1

Group members: Owen Randall and Mahya Jamshidian. I understand that any discussion of this assignment outside my group members is plagiarism. I have not used any resources outside the class notes and textbook.

2

a) \(*2 = \{*0, *1\}, \*3 = \{*0, *1, *2\} \)

b) Using the minimum exclusion principle (MEX), we can see the minimum nimber not in the set \{*0, *1, *2, *5, *11\} is \(*3\). Therefore: \{*0, *1, *2, *5, *11\} = \(*3\).

c) Representing this game in canonical form and using MEX to simplify: \{*0, *1, *2\} = \{*0, *1, *0\} = \{*0, *1\} = \(*2\)

3

\begin{align*}
a) \*3 & \rightarrow 011 \\
\oplus & \oplus \\
\*4 & \rightarrow 100 \\
\oplus & \oplus \\
\*5 & \rightarrow 101 \\
\oplus & \oplus \\
010 & \rightarrow *2 \\
\end{align*}

\begin{align*}
b) \*7 & \rightarrow 0111 \\
\oplus & \oplus \\
\*9 & \rightarrow 1001 \\
\oplus & \oplus \\
\*14 & \rightarrow 1110 \\
\oplus & \oplus \\
\*6 & \rightarrow 0110 \\
\end{align*}

\begin{align*}
c) \*19 & \rightarrow 010011 \\
\oplus & \oplus \\
\*28 & \rightarrow 101100 \\
\oplus & \oplus \\
\*33 & \rightarrow 100001 \\
\end{align*}

\begin{align*}
0110 & \rightarrow \*6 \\
001011 & \rightarrow *11
\end{align*}

a) Take 2 stones from the first heap
b) Take 6 stones from the first heap (there are several correct moves here)
b) Take 5 stones from the third heap
a) \* 1 → 001
\* 4 → 100
\* 5 → 101

b) \* 1 → 0001
\* 9 → 1001
\* 14 → 1110

\* 37 → 100101
\* 23 → 010111

\* 000 → \* 0
\* 0000 → \* 0

4

a)

\[2 \times 3 \text{ Chomp} + 2 \times 4 \text{ Chop} + \* 5\]
\[= \* 4 + (\* 1 + \* 3) + \* 5 = \* 3\]

If we look at the nimber sum as in figure 1, we can see that for example a winning move could be to change Chop 2 \times 4 to 2 \times 1 with nimber \* 1.

\[
\begin{array}{c|c|c}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{array}
\]

Figure 1: Question 4, Part a, nimber sum.

b)

\[\text{Pick Up}(4) + 5 \times 3 \text{ Chop} + \* 7\]
\[= \* 1 + (\* 4 + \* 2) + \* 5\]
\[= \* 0\]

Based on the theorem, this position is \* 0, so it is a losing position. Thus, there is no winning move.

c)

\[\text{Pick Up}(11) + 18 \times 24 \text{ Chop} + \* 20\]
\[= \* 2 + (\* 17 + \* 23) + \* 20\]
\[= \* 2 + (\* 1 + \* 16) + (\* 1 + \* 2 + \* 4 + \* 16) + \* 4 + \* 16\]
\[= \* 16\]

If we look at the nimber sum as in figure 2, we can see that, for example, a winning move could be to change \* 20 to \* 4.

\[
\begin{array}{c|c|c|c|c}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{array}
\]

Figure 2: Question 4, Part c, nimber sum.
5

\[ a + b \text{ is odd if and only if } a \oplus b \text{ is odd.} \]
\[ \implies (a + b \text{ is odd}): \text{W.L.O.G assume that } a \text{ is odd.} \]
Then, \( b \) has to be even.
Note how the xor of these two has to be odd given than the right-most bit is \( 1 \oplus 0 = 1 \).

\[ \iff (a \oplus b \text{ is odd}): \text{W.L.O.G assume that } a \text{ has the left most bit equal to } 1, \text{ meaning that } a \text{ is odd.} \]
Then, \( b \)'s right most bit has to be 0, making \( b \) even.
Note how the sum of \( a \) and \( b \) has to be odd.
Then we know that \( a + b - a \oplus b \) is even.

For any bit at location \( i \) in \( a \) and \( b \), we have \( a_i + b_i \geq a_i \oplus b_i \).
1. \( a_i = b_i \), then \( a_i + b_i = a_i \oplus b_i = 0 \).
2. \( a_i \neq b_i \) and \( b_i \), then \( a_i + b_i = a_i \oplus b_i = 1 \).
Then all the bits in \( a \oplus b \) are equal to those in \( a + b \).
Note how the left most bit of \( a + b \) is at least equal to the left most bit of \( a \oplus b \).
Then, \( a + b \geq a \oplus b \).

