Winning Ways for Your Mathematical Plays

by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy

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Second edition, Wellesley, Massachusetts: A. K. Peters Ltd., 4 vols., 2001–2004; vol. 1: ISBN 1-56881-130-6; vol. 2: ISBN 1-56881-142-X; vol. 3: ISBN 1-56881-143-8; vol. 4: ISBN 1-56881-144-6

On Numbers and Games

by John H. Conway

First Edition, New York: Academic Press, 1976; ISBN 0-12-186350-6

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REVIEWED BY ROBERT A. HEARN

Winning Ways and *On Numbers and Games* capture, between them, the essential connection between the playful and the deeply elegant that so epitomizes Conway's work. Their stories are intimately linked, so it is natural for a retrospective review to consider them together. But I can think of no more appropriate way to begin than to quote Solomon Golomb's 1983 review of Winning Ways:

"Winning ways is a masterpiece. We should have been disappointed were it anything less. Fifteen years in the preparation, and representing the collaboration of three mathematicians of extraordinary talent, the result is the most compelling and comprehensive treatment of mathematical games to appear in this century."

Nearly 40 years on, this evaluation still rings true. We would now say that *Winning Ways* (*WW*) represents the definitive work in *combinatorial game theory*, or CGT (to be distinguished from economic game theory, pioneered by von Neumann and Morgenstern). A combinatorial game is a two-player game with alternating play, no hidden information, and no chance elements. Examples include classic board games such as Go, Chess, and Checkers; paper-and-pencil games such as Tic-Tac-Toe and Dots and Boxes; and many abstract games such as Hackenbush, Col, Snort, and Domineering, all of which are used in both books to develop the theory.

The theory hinges on an amazing discovery of Conway. In a nutshell, the natural unification of Cantor's construction of the ordinals with Dedekind's construction of the reals yields, in one stroke, not only the fabulous number system that Donald Knuth dubbed the *surreal numbers*, but also a natural framework for combinatorial games. It's as if the the mathematics we all know, that underpins all of science, were suddenly revealed to be a just special case of games: the universe is telling us that ultimately, it's all about play.

To see how this happens, let's back up a bit. The theory of combinatorial games can be traced back to the 1930s, when Sprague and Grundy independently showed that every *impartial* game (where the same moves are available to each player) is equivalent to a one-heap game of Nim. Even earlier, Bouton gave a complete analysis of Nim in 1901, perhaps the earliest result in CGT. *WW* was originally planned to be a further development of the theory of impartial games. But Conway had always wondered whether the theory could be extended to the much broader class of *partizan* games (where the players may have different moves available). Inspired by trying to understand the complexities of Go, he observed that in the endgame, the board breaks up into independent regions that it makes sense to add together, as one adds Nim positions.

The idea of game addition is simple. The sum of games G and H, $G+H$, is that game where the player to move may choose to move either in G or in H. (It is normal in CGT to refer to a particular position as a game.) If the move is from G to G', say, then the resulting position is $G'+H$. Concretely, the simplest possible game is called 0, written explicitly as $\{\,|\,\}$. This means a position where neither Left nor Right (the conventional player names) has any move available. Under the "normal play" convention, it's a previous-player win: a $\mathcal{P}$ position. Suppose Left can move to 0, but Right can't move; then we have $\{\,0\,|\,\}$ (or $\{\,\{\,|\,\}\,|\,\}$). It turns out that it makes sense to call this game 1: Left has one free move. Likewise $\{\,0,1\,|\,\}$ is called 2, etc. Negative numbers appear naturally, as do fractions (the concept of half a move is meaningful) — and addition as defined above works as expected: $\frac{1}{2}+\frac{1}{2}=1$, for example. But there is also the simple game $\{\,0\,|\,0\,\}$, called star (*), which is not a number at all (though it is an infinitesimal!). Star is a next-player win, or $\mathcal{N}$ position (because whoever moves leaves the 0 position for the other player).

