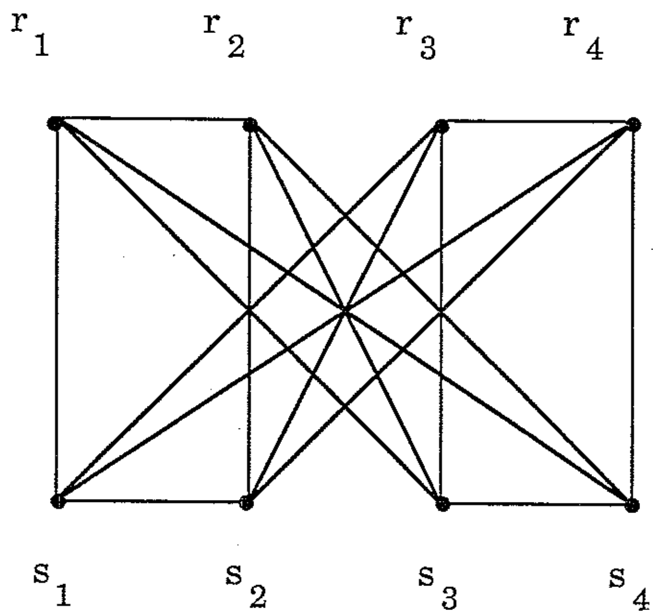


Figure 4.13

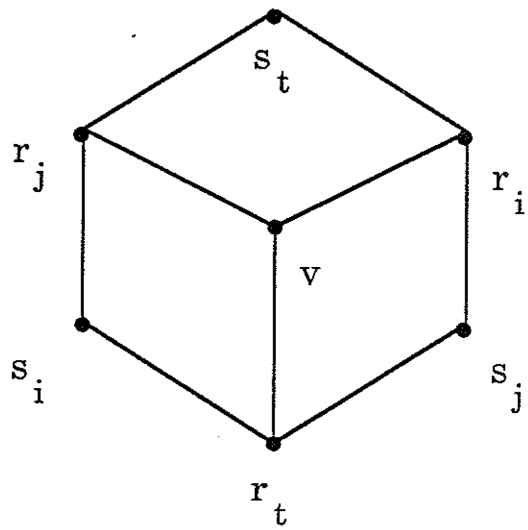
The following two lemmas are both statements of the following form: suppose that G is a strong mirror subgraph of a graph H in M, and suppose that v is some vertex that attaches to G in a certain way; then there is another vertex (or there are other vertices) in G that attach to $H + v$ in another certain way. These lemmas are the last two before the proof of *Theorem 4.1.*

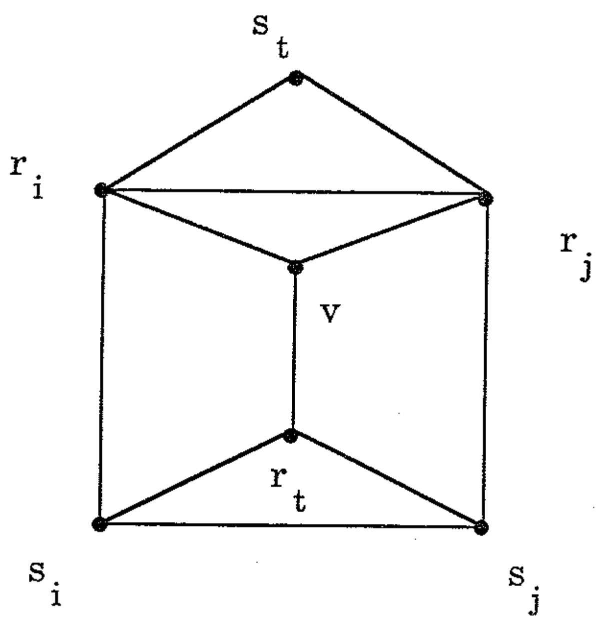
The First Extension Lemma. *Let G be a strong mirror subgraph of a graph H in M, let $[R,S]$ be a strong mirror partition of G, let G_R be disconnected, and let v be a vertex in $H - G$ that is universal on R and null on S. Then there is a vertex w in $H - G$ that misses v, is universal on S, and null on R.*

Proof. We shall argue by induction on the number of vertices in G. If G has precisely six vertices then it is a C_6 and the desired conclusion follows by (2) of the C_6 *Lemma.*

