RECALL

- $\forall$ IMPARTIAL GAME $G \equiv$ SOME NIMBER $*t$ ~~(ALSO WRITTEN $*t$)~~

- HOW TO FIND THIS NIMBER WHEN

$$G = *a_1 + *a_2 + \dots + *a_k \ ?$$

ANS: USE XOR RULE: THM 3.10

$$G \equiv *(a_1 \oplus a_2 \oplus \dots \oplus a_k)$$

E.G. $G = *1 + *2 + *6 \equiv \ ?$

$$G \equiv *(1 \oplus 2 \oplus 6) = *5$$

$$\begin{array}{r} 001 \\ 010 \\ \oplus\ 110 \\ \hline 101 \end{array}$$

☺

- HOW TO FIND THIS NIMBER WHEN

$$G = \underbrace{\{*b_1, *b_2, \dots, *b_k\}}_{\text{OPTIONS}} \ ?$$

ANS: USE MEX RULE

E.G. (A) EXPRESS $G = *1 + *2 + *6$ AS $G = \{*b_1, \dots, *b_k\}$

ANS CAN MOVE TO THESE NIM POSITIONS

$$\begin{array}{ccccccccc} 026, & 116, & 016, & 125, & 124, & 123, & 122, & 112, & 012 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ *4 & *6 & *7 & *6 & *7 & *0 & *1 & *2 & *3 \end{array}$$

MEX

$$G \equiv \{*0, *1, *2, *3, *4, *6, *7\} \equiv *5$$

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NOW WE CONSIDER **PARTIZAN GAMES**

- IS THERE A SIMPLE ADD'N RULE LIKE FOR NIMBERS?

  ~~NO, BUT~~ ...

- IS THERE A SIMPLIFYING RULE BASED ON OPTIONS?

  ~~YES, BUT~~ ...

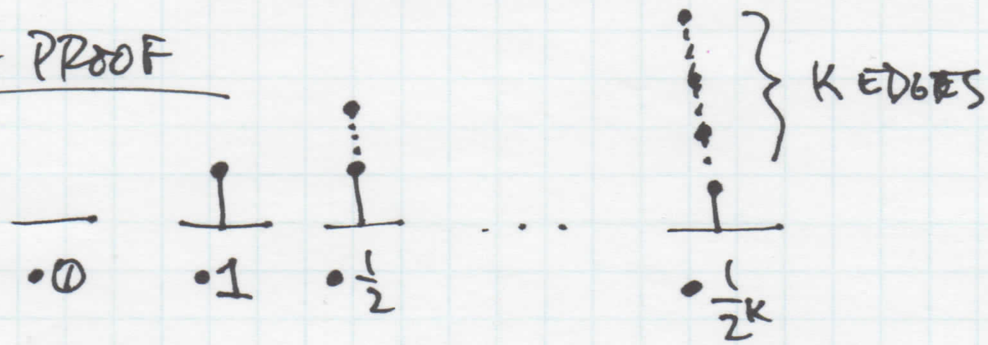
- WHAT ABOUT **DYADIC RATIONALS**?

  E.G. **HACKENBUSH**

CLAIM DYADIC RATIONAL $\Leftrightarrow$ ~~IFF~~ HACKENBUSH

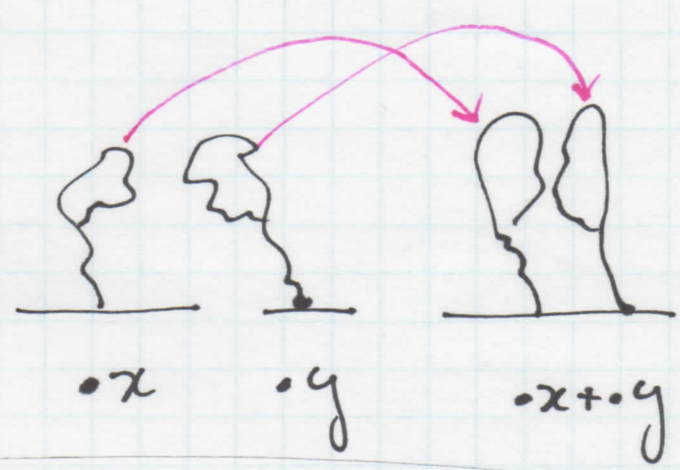
SKETCH OF PROOF

$\Rightarrow$

- 

- NEGATION ✓

- ADDITION ✓



$\Leftarrow$ PROVE

- REMOVING L EDGE FROM HACK x LEAVES HACK x_L AND $x_L \leq x$
- " R " AND $x_R \geq x$
- ~~ASSUME $G = \{\cdot a_1, \cdot a_2 \ldots, \cdot a_t \mid \cdot b_1, \cdot b_2, \ldots, \cdot b_u\}$~~

DYRAT SIMPLICITY THM

- ASSUME $G = \{ \bullet a_1, \bullet a_2, \ldots, \bullet a_t \mid \bullet b_1, \bullet b_2, \ldots \bullet b_u \}$

DYRAT

AND $a_j \leq b_k \quad \forall_{j,k}$.