6

Let \( F \) represent a full cell and \( E \) an empty one.
Note how in a row, consecutive empty cells can be replaced by a single empty cell.

\( k = 0: \)
Board: \( |E| = *0 \), since it is a \( P \)-position.

\( k = 1: \)
Board: \( |F| = |E| = \{*0\} = *1 \).

\( k = 2: \)
Board: \( |F|F| = \{|F|,|E|\} = \{*0,*1\} = *2 \).

\( k = 3: \)
Board: \( |F|E|F| = \{|F|\} = \{*1\} = *0 \).

\( k = 4: \)
Board: \( |F|E|F|F| = \{|F|F|,|F|E|F|,|F|E|F|,|F|E|F|\} = \{*2,*0,*1\} = *3 \).
Board: \( |F|E|F|F|F| = \{|F|F|F|,|F|E|F|F|,|F|E|F|F|,|F|E|F|F|\} = \{*2,*0,*3,*3\} = *1 \).

\( k = 5: \)
Board: \( |F|E|F|F|F| = \{|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|\} = \{*3,*3,*1,*0\} = *2 \).
Board: \( |F|E|F|F|F|F| = \{|F|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|\} = \{*2,*0,*3,*3\} = *0 \).
Board: \( |F|E|F|F|F|F| = \{|F|F|F|F|,|F|E|F|F|F|F|,|F|E|F|F|F|F|,|F|E|F|F|F|\} = \{*1,*2,*0,*3,*3\} = *4 \).

\( k = 6: \)
Board: \( |F|F|F|F|F|F| = \{|F|F|F|F|,|F|E|F|F|F|F|,|F|E|F|F|F|F|,|F|E|F|F|F|F|\} = \{*4,*0,*1,*2,*0\} = *3 \).
Board: \( |F|E|F|F|F|F| = \{|F|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|,|F|E|F|F|F|\} = \{*1,*2,*0,*3,*1\} = *0 \).
Board: \( |F|F|F|F|E|F| = \{|F|F|F|F|,|F|E|F|E|F|,|F|E|F|F|\} = \{3,1,0\} = *2 \).
Board: \( |F|F|F|F|F| = \{|F|F|F|F|,|F|E|F|F|,|F|E|F|F|,|F|E|F|F|\} = *1 \).

Following the same step, the final number value for each \( k \) is:

- \( k = 0: *0 \)
• \(k = 1: *1\)
• \(k = 2: *2\)
• \(k = 3: *3\)
• \(k = 4: *1\)
• \(k = 5: *4\)
• \(k = 6: *3\)
• \(k = 7: *2\)
• \(k = 8: *1\)

7

a) 
\[
\begin{align*}
[1|1] &= \{\} = *0 \\
[1|2] &= \{[1|1]\} = \{*0\} = *1 \\
[1|3] &= \{[1|2]\} = \{*1\} = *0 \\
[2|2] &= \{[1|1]\} = \{*0\} = *1 \\
[1|4] &= \{[1|3],[2|2]\} = \{*0,*1\} = *2 \\
[2|3] &= \{[1|1],[1|2]\} = \{*0,*1\} = *2 \\
[4|5] &= \{[1|3],[2|2],[1|4],[2|3]\} = \{*0,*1,*2,*2\} = \{*0,*1,*2\} = *3
\end{align*}
\]

b) 
\[
\begin{align*}
*3 &\rightarrow 11 & *2 &\rightarrow 10 \\
\oplus & & \oplus \\
*2 &\rightarrow 10 & *2 &\rightarrow 10 \\
= & & = \\
01 &\rightarrow *1 & 00 &\rightarrow *0
\end{align*}
\]

Since \([1|4] = *2\) and \([2|3] = *2\), moving to either of these positions are winning moves. There are no other winning moves.

8

a) 
\[
\begin{align*}
&1 & 2 & 1 & 2 & 1 & 2 \\
&0 & 3 & 0 & 3 & 0 & 1 \\
&0 & 3 & 0 & 1 & 0 & 3 \\
&0 & 3 & 0 & 1 & 0 & 3 \\
&0 & 1 & 0 & 1 & 0 & 1 \\
&0 & 1 & 0 & 1 & 0 & 1
\end{align*}
\]

b) 
For position \((i, j)\), nimber is:

- \(*0\) if \(i, j\) are both odd.
- \(*2\) if \(i, j\) are both even.
- \(*3\) if exactly one of \(i, j\) is odd and it is the larger number.
- \(*1\) if exactly one of \(i, j\) is odd and it is the larger number.