In general, a position $\{\,a,b,c,\ldots\,|\,d,e,f,\ldots\,\}$ has left options $a,b,c,\ldots$, and right options $d,e,f,\ldots$. All (non-loopy) games can be constructed starting from $0=\{\,|\,\}$. If we continue the process above of leaving the right options empty, we will construct the ordinals along the lines of Cantor: the smallest transfinite ordinal is $\omega=\{\,0,1,2,\ldots\,|\,\}$. If we use both left and right options, but ensure that all left options are less than all right options in each new game (where $<$ is appropriately defined), we will construct the reals along the lines of Dedekind (but without the need to assume prior existence of the rationals). If we allow both, we get the surreal numbers, which include such unusual beasts as $\sqrt[3]{\omega+1}-\frac{\pi}{\omega}$. If we remove the ordering constraint, we get general games. Miraculously, the game addition rule defined above, motivated purely by what it means to play in a subcomponent of a game, yields correct results when applied to the surreals. Conway was able to find a multiplication rule as well, which is more complex. With these definitions, the surreals form an ordered Field whose domain is a proper class.

Conway was very taken with the discovery of the surreals, but there was some disagreement about including the theory of surreals in *WW*, as the focus was to be on games. Famously, he then wrote the majority of *On Numbers and Games* (*ONAG*) in one week, to "get it out of the way" so that work on *WW* could progress, then confessed this to his coauthors (much was included that was to be part of *WW*).

With this background, let us take a closer look at each book, and at what further developments they have inspired.

Though this special issue commemorates John Conway (1937–2020), any review of *WW* must also mention the substantial contributions of Elwyn Berlekamp

(1940–2019) and Richard Guy (1916–2020). The loss of all three within one year and two days is a huge blow to mathematics, even apart from CGT: Guy was a giant in number theory, Berlekamp in coding theory, and Conway in group theory, among their many other contributions. In the 1940s, Guy independently rediscovered the impartial-game theory of Sprague and Grundy, and dramatically extended it, most notably to the wide class of Nim-like games called *octal games*. Berlekamp learned of this theory and applied it to many other games, including Dots and Boxes. According to Guy, most of the ideas in *WW* came from Berlekamp and Conway, but were mainly written by Guy, from Conway's dictation. (Of note, however, Guy's work on octal and related games forms a substantial part of volume 1.)

WW begins with explicit examples of simple games, and shows how various increasingly sophisticated concepts naturally emerge. The definitions come later, when they are already obvious, and come across more as observations. It's an unusual and surprisingly effective exposition, aided in no small part by the light, witty, punny tone, as well as plenty of color figures. Overall, the presentation is delightful and engaging. The preface states that *WW* "is not a book on recreational mathematics because there's too much serious mathematics in it". There is, but nonetheless the book is appropriately dedicated to the master of recreational mathematics, Martin Gardner, whose own review calls it "the greatest contribution this century to the burgeoning field of recreational mathematics".

WW consists of two main divisions: *Games in General* (originally volume one), and *Games in Particular* (originally volume two). Each of these has been split further into two volumes in the second edition. Roughly, *Games in General* develops the theory, and *Games in Particular* applies the theory to many actual games. Theory here mostly means effective ways to analyze positions and determine their outcomes, but it can also mean showing that some games are intractable (NP-hard or harder) from a computational complexity standpoint.