Another case that will be treated separately is that of $G = 2K_2$. Assume that G is labelled as in *Figure 4.13.* Applying the P_5 *Lemma* to $s_4 r_2 v r_3 s_1$, we find a vertex w that sees s_4, s_1 and misses r_2, r_3, v. Now w must see s_2 and s_3 (to avoid a C_5 on $w s_1 s_2 r_2 s_4$ and $w s_4 s_3 r_3 s_1$, respectively), and w must miss r_1 and r_4 (to avoid a $\overline{P}_6$ on $w r_2 s_1 s_4 s_2 r_1$ and $w r_3 s_4 s_1 s_3 r_4$, respectively).

Now we may assume that G has at least eight vertices and that $G \neq 2K_2$. Let r_i, r_j be as in the *Reduction Lemma,* and let s_i, s_j be their respective mates with respect to the partition $[R, S]$. Observe that G_R has a component R_0 that includes neither r_i nor r_j. Let S_0 be the corresponding component of G_S. Let r_t be any vertex in R_0. If $R_0 = \{r_t\}$ then let s_t be the mate of r_t, else let s_t be the mate of any neighbour of r_t in G_R. Note that s_t is in S_0 and misses r_t. If r_i, r_j are adjacent then the subgraph of



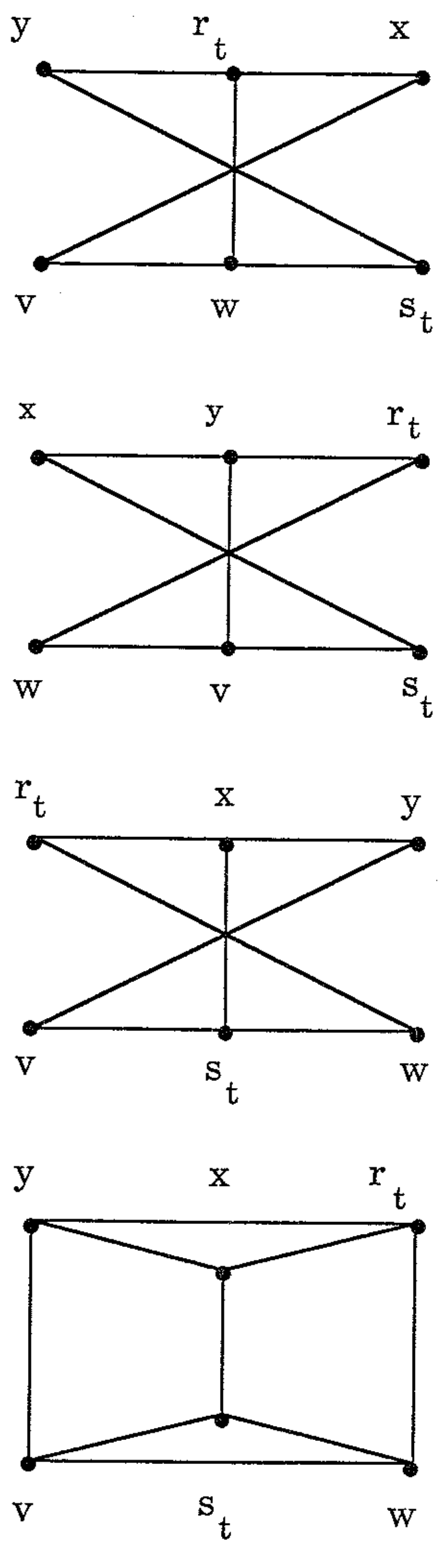


Figures 4.14.1 and 4.14.2 (top and bottom)

G induced by $\{r_i, r_j, r_t, s_i, s_j, s_t\}$ is as in *Figure 4.14.1*, else it is as in *Figure 4.14.2*.

If conclusion (a) of the *Reduction Lemma* holds then, by the induction hypothesis, we find vertices w_i and w_j non-adjacent to v such that w_i is universal on $S - s_i$ and null on $R - r_i$, and w_j is universal on $S - s_j$ and null on $R - r_j$. In case r_i, r_j are non-adjacent, case (1A) of the *Little Local Lemma* guarantees that one of w_i, w_j is universal on $\{s_i, s_j, s_t\}$ (and therefore on S), and null on $\{r_i, r_j, r_t\}$ (and therefore on R). In case r_i, r_j are adjacent, case (2A) of the *Little Local Lemma* guarantees that w_i in universal on $\{s_i, s_j, s_t\}$ and null on $\{r_i, r_j, r_t\}$.

