games that are numbers

finite:

- **integers**
- **dyadic rationals**

infinite:

- **integers**
- **rationals**
- **reals**
- **surreals**

integers (definition)

$$\begin{aligned}
0 &= \{\ \mid\ \} \\
1 &= \{\, 0 \mid\ \} \\
2 &= \{\, 1 \mid\ \} \\
\ldots \\
-1 &= \{\ \mid 0\, \} \\
-2 &= \{\ \mid -1\, \} \\
\ldots \\
k &= \{\, k-1 \mid\ \} \\
-k &= \{\ \mid -(k-1)\, \}
\end{aligned}$$

dyadic rationals (definition)

for integers m **odd,** t **positive,**

$$m\,/\,2^t \;=\; \{\,(m-1)/2^t \;|\; (m+1)/2^t\,\}$$

e.g. $9\,/\,4 \;=\; \{\,8/4 \;|\; 10/4\,\}$

$$=\; \{\,2 \;|\; 5/2\,\}$$

exercise: let $g = 7/8$**,** $h = 1/2 + 1/4 + 1/8$ **prove** $h = g$

hint: $h^L = \{1/4 + 1/8,\; 1/2 + 1/8,\; 1/2 + 1/4\,\}$

$$h^R = \{\,1 + 1/4 + 1/8,\; 1/2 + 1/2 + 1/8,\; 1/2 + 1/4 + 1/4\,\}$$

simplest number (definition)

let $g = \{\, x^L \mid y^R \,\}$ **with each** x_j, y_k **number and** $x^j < y_k$

let $\quad x_+ = \textbf{max}\ \{x^L\} \qquad\qquad y_- = \textbf{min}\ \{y^R\}$

simplest number $s(g)$:

