last lecture: using LP to find matrix game value

• our story so far . . .

• classical GT: matrix games

• Von Neumann’s theorem: every matrix game has a value . . .

• today: how to use linear programming to find that value

• lecture assumes you have read and understood text chapter 5.1

• details come out of Appendix B
• example matrix game from Ch. 5

\[
\begin{array}{cc}
C & \text{what happens with pure strat} \\
2 & r1 \text{ vs } c1 \\
-1 & r2 \text{ vs } c1 \\
R & r1 \text{ vs } c2 \\
1 & r2 \text{ vs } c2 \\
2 & r1 \text{ vs } c2 \\
\end{array}
\]

now consider R plays mixed strategy \( x \cdot r1 + y \cdot r2 \),
where \( x, y \) are probabilities (non-negative rationals, \( x + y = 1 \))

\[
\begin{align*}
\text{case 1)} & \quad x \cdot r1 + y \cdot r2 \text{ versus } c1 \\
\text{case 2)} & \quad x \cdot r1 + y \cdot r2 \text{ versus } c2 \\
\text{case 1)} & \quad R \text{ exp. payout } x \cdot 2 + y \cdot 1 \\
\text{case 2)} & \quad R \text{ exp. payout } x \cdot (-1) + y \cdot 2
\end{align*}
\]
• Rose wants to maximize her guaranteed expected payout

• for any mixed Rose-strat \((x, y)\), we assume Colin will play the minimizing pure C-strat

• thus Rose wants to maximize \(\min\{ 2x + y, -x + 2y \} \)

• Rose wants to

\[
\text{maximize } z \text{ such that } \begin{align*}
z & \leq 2x + y \\
z & \leq -x + 2y \\
0 & \leq x, y \leq 1 \\
x + y & = 1
\end{align*}
\]
evaluate this program at https://sagecell.sagemath.org/

```python
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
x, y, z = v['x'], v['y'], v['z']
p.set_objective(z)
p.add_constraint(z <= 2*x + y)
p.add_constraint(z <= -x + 2*y)
p.add_constraint(x + y == 1)
p.add_constraint(x >= 0)
p.add_constraint(y >= 0)
p.solve()
p.get_values(z, x, y)
```

you should get this output

```
[1.25, 0.25, 0.75]
```
• this tells us that R is guaranteed expected payout 1.25 when she plays row 1 with probability .25 and row 2 with probability .75

• check this: her expected payout against pure strat column 1? column 2?

• can we verify that this is her maximum guaranteed expected payout?

• yes: Von Neumann’s theorem, which tells us that there will be a mixed strategy for Colin with guaranteed expected payout (owing to Rose) at most 1.25

• let’s use the same method as above to find an optimizing mixed strategy for Colin
• Colin wants to minimize his guaranteed expected payout

• for any mixed Colin-strat \((s,t)\) we assume Rose will play the maximizing pure R-strat

• Colin wants to minimize \(\max\{ 2s - t, s + 2t \} \)

• reformulate this as a maximization problem (for SageMath)

• Colin wants to maximize \(\min\{ -2s + t, -s - 2t \} \)

maximize \(z\) such that

\[
\begin{align*}
z &\leq -2s + t \\
z &\leq -s - 2t \\
0 &\leq s, t \leq 1 \\
s + t & = 1
\end{align*}
\]
• evaluate this program at https://sagecell.sagemath.org/

```python
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
s, t, z = v['s'], v['t'], v['z']
p.set_objective(z)
p.add_constraint(z <= -2*s + t)
p.add_constraint(z <= -s - 2*t)
p.add_constraint(s + t == 1)
p.add_constraint(s >= 0)
p.add_constraint(t >= 0)
p.solve()
p.get_values(z,s,t)
```

• you should get this output

`[-1.25, 0.75, 0.25]`
this tells us that C is guaranteed expected payout $-1.25$ when he plays col 1 with prob .75 and col 2 with prob .25

check this: his expected payout against pure strat row 1? row 2?

can we verify that this is his guaranteed expected payout?

yes, because R has a guaranteed expected payout exactly the negative of this amount

we have found, and verified, that value for this game is 1.25

– in expected value, by following her mixed (.25, .75) strategy, R is guaranteed to win at least this amount against any pure col strat

– in expected value, by following his mixed (.75, .25) strategy, C is guaranteed to lose at most this amount against any pure row strat
• another example

\[
\begin{array}{ccc}
C \\
1 & 2 & 1 \\
R & 1 & 0 & 2 \\
3 & 1 & 0
\end{array}
\]
• R wants to maximize her guaranteed expected payout

• for any mixed R-strat (a,b,c), assume C plays the minimizing pure C-strat

• R wants to maximize \( \min\{ a + b + 3c, 2a + c, a + 2b\} \)

• R wants to maximize \( z \) such that

\[
\begin{align*}
  z &\leq a + b + 3c \\
  z &\leq 2a + c \\
  z &\leq a + 2b \\
  0 &\leq a, b, c \leq 1 \\
  a + b + c &= 1
\end{align*}
\]
• use sagemath to find
  – value of this matrix game
  – a minimax strategy for R
  – a minimax strategy for C