If conclusion (b) of the *Reduction Lemma* holds, then by the induction hypothesis, we find a vertex w non-adjacent to v such that w is universal on $S - s_j$ and null on $R - r_i$. But now, by the *Little Local Lemma* (apply cases (1B) and (2B) if r_i, r_j are respectively non-adjacent and adjacent), w is universal on $\{s_i, s_j, s_t\}$ and null on $\{r_i, r_j, r_t\}$. ∎



Figures 4.15A to 4.15D (top to bottom)

The Second Extension Lemma. *Let G be a strong mirror subgraph of a graph H in M, let $[R,S]$ be a strong mirror partition of G, let r_t be isolated in G_R, and let v be a vertex in $H-G$ that is universal on $R-r_t$ and null on $S-s_t$.*

(1) *If v sees both r_t and s_t*

then some vertex w misses v, sees both r_t and s_t, is universal on $S-s_t$ and null on $R-r_t$.

(2) *If v sees r_t and misses s_t*

then some vertex w misses v, is universal on S and null on R.

(3) *If v misses both r_t and s_t*

then there are vertices w,x,y such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15A, and such that

both x and y are universal on $S-s_t$ and null on $R-r_t$, and w is universal on $R-r_t$ and universal on $S-s_t$.

(4) *If v misses r_t and sees s_t*

then there are vertices w,x,y such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in one of Figures 4.15B, 4.15C, 4.15D and such that

both x and y are universal on $S-s_t$ and null on $R-r_t$, and w is universal on $R-r_t$ and universal on $S-s_t$.

Proof. In all four cases, we shall argue by induction on the number of vertices in G. If G has precisely six vertices then, in each of the four cases, the desired conclusion follows from the corresponding case of the *C_6 Lemma*; see *Figure 4.7.* Now assume that G has at least eight vertices. Note that (since r_t is isolated in G_R) neither G_R nor $\overline{G}_R$ is $2K_2$; let r_i, r_j be as in the *Reduction Lemma.*

If case (a) of the *Reduction Lemma* applies, set $G' = G - \{r_i, s_i\}$, $R' = R - r_i$, $S' = S - s_i$;

set $G'' = G - \{r_j, s_j\}$, $R'' = R - r_j$, $S'' = S - s_j$.

If case (b) applies,

set $G' = G - \{r_i, s_j\}$, $R' = R - r_i$, $S' = S - s_j$;

set $G'' = G - \{r_j, s_i\}$, $R'' = R - r_j$, $S'' = S - s_i$.

Proof of (1). By the induction hypothesis, there is a vertex w that misses v, sees both r_t and s_t, and is universal on $S' - s_t$ and null on $R' - r_t$. Since w is universal on $\{r_t, s_t\}$, the *Zero-Two Lemma* guarantees that w is either universal on $S - s_t$ and null on $R - r_t$ or null on $S - s_t$ and universal on $R - r_t$. To exclude the latter alternative, we only need recall that w is universal on $S' - s_t$.

Proof of (2). By the induction hypothesis, there is a vertex w' that misses v, is universal on S' and null on R'; there is also a vertex w'' that misses v, is universal on S'' and null on R''. By the *Little Local Lemma,* one of w', w'' has the properties required of w.

Proof of (3). By the induction hypothesis, there are vertices w, x, y such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as in *Figure 4.15A,* and such that x and y are both universal on $S' - s_t$ and null on $R' - r_t$, and w is universal on $R' - r_t$ and null on $S' - s_t$. Since w is universal on $\{r_t, s_t\}$, the *Zero-Two Lemma* guarantees that w is either universal on $R - r_t$ and null on $S - s_t$ or universal on $S - s_t$ and null on $R - r_t$. To exclude the latter alternative, we only need recall that w is universal on $R' - r_t$. The same argument shows that y is universal on $S - s_t$ and null on $R - r_t$. Finally, since v and s_t are twins with respect to G, and since x is universal on $\{r_t, v\}$, the same argument used once again shows that x is universal on $S - s_t$ and null on $R - r_t$.

