CGT class notes for 2020 Mar 26

In this lecture we show how, for the game of Hex, a player strategy can be presented in compact form. Roughly, the idea is this:

- start with the usual 2-player alternate-turn game tree representation for a strategy (when describing a strategy for player P with opponent O, at each node in the tree with O to play, give all O-move; at each node in the tree with P to play, give only the one move from P’s strategy)

- compress the tree in these two ways
  - omit all reference to opponent moves, so go from an explicit tree to an implicit tree, showing only P-moves (we will explain how to use this notation later)
  - whenever the remaining strategy can be described by independent subgames (in CGT, think of this as a sum of games), represent each subgame separately

To illustrate, consider 3x3 Hex on the board shown below. How would we describe the simplest 1st-player-Black winning strategy in the usual way?

```
* * *
1 . . .
2 . . .
3 . . .
* * *
 a b c
```

Well, the root of the tree is the first move b2. The root has 8 children, one for each possible white reply. For each of these root-children, the tree shows Black’s next move, and each of those 8 root-grandchildren has 6 children, one for each possible white reply. The final tree has 1, 8, 8, 48, 48 nodes respectively at levels depth 0 through 4, and that is all, because we are describing a strategy where Black can win by move 3. So this tree has 113 nodes.

Here is how to describe Black’s winning strategy more compactly: play at b2, then maintain the virtual connection from b2 to the top using empty cells \{b1,c1\} (whenever the opponent takes one, take the other) and a similar connection to the bottom using \{a3,b3\}. We write this strategy \(S\) as \(b2\{b1,c1\}\{a3,b3\}\).

Here’s how we define such strategies. A first-player Hex strategy is a cell (indicating the first move of the strategy) together with 0 or more second-player strategies. A second-player Hex strategy is two or more first-player strategies, with the property that the intersection of the cells used by these first-player substrategies is empty.
Here is how these implicit strategies are used. Assume that player $P$ is following a second-player strategy $S = s_1, \ldots, s_t$ for player $P$ and that the opponent moves to a cell $c$. Then $P$ replies by picking any substrategy $s_j$ that does not contain $c$. (There must be some such strategy, because the intersection of the cell sets of $s_1, \ldots, s_t$ is empty.)

Here is an example 3x3 Hex game in which Black follows $S = b2 \{b1, c1\} \{a3, b3\}$.

1. B[b2]. Now B follows $S' = \{b1, c1\} \{a3, b3\}$.

   say 2. W[a3].

   This move hits 1st-player strategy a3 (just a single move) in 2nd-player strategy \{a3, b3\}, so B must reply by following the unhit strategy b3 (just a single move)

   3. B[b3]. This leaves $S'' = \{b1, c1\}$.

   say 4. W[a1]

   This hits neither 1st-player strategy, so B can follow either 1st-player strategy. (In fact, B can still win by playing anywhere, but it would be silly to not win as quickly as possible.)

Notice that this move leaves B with the empty set as the remaining strategy to follow: the game is over, B has won.

Another 1st-player-black 3x3 winning strategy: a3 \{ a2 \{a1, b1\}, c1 \{b2, c2 \{b3, c3\} \} \}.

Here is what happens after 1.B[a3] 2.W[x] where $x$ is any of \{b2, b3, c1, c2, c3\}.

Here is what happens after 1.B[a3] 2.W[y] where $y$ is any of a1, a2, b1.
Can you find the other (non-isomorphic to both examples so far) 1st-player-black winning strategy?

Answer: \( S = a_2 \{a_1 \ b_1\} \{a_3, \ c_2\{b_2 \ c_1\}{b_3 \ c_3}\} \)

Let’s write this as \( S = a_2 \alpha \beta \) where \( \alpha = \{a_1, b_1\} \) and \( \beta = \{\gamma_1, \gamma_2\} \), where \( \gamma_1 = a_3 \) and \( \gamma_2 = c_2 \{b_2, c_1\}{b_3, c_3}\).

Here is the picture after 1.B[a2]. Cells marked + are used by \( \alpha \), cells marked − are used by \( \beta \).

\[
\begin{array}{ccc}
\ast & \ast & \ast \\
1 & + & + & - \\
2 & ! & - & - \\
3 & - & - & - \\
\ast & \ast & \ast \\
 a & b & c \\
\end{array}
\]

Example: where does B play after 1.B[a2] 2.W[b2]?

Answer: 2.W[b2] hits strategy \( \gamma_2 \) of \( \beta \), so B must follow \( \gamma_1 \), so 3.B[a3], leaving \( S' = \{a_1, b_1\} \).

Example: where does B play after 1.B[a2] 2.W[a3]?

Answer: 2.W[a3] hits strategy \( \gamma_1 \) of \( \beta \), so B must follow \( \gamma_2 \), so 3.B[c1], leaving \( S' = \{a_1, b_1\}{b_2, c_1\}{b_3, c_3}\).

Next: try this for 4x4. Find all first-player-black winning moves, and for each give a winning strategy in this notation.

By the way, Jing Yang did this by hand for 9x9! Some day I hope that a computer will find a strategy in this notation for 10x10. It might be a way to eventually find a winning 11x11 strategy.