lecture 3

alternating linear clobber

		outcome class	comment
game A2	ox	N	ox = { 0 | 0 } called *
game A4	oxox	N	= { up, 0, * | down, 0, *}
game A6	oxoxox	P	
game A8	oxoxoxox	?	

outcome class for A6 ? show Left options at root

A6

```
xxoxo     oxo_xx     oxx_ox     o_xxox     x_oxox
```

good move for Right?

```
xo_xo     _oo_xx     _ox_ox     o_xx_o     o_xx_o
```

so A6 is a P-position

alternating linear clobber conjecture:

A2t (for all positive 2t, except 2t = 6) is a N-position

with your neighbour: try to prove that A8 is an N-position

motivation: how can we reduce our work

in trying to prove that A8 is an N-position?

first moves from A8   xoxoxoxo

say x plays first, can move to

xoxoxo, xo xxoxo, xoxo xxo, xoxoxo x, xx oxoxo, xoxx oxo, xoxoxx o

definition of games equal to 0 (in cgt)

we say a game equals 0 if and only if it is a P-position

why ?

suppose we have a comb. game g and it decomposes

into 2 independent parts, x and y, and x is a P-psn

we write $g = x + y$

now assume y is an N position.

what can we say about g?

answer: g is an N-position.

proof?

proof: assume L plays first.

make a winning move on y.

if the opponent replies on y, continue with a winning move on y.

whenever the opponent replies on x, play a 2nd-player wins

strategy on x. this must leave (on x) either 0 or a P-psn (why).

so argue by induction.

recall: in game notation, 0 is the game { | }

now we can simplify: ignore P-positions

outcome class for A6 ? show Left options at root

```
                          A6
        xxoxo   oxo_xx  oxx_ox     o_xxox  x_oxox

simpler: xxoxo     oxo   oxx + ox    xxox    oxox


blah blah blah       ....   A4 + up + oxoxo + A6 + *

                      =     A4 + up + oxoxo + *
```

- **Devos/Kent chap 1**

  https://webdocs.cs.ualberta.ca/~hayward/cgt/asn/devoschap1.pdf

- **rules of go**

  https://webdocs.cs.ualberta.ca/~hayward/355/rules.pdf

lecture 4

- **rules of hex**
  https://webdocs.cs.ualberta.ca/~hayward/355/hexgame.pdf

hex

- no draw
- extra stones
- 1pw
- 2pw (irregular)
- virtual connections, mustplay

hex talk

https://webdocs.cs.ualberta.ca/~hayward/talks/twist2.pdf

hex page on CMPUT 355

https://webdocs.cs.ualberta.ca/~hayward/355/jem/hex.html

hex-the-full-story via ualberta ccid

https://www.taylorfrancis.com/books/hex-inside-ryan-hayward-bjarne-toft/10.1201/9780429031960

chomp

Fred Schuh 1952, David Gale 1974, Martin Gardner 1973

https://en.wikipedia.org/wiki/Chomp

https://www.win.tue.nl/~aeb/games/chomp.html

boxoff

https://www.abstractgames.org/boxoff.html

nim

two players. a *pile* is a non-empty set of stones. a *position* is a (possibly empty) set of piles of stones. on a turn, the player-to-move removes a positive number of stones from some pile. whoever cannot make a move loses and the other player wins.

we can represent a position as a (possibly empty) multiset of positive integers. here is a game between A and B starting from `{3,5,5}` **:**

```
  {3,5,5}
A {3,1,5}={1,3,5}
B {1,3}
A {1,1}
B {1}
A {}
B loses
```

https://en.wikipedia.org/wiki/Nim

https://webdocs.cs.ualberta.ca/~hayward/355/jem/nim.html

normal play end condition: whoever cannot move is loser

***misère* end condition: whoever cannot move is winner**

N (next) is the player-to-move next. P (previous) is the player who is not N, i.e. the player-who-moved previously, if there was a previous move.

who wins nim {0,0}?

N has no legal moves, so loses, so P wins.

who wins nim {0,1} = {1,0}?

N: she can move to the P-position {0,0}

who wins nim {0,n} for any positive n? **N**

who wins nim {n,n} for any positive n? **P**

defn: tweedledee-tweedledum strategy a 2nd-player strategy for player B in which B always mirrors A's move. e.g. in 2-pile nim from $\{n,n\}$, here is such a strategy for second player B: if A just removed t stones from a pile, remove t stones from the other pile

how many moves from nim (1, 3, 3)? $1+3+3=7$

how many *pairwise non-isomorphic* moves from nim (1, 3, 3)?