Proof of (4). Let F be the subgraph of G induced by $\{r_t, s_t, v, x, x, y\}$. By the induction hypothesis, there are vertices w, x, y such that F is as in one of *Figures 4.15B, 4.15C, 4.15D,* and such that x and y are both universal on $S' - s_t$ and null on $R' - r_t$, and w is universal on $R' - r_t$ and null on $S' - s_t$. There are three cases to consider.

Case B: the subgraph F is as in *Figure 4.15B.*

Since s_t and v are twins with respect to G, and since w is universal on $\{r_t, v\}$, by the *Zero-Two Lemma* it follows that w is either universal on $R - r_t$ and null on $S - s_t$ or null on $R - r_t$ and universal on $S - s_t$; to exclude the latter alternative, note that w is universal on $R' - r_t$. Since x and y are respectively null and universal on $\{r_t, v\}$, the same argument shows that both x and y are universal on $S - s_t$ and null on $R - r_t$.

Case C: the subgraph F is as in *Figure 4.15C.*

Since x, y, w are each either universal or null on $\{r_t, s_t\}$, the *Zero-Two Lemma* together with x and y being universal on $S' - s_t$ and w being universal on $R' - r_t$ imply that x and y are both universal on $S - s_t$ and null on $R - r_t$, and w is universal on $R - r_t$ and null on $S - s_t$.

Case D: the subgraph F is as in *Figure 4.15D.*

Since x, w are both universal on $\{r_t, s_t\}$, the *Zero-Two Lemma* together with x being universal on $S' - s_t$ and w being universal on $R' - r_t$ imply that x is universal on $S - s_t$ and null on $R - r_t$, and w is universal on $R - r_t$ and null on $S - s_t$. Finally, note that v and s_t are twins with respect to G, and that y is universal on $\{r_t, v\}$. Now the *Zero-Two Lemma* together with y being universal on $S' - s_t$ implies that y is universal on $S - s_t$ and null on $R - r_t$. This concludes the proof of the *Second Extension Lemma.* ∎

We now prove the main result of this chapter, namely, that the only unbreakable murky graphs are L_8, L_9, and strong mirror graphs.

Proof of Theorem 4.1. Let H be an unbreakable murky graph. If H contains L_8 as an induced subgraph, then by the L_8 *Lemma,* H is either L_8 or L_9.

Thus we may assume that H does not contain L_8, and so H is in M. Now note that the *WT Star Cutset Theorem* of Chapter 3 guarantees that H contains a chordless cycle with at least five vertices, or the complement of such a cycle. Since H is murky, H does not contain C_5, C_k, or $\overline{C}_k$, for $k \geq 7$. Thus H contains either C_6 or $\overline{C}_6$ as an induced subgraph; note that both C_6 and $\overline{C}_6$ are strong mirror graphs.

Now let G be any strong mirror subgraph of H with the greatest number of vertices. If $G = H$ then we are done, so assume that G is a proper subgraph of H; we will show that this leads to a contradiction.

Since H is unbreakable, there is some vertex v in $H - G$ that is partial on G. By the *Complement Lemma,* by taking the complement if necessary, we may assume that G_R is disconnected (note that v is partial on G if and only if v is partial on $\overline{G}$ in $\overline{H}$). By the *Attachment Lemma,* there are two possible cases.

Case (i): *there is a strong mirror partition [R,S] of G such that v is universal on R and null on S.*

In this case, by the *First Extension Lemma,* there is a vertex w that misses v, is null on R, and universal on S. Let $R' = R + w$ and $S' = S + v$. Now we claim that the partition $[R',S']$, whose couples are $\{w,v\}$ and all couples of $[R,S]$, is a strong mirror partition. To justify this claim, we need only show that R' and S' are strong P_4-free graphs, that $f(G_{R'},w) = f(G_{S'},v)$, and that for every couple $\{r_j,s_j\}$ of $[R,S]$,

$f(G_{R'}, r_j) = f(G_R, r_j)$ and $f(G_{S'}, s_j) = f(G_S, s_j)$.