Referring now to the second edition, volume one presents the general framework, motivated by extensive examples; and develops the theory, introducing such analytical tools as *temperature*, *thermography*, and *cooling*. Volume two explores the consequences of changing the framework in various ways: What happens when you may play in multiple components of a sum, instead of only one? What if there are infinitely many positions? What if the positions can repeat (*loopy* games)? What if the last player to play loses instead of wins (*misère* play)? Volume three is full of applications of the theory to all kinds of games: games with coins, pencil-and-paper games, games of pursuit, board games. Some games are presented for which there is no application of the theory, but are nonetheless interesting original contributions. (The chief of these would be *Philosopher's Football*, or *Phutball*, a favorite invention of Conway's, in which I was privileged to have him instruct me.) Volume three is the largest of the volumes. Volume four is properly not connected to the main body of CGT, but fits into the broader category of game-like recreations: it discusses several one-player games (puzzles), such as Peg Solitaire and Rubik's Cube, and even devotes substantial space to a zero-player game, Conway's famous Game of Life. Though CGT is mostly applied in order to "solve" games (achieve "positive" results), showing the intractability of a game (a "negative" result) can be equally beautiful. This is exemplified by the intricate, delightful construction of a universal computer in the Game of Life, proving that simple questions about some Life positions have no simple answers.

Since *WW*, progress in CGT has continued on multiple fronts. There have been several CGT workshops, attended by various subsets of the *WW* authors, introducing a new generation of researchers to the subject. The results of these workshops are collected in the books *Games of No Chance*, volumes 1–5 (a sixth is in preparation). Conway's original interest in Go endgames was further developed by Berlekamp and David Wolfe, resulting in the book *Mathematical Go: Chilling Gets the Last Point*. Two CGT textbooks have appeared: *Lessons in Play*, by Michael Albert, Richard Nowakowski, and David Wolfe is an outstanding undergraduate text; and *Combinatorial Game Theory*, by Aaron Siegel, is a comprehensive graduate text, incorporating essentially every new development in CGT up to 2013 (as well as a thorough history of CGT). Work on the theory of intractability of mathematical games is summarized in the paper *Playing Games with Algorithms: Algorithmic Combinatorial Game Theory*, by Erik Demaine and Robert Hearn (in *Games of No Chance 3*), and the book *Games, Puzzles, and Computation*, by Hearn and Demaine. Other CGT research includes the useful notion of *invariant games*, developed by Eric Duchêne and Michel Rigo.

Returning now to *ONAG*, it is, like *WW*, divided into two parts: *. . . On Numbers*, and *. . . and Games*. The latter is essentially an early, abbreviated version of *WW*, so I will focus on the former, which develops the surreals. In contrast to *WW*, *ONAG* develops the surreals beginning with definitions, exploring their consequences, in a concise, formal development. In short order Conway proves that the surreals form a Field (called **No**), then discusses their relationship to the ordinals and to the reals. Next, the *normal form* of a general surreal is defined, which proves necessary for a discussion of algebra and analysis with the surreals. The concept of "gaps" in the surreals (as reals are gaps in the rationals) leads to the amazing equation $\infty = \sqrt[On]{\omega}$, relating three different infinities. The subclass of the surreals called *omnific integers* (or **Oz**) is considered from a number-theoretic perspective. Finally, in an analogy with impartial games, abolishing sign distinctions leads to the "simplest" way of turning the ordinals into a Field, which is called $\mathbf{On}_2$, and obeys the Nim addition rule discovered by Bouton.

Before discussing developments since *ONAG*, a prior development must be mentioned: namely Donald Knuth's charming book *Surreal Numbers*, based on Conway's pre-*ONAG* lectures. Like Conway, Knuth wrote his book in one week — and then rewrote it in another week when Conway pointed out he had started with an incorrect axiom! Post-*ONAG* work on the surreals includes the books *An Introduction to The Theory of Surreal Numbers*, by Harry Gonshor, *Foundations of Analysis over Surreal Number Fields*, by Norman Alling, and *Real Numbers, Generalizations of the Reals, and Theories of Continua*, by Philip

Ehrlich. An accessible popular account can also be found in Rudy Rucker's fascinating *Infinity and the Mind*, in the larger context of kinds of infinities.

WW and *ONAG* are gems of both recreational and serious mathematics, and the reader who is unacquainted with them has much enjoyment in store.

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