4: to any of {0,3,3}, {1,2,3}, {1,1,3}, {0,1,3}

how many moves from nim {1, 3, 3}? **4**

who wins nim {1, 2, 3}? **solve from bottom up**

{0, 0, 0}	**P-position**	**no legal moves**
{0, 0, 1}	**N-position**	**exists move to P-position {0,0,0}**
{0, 0, 2}	**N**	**exists move to P-position**
{0, 1, 1}	**P**	**only move is to N-position**
{0, 0, 3}	**N**	**exists move to P-position**
{0, 1, 2}	**N**	**exists move to P-position**
{1, 1, 1}	**N**	**exists move to P-position**
{0, 1, 3}	**N**	**exists move to P-position**

{0, 2, 2}	**P**	**each move is to N-position**
{1, 1, 2}	**N**	**exists move to P-position**
{0, 2, 3}	**N**	**exists move to P-position**
{1, 1, 3}	**N**	**exists move to P-position**
{1, 2, 2}	**N**	**exists move to P-position**

{1, 2, 3} ?

nim-sum of multiset of non-neg. integers is xor-sum

e.g. $1 \oplus 3 \oplus 6 \oplus 7 = 3$

```
1                             1
3                           1 1
6                         1 1 0
7                         1 1 1
                          -----
             nim-sum      0 1 1
```

multiset $M = \{6, 13, 13, 24, 30\}$ **has nim-sum 0:**

```
 6                          1 1 0
13                        1 1 0 1
13                        1 1 0 1
24                      1 1 0 0 0
30                      1 1 1 1 0
                          -----
             nim-sum    0 0 0 0 0
```

if we change (decrease, increase) one integer in M**?**

nim-sum will no longer be 0. do you see why?

```
 6                     1 1 0
10                   1 0 1 0   <- changed
13                   1 1 0 1
24                 1 1 0 0 0
30                 1 1 1 1 0
                     -----
      nim-sum      0 0 1 1 1   <- no longer 0
```

nimsum of a multiset of positive integers is the columnwise mod-2 sum of the binary representations. Bouton's theorem: a nim position is winning if and only if its nimsum is positive.

e.g. nimsum of {3,5,5} is `0b 011` **which is 3 which is positive, so Bouton's theorem says that this is position is winning (for the player-to-move), i.e. that the ptm has a winning strategy.**

```
3         0 1 1
5         1 0 1
5         1 0 1
nimsum    0 1 1
```

We can use Bouton's theorem to find all winning moves. nim{3,5,5} is winning, so there must be a move that leaves a losing position, i.e. a position with nimsum 0. here are all positions that ptm can move to from nim{3,5,5}: 55 155 255 35 135 235 335 345. compute the nimsum of each. which have nimsum 0?

Here is a faster way to use Bouton's theorem to find all winning moves from a nim position:

- in the nimsum array, find the left-most column c whose mod-2 sum is positive (if the nimsum is positive, why must such a column exist?)
- pick any row r with a one in column c. (why must such a row exist?)
- change the 1 at (row r, column c) to 0
- for each location Z = (r,c') with c' ¿ c, set Z to be the mod-2 sum of all other elements of that column
- why does this process leave a position with nimsum 0?
- why does process correspond to a legal move in nim?

e.g. nim{3,5,5}, the only column with mod-2 sum 1 is the middle column, so c is 2 (middle column). the only row with a 1 in that

column is column 1 (top), we we set that location to 0. now we set location (row 1, column 3) to the mod-2 sum of the rest of that column which is +=0*. So the only winning move from {3,5,5} is to {5,5}, i.e. from the pile with 3 stones, remove 3 stones.

```
3          1 1    -->    0 0     0
5        1 0 1         1 0 1     5
5        1 0 1         1 0 1     5
nimsum   0 1 0         0 0 0
```

another example: from {3,5,7}, three winning moves

```
3    1 1          1 1   3          1 1  3  -->    1 0   2
5  1 0 1        1 0 1   5  --> 1 0 0  4        1 0 1   5
7  1 1 1   --> 1 1 0   6       1 1 1  7        1 1 1   7
   0 0 1        0 0 0          0 0 0           0 0 0
```

nim-sum theorem (Bouton's theorem) nim state winning iff its nim-sum is positive

- **Let M be a multiset of non-negative integers including a positive integer m. Let M' be the multiset obtained from M by replacing m (just one instance) with an integer $0 \leq m' < m$. If nim-sum$(M) = 0$, then nim-sum$(M') \neq 0$.**

- **Let M be a multiset of non-negative integers such that nim-sum$(M) \neq 0$. Then some m in M can be replaced with a smaller integer m' so that nim-sum$(M') = 0$.**

  How to find m and m'? Let j be the position of the leftmost 1 in binary(nim-sum(M)). Let m be any integer in M whose position-j digit in binary(m) is also 1. Let $M - m$ be the multiset obtained by removing (one instance of m) from M. Let $q =$ nim-sum($M - m$). Let M' be the multiset obtained from M by replacing (one instance of) m with q. Then nim-sum$(M') = 0$.

example of second part of theorem.

```
 6                    1 1 0
10                  1 0 1 0
13                  1 1 0 1   <- change
24                1 1 0 0 0
30                1 1 1 1 0
                    -----
      nim-sum     0 0 1 1 1
                      j
```

j **is position 3 (counting from right), so** m **can be 6, 13, or 30. Pick** $m = 13$**. Then** $M - m = \{6, 10, 24, 30\}$**, nim-sum**$(M - m) =$ `01010_b` $= 10$**,** $M' = \{6, 10, 10, 24, 30\}$**.**

```
 6                    1 1 0
10                  1 0 1 0
10                  1 0 1 0
24                1 1 0 0 0
30                1 1 1 1 0
                    -----
```

nim-sum	0 0 0 0 0

nim: Bouton's corollary

recall Bouton's theorem:

nimsum-0, all moves to nimsum-pos

nimsum-pos, some move to nimsum-0

corollary nim position winning iff nimsum-pos

proof induction on pile-sum.