Since G_R is a disconnected strong P_4-free graph, it has at least two components. $G_{R'}$ is formed by adding the isolated vertex w to G_R; thus $G_{R'}$ is P_4-free, and has at least three components; thus $G_{R'}$ is strong. Since w is an isolated vertex in $G_{R'}$, $f(G_{R'}, w) = 0$. Similarly, $G_{S'}$ is a strong P_4-free graph, and $f(G_{S'}, v) = 0$.

Finally, let r_j be any vertex of R, and let X be the vertex set of the component of G_R containing r_j. Note that X is also the vertex set of the component of $G_{R'}$ containing r_j. If $|X| \geq 2$, then X is a homogeneous set of both G_R and $G_{R'}$, and $f(G_R, r_j) = f(G_X, r_j) = f(G_{R'}, r_j)$, by the *Localization Lemma.* On the other hand, if $|X| = 1$, then r_j is a singleton in both G_R and $G_{R'}$, and $f(G_R, r_j) = 0 = f(G_{R'}, r_j)$. Similarly, $f(G_S, s_j) = f(G_{S'}, s_j)$ for all s_j in S. Thus the claim holds in this case, and $[R', S']$ is a strong mirror partition, contradicting the assumption that G was a largest strong mirror subgraph of H.

Case (ii): *in every strong mirror partition [R,S] of G there is a couple* $\{r_t, s_t\}$ *such that v is a twin of one of* r_t, s_t *with respect to* $G - \{r_t, s_t\}$.

By the *Isolation Lemma,* there is a strong mirror partition of G such that r_t is a singleton in whichever of G_R or $\overline{G}_R$ is disconnected. By the *Complement Lemma,* $\overline{G}$ is also a strong mirror graph, with the same partition; v is partial on G in H if and only if v is partial on $\overline{G}$ in $\overline{H}$. Thus, by taking the complement of H if necessary, we may assume that r_t is isolated in G_R. Now v is a twin of either r_t or s_t with respect $G - \{r_t, s_t\}$; by swapping R and S if necessary, we may assume that v is a twin of s_t. Thus v is universal on $R - r_t$ and null on $S - s_t$. Now the *Second Extension Lemma* applies, and there are four subcases to consider. (See *Figure 4.7.*)

Subcase (1): *v sees both r_t and s_t, and some vertex w misses v, sees both r_t and s_t, and is universal on $S - s_t$ and null on $R - r_t$.*

Let $R' = R + w$ and $S' = S + v$. It is a routine exercise to show that the partition $[R',S']$, whose couples are $\{w, s_t\}$, $\{r_t, v\}$, and all couples of $[R - r_t, S - s_t]$, is a strong mirror partition.

Subcase (2): *v sees r_t and misses s_t, and some vertex w misses v, is universal on S, and null on R.*

Let $R' = R + w$ and $S' = S + v$. It is a routine exercise to show that the partition $[R',S']$, whose couples are $\{w, v\}$, and all couples of $[R, S]$, is a strong mirror partition.

Subcase (3): *v misses both r_t and s_t, and there are vertices w,x,y such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15A, and such that x and y are both universal on $S - s_t$ and null on $R - r_t$, and w is universal on $R - r_t$ and null on $S - s_t$.*

Let $R' = R + \{x, y\}$ and let $S' = S + \{v, w\}$. It is a routine exercise to show that the partition $[R',S']$, whose couples are $\{r_t, w\}$, $\{x, s_t\}$ and $\{y, v\}$ and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition.

Subcase (4A): *v misses r_t and sees s_t, and there are vertices w,x,y such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15B, Figure 4.15C, or Figure 4.15D, and such that x and y are both universal on $S - s_t$ and null on $R - r_t$, and w is universal on $R - r_t$ and null on $S - s_t$.*

Let $R' = R + \{x, y\}$ and let $S' = S + \{v, w\}$. If the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in *Figure 4.15B*, then the partition $[R',S']$ with couples $\{r_t, s_t\}$, $\{x, w\}$, $\{y, v\}$, and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition; if the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in *Figure 4.15C*, then the partition $[R',S']$ with couples $\{r_t, v\}$, $\{y, w\}$, $\{x, s_t\}$ and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition;

if the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in *Figure 4.15D*, then the partition $[R',S']$ with couples $\{r_t, w\}$, $\{y, v\}$, $\{x, s_t\}$ and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition;

Thus, in all four subcases there exists in H a strong mirror subgraph with more vertices than G; this contradiction completes the proof of *Theorem 4.1.* ■

4.4 A Characterization of Unbreakable Murky Graphs

In the previous section we showed that if a murky graph is unbreakable, then it must be L_8, L_9 or a strong mirror graph. In this section, we will prove the converse, namely, that L_8, L_9 and strong mirror graphs are murky and unbreakable. These two results combine to give the following characterization of unbreakable murky graphs.

