Each student submits their own assignment. Discuss this assignment only within your group. You can post questions asking for question clarification on the eclass discussion forum, but do not post answers or hints or solution suggestions there or anywhere else.

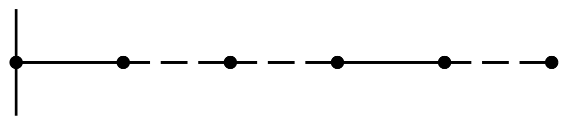
1. List all members of your discussion group, including yourself. Acknowledge that you understand that discussion of this assignment with anyone outside your group is considered plagiarism. Acknowledge any resources (except for class notes or text) you accessed when working on this assignment.

2. Chapter 4 Exercise 4 (p 83)

3. Chapter 4 Exercise 5

hackenbush string formulas

Below, drawn on its side, is a hackenbush string H. From the ground, its edges are respectively L, R, R, L, R. We can represent this string with binary vector `(1 0 0 1 0)`: each 1 is an L-edge, each 0 is an R-edge.



Thea's Lemma. Define binary vectors $b = (b_1, \ldots, b_{t-1}, b_t)$ and $b' = (b_1, \ldots, b_{t-1})$ with $b_1 = 1$, $b_2 = 0$, and $t \geq 2$.

Case 0. If $b_t = 0$ then $\text{hack}(b) \equiv \text{hack}(b') - \bullet\ 1/2^{t-1}$.

Case 1. if $b_t = 1$ then $\text{hack}(b) \equiv \text{hack}(b') + \bullet\ 1/2^{t-1}$.

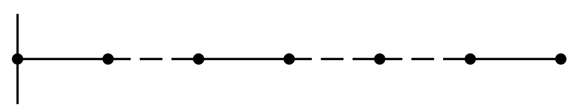
Example. We proved in a lecture that hack(1 0) $\equiv \bullet\, 1/2$, so by the lemma

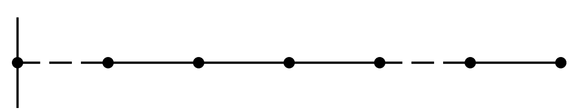
hack(1 0 0) $\equiv \bullet\, 1/2 - \bullet\, 1/4 \equiv \bullet\, 1/4$, and

hack(1 0 0 1) $\equiv \bullet\, 1/4 + \bullet\, 1/8 \equiv \bullet\, 3/8$, and

hack(1 0 0 1 0) $\equiv \bullet\, 3/8 - \bullet\, 1/16 \equiv \bullet\, 5/16$.

4. Using the lemma, for each hackenbush string below, find an equivalent dyadic rational. Show your work.





5. Using the lemma, answer Chapter 4 Exercise 11 (p 85). Show your work. You do not need to prove your answer.

Elwyn's Lemma. Define binary vector $b = (b_1, \ldots, b_{t-1}, b_t)$ as before: $b_1 = 1$, $b_2 = 0$, and $t \geq 2$. Then hack(b) $\equiv \bullet\, x$, where x is the binary fraction $(0.b_3 \ldots b_t\ 1)$.

Examples. hack(1 0) $\equiv \bullet\, 0.1 = \bullet\, 1/2$.

hack(1 0 0) $\equiv \bullet\, 0.01 = \bullet\, 1/4$, and

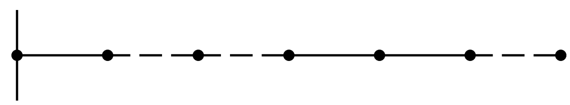
hack(1 0 0 1) $\equiv \bullet\, 0.011 = \bullet\, 3/8$, and

hack(1 0 0 1 0) $\equiv \bullet\, 0.0101 = \bullet\, 5/16$.

