cmput 497/670 2024   homework 1

1. Give the outline of the proof of Zermelo's theorem from the lectures. Recall: in the proof, once we prove the theorem for a sub-dag of the original dag, we label the root of that sub-dag -1, 0 or 1. Explain the meaning of these labels.

2. In case 2 of Z's theorem, we consider a node v whose children are roots of sub-dags that have been correctly labelled. What else do we assume in case 2?

3. Finish the proof of case 2 of Z's thm: prove that for the sub-dag rooted at v,

   (a) ptm has a strategy that wins or draws against all possible optm strategies,

   (b) optm has a strategy that wins or draws against all possible ptm strategies.

4. Finish the proof of case 1 of Z's theorem.

5. Finish the proof of case 3 of Z's theorem.

6. Prove Zermelo's theorem for no-draw games (each terminal position is win or loss).

7. For a normal-play CGT, define Left, Right, outcome classes P, N, L, R hint: lec. 1

8. prove that clobber position `oxoxox` is in P hint: lec. 1

9. for clobber position `oxox`, assuming `x` plays first, give the tree of all possible continuations of the game hint: lec. 1

10. course text Chapter 1 Exercise 3 (page 20)

11. course text Chapter 1 Exercise 5

12. course text Chapter 1 Exercise 18

13. Sketch the proof that there exists a winning first-player strategy for $n \times n$ hex. Where in your proof do you use the property (a) that hex has no draws? (b) that if player P has a winning strategy for a position X, then then have a winning strategy for the position X' obtained by adding a P-stone to X?

14. (a) Can the strategy-stealing argument used to prove that $n \times n$ hex is a first-player-win be used to prove that 19×19 go with komi 0 is a first-player-at-least-draw? Explain carefully. (b) Repeat (a) for komi 1.

15. Consider UR-hex (all moves are made uniform-randomly, over all possible moves). (a) Explain why the first player win-probability in $2t \times 2t$ UR-hex is .5. (b) Explain why the first player win-probability in $2t{+}1 \times 2t{+}1$ UR-hex is greater than .5. (c) **challenge problem: publishable** does the limit, as $t \to \infty$, of the win-prob for $2t{+}1 \times 2t{+}1$ UR-hex exist? if yes, is it greater than .5?

16. (a) Let $B(k, n, n)$ be a uniform-random, k-color (same number of stones of each color) $n \times n$ box-off position. Let $s(k, n, n)$ be the probability that $B(k, n, n)$ is solvable.

    (a) prove that $s(2, 2, 2) = 2/3$.

    (b) find $s(2, 2, 4)$.

    (c) **(challenge problem.)** find a polytime algorithm for solving 2-color boxoff.

    (d) **(challenge problem.)** give bounds on $s(5, 12, 15)$. this might be a fun course project.

17. Let M be a multiset of positive integers whose xor-sum $s(M)$ is non-zero. Let j be the non-negative integer that is the xor-sum of $s(M)$ and M.

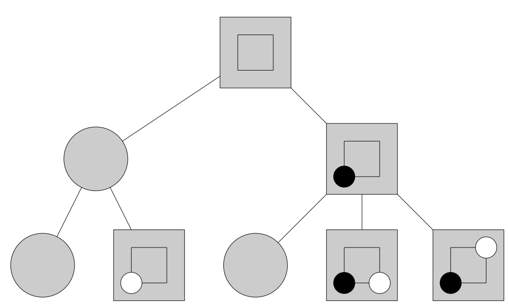
    (a) Find j when $M = \{5, 5, 6, 13, 24, 30\}$.

    (b) Prove that $j < s(M)$.

    (c) Explain why M is winning. Use (b).