Theorem 4.3. *A graph is murky and unbreakable if and only if it is either L_8, L_9, or a strong mirror graph.*

The necessary half (i.e. the "only if" part) of the theorem is *Theorem 4.1*; thus to prove *Theorem 4.3*, we need only prove the sufficiency half of the theorem. This half of the theorem is proved as the following four propositions.

Proposition 1. *The graphs L_8 and L_9 are murky.*

Proposition 2. *Mirror graphs are murky.*

Proposition 3. *The graphs L_8 and L_9 are unbreakable.*

Proposition 4. *Strong mirror graphs are unbreakable.*

Proof of Proposition 1. Since removing a vertex from L_9 corresponds to removing an edge from $K_{3,3}$, it follows that every eight-vertex induced subgraph of L_9 is L_8. Also, removing a vertex of degree four from L_8 leaves L_7; removing a vertex of degree three leaves $\overline{L}_7$. Thus every seven-vertex subgraph of L_8, and L_9, is L_7 or $\overline{L}_7$. It is a routine matter to verify that L_7 is murky; since the complement of a murky graph is murky, $\overline{L}_7$ is murky. Thus both L_8 and L_9 are murky. ∎

Proof of Proposition 2. Recall the *Mirror Proposition* of Section 4.2: every induced subgraph of a mirror graph has twins or anti-twins. Since neither C_5, P_6, nor $\overline{P}_6$ have either twins or anti-twins, mirror graphs cannot have C_5, P_6, or $\overline{P}_6$ as induced subgraphs; thus mirror graphs are murky. ■

Recall that the neighbourhood $N(v)$ of a vertex v in a graph G is the set of all vertices of $G - v$ that see v, and that the non-neighbourhood $M(v)$ is the set of all vertices of $G - v$ that miss v. A *pure star cutset* of a graph G is a set $S = v \bigcup N(v)$, for some vertex v in G, such that $G - S$ is disconnected. The difference between a pure star cutset and a star cutset is that a pure star cutset consists of a vertex together with *all* of its neighbours, whereas a star cutset consists of a vertex together with *any subset* of its neighbours. We will call a graph G with at least three vertices *breakable* if either G or $\overline{G}$ has a star cutset. The following claim helps to shorten the proof of the final two propositions.

Claim (Chvátal, private communication). *Let G be a breakable graph with at least five vertices. Then either G or $\overline{G}$ has a pure star cutset.*

Proof of Claim. Let G be a breakable graph with no pure star cutset. Chvátal observed [1985a] that this implies the existence of vertices v, w in G, such that v sees w, and v dominates w. Now, if v and w have any common neighbour x in G, then, in $\overline{G}$, $w \bigcup N(w)$ is a pure star cutset of $\overline{G}$. (In $\overline{G}$, removing w and all its neighbours leaves a graph in which v is a singleton, and x is in some other component.) Thus we may assume that the only neighbour of w in G is v. Let $H = G - \{v, w\}$. Now there are two cases to consider.

Case 1: some vertex z (other than w) sees v and misses some $h \in H$.
In this case, we are done: in G, $z \bigcup N(z)$ is a pure star cutset.

Case 2: every vertex z (other than z) that sees v sees all vertices in H.
Let S be the set of vertices of H that see v, and let T be all other vertices of H. Note that the hypothesis of Case 2 implies that S is a clique, and that every vertex in S sees

every vertex in T. Now, if there are any two non-adjacent vertices a, b in T, then, in G, $a \bigcup N(a)$ (which includes all of S) is a pure star cutset. Otherwise, T is a clique, and therefore the vertices of H form a clique. But now, there is a vertex $h \in H$ such that in $\overline{G}$, $h \bigcup N(h)$ is a pure star cutset: if T is non-empty, pick h any vertex in T; else, pick h any vertex in S (in each case, in $\overline{G}$ $N(h)$ is a subset of $\{v, w\}$, and the vertices in $M(h)$ form a stable set; since G has at least five vertices, the stable set has at least two.) This completes the justification of the *Claim.* ∎