6. Use Elwyn's lemma to answer question 4 again. Show your work.

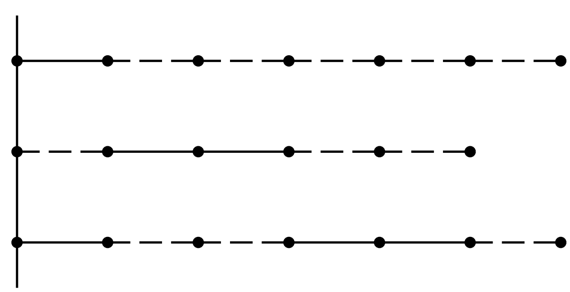
proving the hackenbush string formulas

The two lemmas give us the formulas. We now sketch of proof by induction on t of the lemmas. In the sketch, we use the hackenbush string K below.



For K, $b = (1\ 0\ 0\ 1\ 1\ 0)$, $t = 6$, and $b' = (1\ 0\ 0\ 1\ 1)$. Assume both lemmas hold for all shorter hackenbush strings. By induction on E's lemma, hack(b') $\equiv \bullet\, 0.0111 = \bullet\, 7/16$. We want to show (WTS) that both lemmas hold for K. For this example, we only care about case 0 of Thea's lemma, since the last edge of K is an R-edge.

For T's lemma, we want to show that hack(b) $\equiv$ hack(b') $-\bullet\, 1/2^{t-1} =$ hack(b') $-\bullet\, 1/32$. For E's lemma, we want to show that hack(b) $\equiv \bullet\, 0.01101$. This will follow once we have proved T's lemma for K, since $0.01101 + 1/32 = 0.01101 + 0.00001 = 0.0111$. So let's show that T's lemma holds for K. Adding $\bullet\, 1/32$ to both sides of the equivalence, WTS hack(b) + $\bullet\, 1/32 \equiv$ hack(b'). It suffices to show that the game (shown below) $\quad J = \bullet\, 1/32\ +\ -$hack($b'$) + hack($b$) $\quad$ is a P-position.



7. Explain in your own words why it suffices to show this.

Now let's prove that J is a P-position. Assume L plays first. We want to show that R wins. There are three cases: L plays first on $\bullet\, 1/32$, on $-\mathrm{hack}(b')$, or on $\mathrm{hack}(b)$.

Case 2: L plays first on $-\mathrm{hack}(b')$. By induction, $-\mathrm{hack}(b') \equiv \bullet\, - 0.0111$. L can play on the lower L-edge, replacing $-\mathrm{hack}(b')$ with $\mathrm{hack}((0)) \equiv \bullet\, -1$, leaving the game $\bullet\, 1/32 + \bullet\, -1 + \mathrm{hack}(b)$; or on the upper L-edge, replacing $-\mathrm{hack}(b')$ with $\mathrm{hack}((0\ 1)) \equiv \bullet\, -1/2$, leaving the game $\bullet\, 1/32 + \bullet\, -1/2 + \mathrm{hack}(b)$. L prefers to leave larger numbers, so of these two options L prefers the latter.
But then R can play on the last edge of $\mathrm{hack}(b)$, leaving $Z = \bullet\, 1/32 + \bullet\, -1/2 + \mathrm{hack}(b')$. By induction, $\mathrm{hack}(b-) \equiv \bullet\, (0.0111) = \bullet\, 7/16$. so now it is L's turn and the game is equivalent to $\bullet\, 1/32 + \bullet\, -1/2 + \bullet\, 7/16 \equiv \bullet\, 1/32 - 1/2 + 7/16 \equiv \bullet\, -1/32$, so L loses.

8. Give the argument for Case 1 (easy) and Case 3 (a bit more work). That concludes the sketch of the proof!

9. Now that we believe that the lemmas hold, go back to game J. When L plays, a best move for L leaves the largest possible dyadic rational number. When R plays, a best move for R leaves the smallest possibly dyadic rational. Using either of the lemmas, give all best moves for L, and all best moves for R. Explain briefly.

10. Chapter 4 Exercise 16