in $(x_+,\ y_-)$, **some int ?** $\qquad\qquad s(g) =$ **min.abs.val. int**

else: $\quad$ **in** $(x_+,\ y_-)$, $\quad s(g) = j/2^t$ **with min** t

theorem: $g = s(g)$

e.g. $\quad g = \{\, -15/8 \mid -7/16 \,\} \quad s(g) = -1$

$g = \{\, 3/8 \mid 3/4 \,\} \qquad s(g) = 1/2$

$g = \{\, 3/8 \mid 5/2 \,\} \qquad s(g) = 1$

$g = \{\, 27/16 \mid 15/8 \,\} \qquad s(g) = 7/4$

lemma: int $n \geq 0$**,** **depth of** $\pm n$ **is** n

$t \geq 1$ $0 < j$ **odd** $< 2^t$ **depth of** $\pm(n + j/2^t)$ **is** $n + t$

theorem: g **number iff** $g^L < g < g^R$ **and** g^L**,** g^R **numbers**

for finite games, above theorem holds for ints, dyadic rationals

Conway: for infinite games, above theorem holds for

ints, rationals, reals, surreals

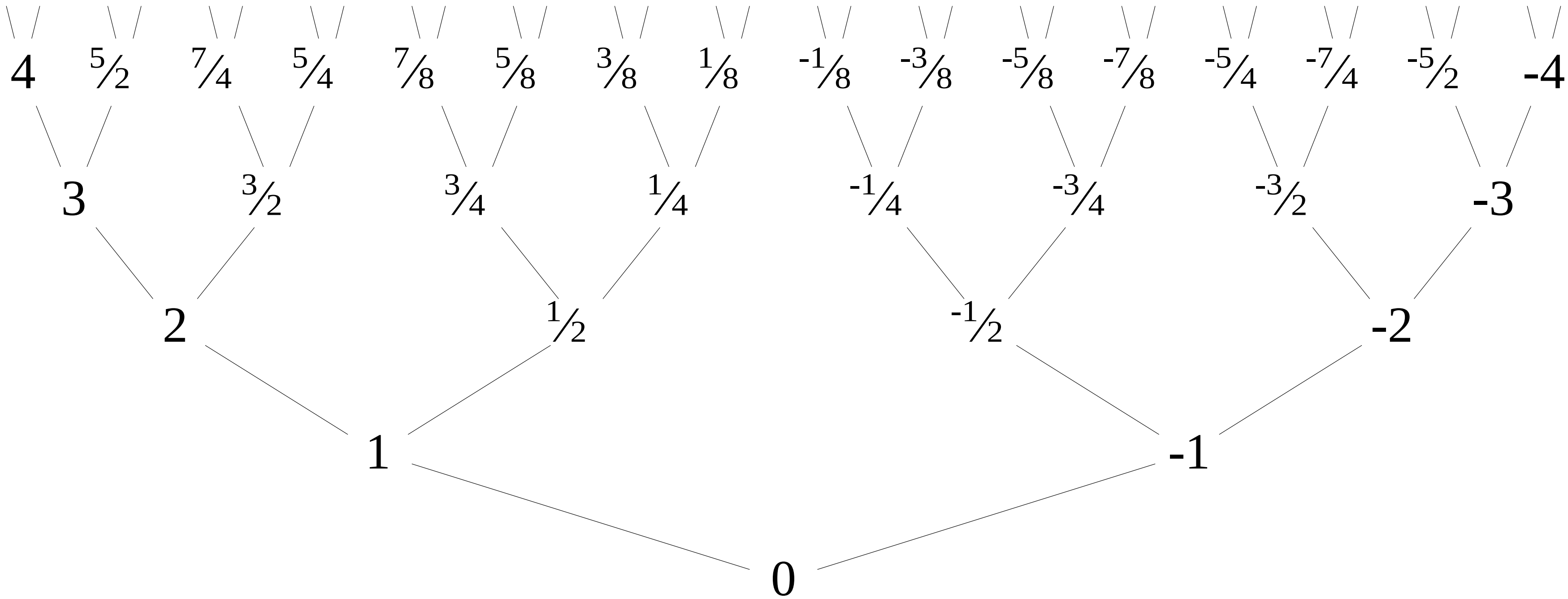
number avoidance theorem

game $g = x + h$ x **number,** h **not number**

L **has winning move on** g **?**

then L **can win by playing on** h

e.g. find a winning L**-move on** $g = 2 + *$



4
5/2
7/4
5/4
7/8
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1/8
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3
3/2
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-3/4
-3/2
-3
2
1/2
-1/2
-2
1
-1
0

exercise

find $s(g)$**, the simplest number equal to** $g = \{\, 3/8 \,|\, 7/8 \,\}$

answer

in interval (3/8, 7/8), we seek $m/2^t$ **with smallest** 2^t

candidates: 4/8=1/2 5/8 6/8=3/4 7/16 . . .

$s(g) =$ **1/2**

exercise **check previous answer with canonical form theorem**

$g = \{$ **3/8** $|$ **7/8** $\}$ **can we prune any dominated option?**

no **only 1 L-option, only 1 R-option**

$g = \{$ **3/8** $|$ **7/8** $\}$ **is either option reversible ?**

consider L-option

R can reverse 3/8 to 1/2 **R prefers 1/2 to** g **?**

1/2 $\geq g$ **?** **1/2** $= g$ **?** **1/2** $< g$ **?**

1/2 $=g$ **?** $\quad s=g+$ **{ −1 | 0 } is P-position?**

L can play s **to 3/8–1/2: R will win**

L can play s **to** g**–1: R will win**

R can play s **to** g **: L will win (exercise)**

R can play s **to 7/8–1/2 : L will win (exercise)**

so s **is P-position, woo hoo :)**

so R reverses 3/8 to 1/2 $\quad g=$ **{ 3/8 | 7/8 }** $=$ **{ 0 | 7/8 }**

final exercise: show right option of g **reverses to 1**

so L reverses 7/8 to 1 $\quad g=$ **{ 0 | 7/8 }** $=$ **{ 0 | 1 }**

conclusion: $s(g)=$ **1/2** $\quad$ **woo hoo :)**