Proof of Proposition 3. To prove that L_8 is unbreakable, by the preceding claim and the fact that L_8 is self-complementary, we need only prove that L_8 has no pure star cutset: we need only prove that, for each vertex $v \in L_8$, $M(v)$ is connected. An *automorphism* of a graph G is a permutation P of the vertices such that x and y are adjacent if and only if $P(x)$ and $P(y)$ are adjacent, for all pairs of vertices x and y. Note that for every pair of vertices of L_8 with the same degree, there is an automorphism which maps one vertex to the other. Label the vertices of L_8 as in *Figure 4.3.* Vertex 1 has degree 4; the subgraph induced by $M(1)$ is a P_3, and is hence connected. Vertex 5 has degree 3; the subgraph induced by $M(5)$ is a C_4, and is hence connected. Thus L_8 is unbreakable.

To prove that L_9 is unbreakable, by the preceding claim and the fact that L_9 is self-complementary, we need only prove that L_9 has no pure star cutset; i.e. we need only prove that, for each vertex $v \in L_9$, $M(v)$ is connected. Note that for any two vertices in L_9 there is an automorphism which maps one vertex to the other. Thus we need only show that, for any vertex v of L_9, $M(v)$ is connected. Pick any vertex of L_9; its non-neighbourhood induces a C_4, and is hence connected. Thus L_9 is unbreakable. ∎

Proof of Proposition 4. To prove that a strong mirror graph is unbreakable, by the preceding claim and the fact that $\overline{G}$ is a strong mirror graph (see the *Complement Lemma*) we need only show that no vertex in G has a pure star cutset. By the *Isolation Lemma,* there is a strong mirror partition $[R, S]$ such that (v is in R and) v is a singleton in whichever of G_R or $\overline{G}_R$ is disconnected. Let w be the mate of v.

Case 1: G_R is disconnected.

In this case $f(G_R, v) = 0$, and v misses w. Thus $M(v) = R - v + w$; since w sees all of $R - v$, $M(v)$ is connected.

Case 2: $\overline{G}_R$ is disconnected.

In this case $f(G_R, v) = 1$, and v sees w. Thus $M(v) = R - v$. But since G is strong, the fact that v is a singleton in $\overline{G}_R$ implies that $\overline{G}_R$ has at least three components, and so $\overline{G}_{R-v}$ is disconnected, and therefore G_{R-v} is connected. This concludes the proof of *Proposition 4*, and also the proof of *Theorem 4.3.* ■

Appendix

The following result appears as *Theorem 2.2.1* in Olariu [1986].

Theorem (Olariu). *No minimal imperfect graph contains anti-twins.*

Proof. Assume the statement false: some minimal imperfect graph G contains anti-twins u and v. Let A denote the set of all neighbours of u other than v, and let B denote the set of neighbours of v other than u.

Let α and ω denote the number of vertices in a largest stable set and clique respectively of G. Now

B contains a clique of size $\omega - 1$ that extends
into no clique of size ω in $A \bigcup B$. (*)

To justify (*), colour $G - v$ by ω colours and let S be the colour class that includes u. Since $G - S$ cannot be coloured by $\omega - 1$ colours, it contains a clique of size ω; since $G - S - v$ is coloured by $\omega - 1$ colours, it must be that $v \in C$. Hence $C - v$ is a clique in B of size $\omega - 1$. If a vertex x extends $C - v$ into a clique of size ω then $x \notin B$ (since otherwise x would extend C into a clique of size $\omega + 1$). Thus (*) is justified.

The *Perfect Graph Theorem* guarantees that the complement of G is minimal imperfect; thus (*) implies that

A contains a stable set of size $\alpha - 1$ that extends
into no stable set of size α in $A \bigcup B$. (**)

Now let C be the clique featured in (*) and let S be the stable set featured in (**); let X be a vertex in C that has the smallest number of neighbours in S. By (**), x has a neighbour z in S; by (*), z is non-adjacent to some y in C. Since y has at least as many neighbours in S as x, it must have a neighbour w in S that is non-adjacent to x. Now u,z,x,y,w induce in G a chordless cycle. Thus G is not minimal imperfect. ■

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