THEN $G \equiv \{ \bullet \max\{a_j\} \mid \bullet \min\{b_k\} \}$

$= \{ \bullet a_{t+1} \mid \bullet b_{u+1} \} \equiv \bullet c$

WHERE c IS THE OLDEST DYRAT IN THE INTERVAL ~~(b_{u+1}, a_{t+1})~~ (a_{t+1}, b_{u+1})

IF $a_{t+1} = b_{u+1}$ THEN $c = a_{t+1}$

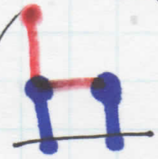
E.G. ASSUME $G = \{ \bullet \frac{3}{4}, \bullet \frac{7}{8}, \bullet \frac{15}{16} \mid \bullet \frac{1}{4}, \bullet \frac{3}{8}, \bullet \frac{5}{8} \}$

THEN $G \equiv \{ \bullet \frac{15}{16} \mid \bullet \frac{1}{4} \}$

ASSUME $H = \{ \bullet \frac{1}{4}, \bullet \frac{3}{8}, \bullet \frac{5}{8} \mid \bullet \frac{3}{4}, \bullet \frac{7}{8}, \bullet \frac{15}{16} \}$

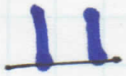
THEN $H = \{ \bullet \frac{5}{8} \mid \bullet \frac{3}{4} \} \equiv \bullet \frac{9}{16}$

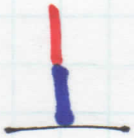
EXAMPLE

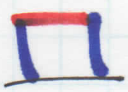
~~[scribbled]~~ WHAT DYRAT IS HACK()?

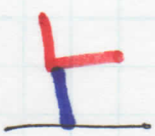
 $\equiv \{\,|\,\} \equiv \cdot 0$

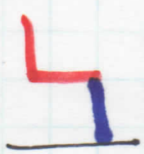
 $\equiv \{\cdot 0\,|\,\} \equiv \cdot 1$

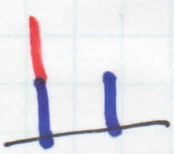
 $\equiv \{\cdot 1\,|\,\} \equiv \cdot 2$

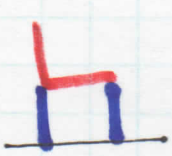
 $\equiv \{\cdot 0 \,|\, \cdot 1\} \equiv \cdot \tfrac{1}{2}$

 $\equiv \{\cdot \tfrac{1}{2} \,|\, \cdot 2\} \equiv \cdot 1$

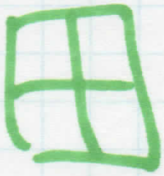
 $\equiv \{\cdot 0 \,|\, \cdot \tfrac{1}{2}\} \equiv \cdot \tfrac{1}{4}$

 $\equiv \{\cdot 0 \,|\, \cdot \tfrac{1}{2}, \cdot \cancel{\;\;}\} \equiv \{\cdot 0 \,|\, \cdot \tfrac{1}{4}\} \equiv \cdot \tfrac{1}{4}$

 $\equiv \{\cdot 1, \cdot \tfrac{1}{2} \,|\, \cdot 2\} \equiv \{\cdot 1 \,|\, \cdot 2\} \equiv \cdot \tfrac{3}{2}$

 $\equiv \{\cdot \tfrac{1}{4}, \,|\, \cdot \tfrac{3}{2}, \cdot 1\} \equiv \{\cdot \tfrac{1}{4} \,|\, \cdot 1\} \equiv \cdot \tfrac{1}{2}$

ARE THERE PARTIZAN GAMES THAT ARE NOT DYRAT?

DOMINEERING  $\equiv \left\{ \cdot 1 \;\middle|\; \cdot -1 \right\}$

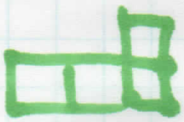
CALLED ± 1

SOME DOM. ARE DYRAT

 $\equiv \cdot 0$

 $\equiv \cdot -1$

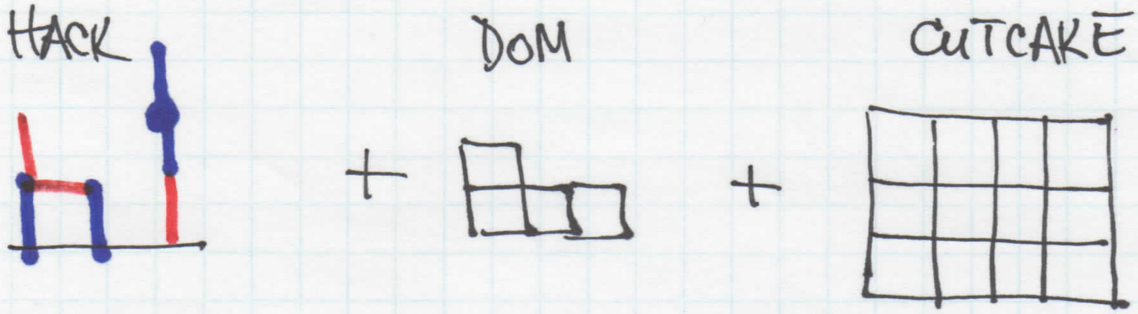
 $\equiv \cdot 1$

 $\equiv \left\{ \cdot -1 \;\middle|\; \cdot 0, \cdot 1 \right\}$

$\equiv \left\{ \cdot -1 \;\middle|\; \cdot 0 \right\}$

$\equiv \cdot -\frac{1}{2}$

WHAT GAME IS THIS ?



$$\cdot\tfrac{1}{2} + \cdot -\tfrac{1}{4} \quad + \quad \cdot -\tfrac{1}{2} \quad + \quad \cdot 1 \quad \equiv \cdot\tfrac{3}{4}$$

NOTE (NOT IN TEXT)

THERE IS A POWERFUL SIMPL. THM. FOR PARTIZAN GAMES USING DOMINATION + REVERSING.

E.G. $$G = \left\{ \cdot\tfrac{1}{2}, \cdot 1, a_3, a_4 \;\middle|\; \cdot\tfrac{1}{2}, \cdot\tfrac{3}{4}, b_3, b_4, b_5 \right\}$$

$$\equiv \left\{ \cdot 1, a_3, a_4 \;\middle|\; \cdot\tfrac{1}{2}, b_3, b_4, b_5 \right\}$$