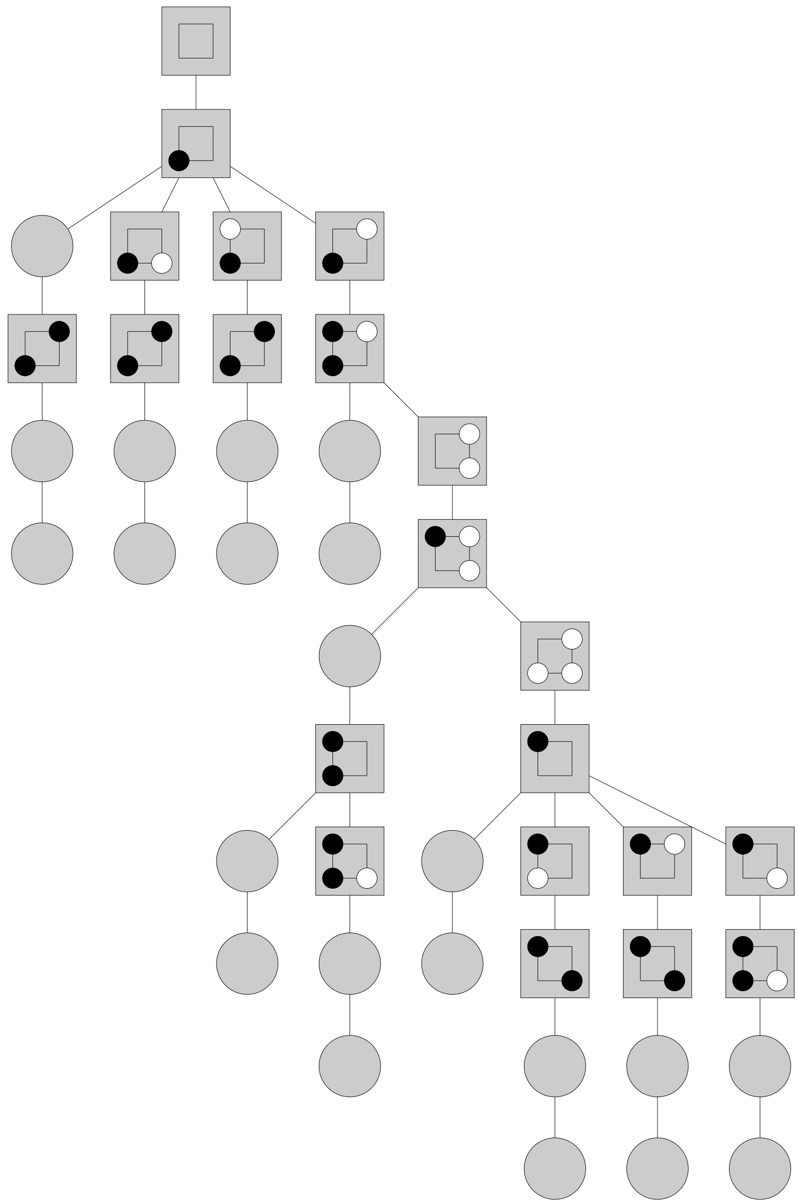
    (d) Sketch a proof of Bouton's theorem.

18. Below are the top 3 levels of the game tree (tree of all continuations) of 2x2 go, with symmetry pruning. Each pass move is shown as an empty circle. Draw the next two levels of the game tree (you can prune for symmetry).

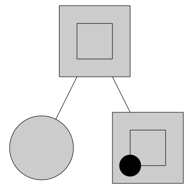


19. Here is a 2x2 go first-player (Black) strategy that shows how Black can win by at least 1 point. In this tree, each node with Black to play shows Black's strategy move, and each node with White to play shows all White moves (with some strategy pruning).

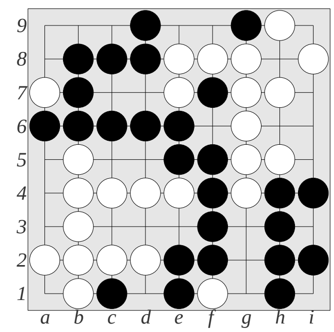
    (a) After 1.B[a1] 2.W[b2] 3.B[a2] 4.W[b1] 5.B[a2] 6.W[pass] 7.B[a1], the only White responses shown are 8.W[pass] and 8.W[b1]: why is 8.W[b2] not shown after these 7 moves?



(b) Draw a 2x2 go second-player (White) strategy that shows how White can lose by at most 1. You can prune symmetric subtrees, so the top 2 levels of your strategy tree will look like this:



20. For a go position and a color, a *block* is a maximal connected set of stones of that color. Answer these questions for this position, from Figure 3.1 in *Mathematical Go: Chilling Gets the Last Point* by Berlekamp and Wolfe. (i) Give the number of black blocks, white blocks, black stones, white stones, black territory, white territory and the current score (e.g. Black by x or White by y or tied). (ii) Assume White now makes a non-pass move, and then both players pass. What is a best move for White? Explain briefly. (iii) From the position, assume White makes some number of non-pass moves and that Black passes after each and then White passes. What is the best score that White can achieve? Explain briefly.



21. In game dag form, give a winning second-player strategy for clobber `oxoxox`. Assume `x` plays first. (The root `oxoxox` has five children.)