CGT class notes 2021

this is a brief lecture summary.

I suggest that you also take your own notes

week 10. all numbers

• integers	done
• dyadic rationals	done
• non-dyadic rationals	?
• irrationals	?

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/rationals.pdf

week 9. numbers plus

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/mar23.pdf

week 8. numbers

- warmup

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/wk9.pdf

NUMBERS

DEF'N (CONWAY) • ALL L, R OPTIONS ARE NUMBERS

• ∀ OPTIONS L_x R_y L_x (NOT $\geq$) R_y

EXERCISE: PROVE $0, 1, -1, \frac{1}{2}, -\frac{1}{2}$ ARE NUMBERS

OBSERVE: ~~∀ NUMBER $x = \{x^L \mid x^R\}$, x NOT FUZZY.~~

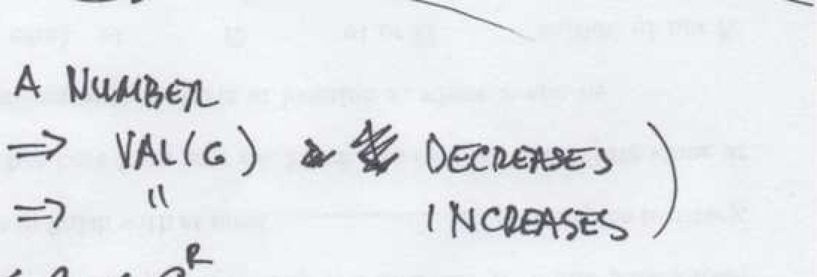
$\Longrightarrow$

G FUZZY. G NOT A NUMBER.

PF. G FUZZY $\Rightarrow \exists\, G^L$, s.t. $G^L \geq 0$.
$\Rightarrow \exists\, G^R$, $G^R \not\leq 0$ $\Big\} \Rightarrow G^L \geq G^R$. ✓

COROLLARY G NUMBER $\Rightarrow$ ONE OF $G < 0$, $G = 0$, OR $G > 0$

RECALL BLUE-RED HACK.



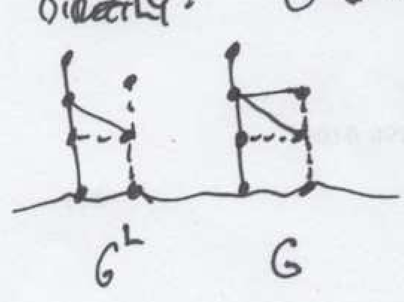
THM: – EACH R-B HACK IS A NUMBER

(ERASE ANY L-EDGE $\Rightarrow$ VAL(G) DECREASES
R-" $\Rightarrow$ " INCREASES)

– ∀ R-B G, $G^L < G < G^R$

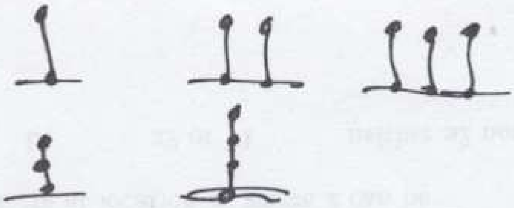
EXERCISES x NUMBER $\Rightarrow x^L < x < x^R$
$\Rightarrow x^{LL} < x^L < x^{LR}$

PROVE DIRECTLY: $G^L < G$



INTEGERS

WHAT NUMBERS THESE?



DEF'N $\forall n \geq 1$

$$n = \{ n-1 \mid \} \qquad -n = \{ \mid -(n-1) \}$$

EXERCISE: PROVE DIRECTLY $1 < 2$

SIMPLEST FORM:

WHAT GAME IS $\{-3 \mid 10\}$?

ANS: 0

$\forall^{\text{INT}}$ $a, b \geq 1$,

$$\{-a \mid b\} = \{-a \mid \} = \{ \mid b\} = 0$$

THM: LET NUMBER $x = \{a_1, a_2 \mid b_1, b_2\}$ WITH $a_1 \geq a_2$, $b_1 \leq b_2$

THEN $x = \{a_1 \mid b_1\}$

THM: $\forall$ INTEGERS a, b ST $-1 \leq a$, $a+1 < b$

$$\{a \mid b\} = a+1$$

EXERCISES

4.4 DYADIC RATIONALS

RATIONAL $\dfrac{\text{INT. } p}{\text{INT. } q \neq 0}$ DYADIC RATIONAL $\dfrac{\text{INT } p}{2^{\text{INT } n}}$

DYADIC RATIONA $\dfrac{\text{INT } p}{2^{\text{INT } n}}$

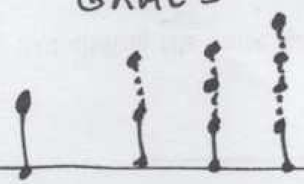
THM:

$1 > \frac{1}{2} > \frac{1}{4} \ldots$

$\frac{2}{2^n} = \frac{1}{2^{n-1}}$

$\forall$ INT $0 \le m \le n$, $\frac{2^m}{2^n} = \frac{1}{2^{n-m}}$

WHAT GAMES ARE THESE?



DEF'N $\forall n \ge 1$

$$\frac{1}{2^n} = \left\{ 0 \;\middle|\; \frac{1}{2^{n-1}} \right\} \qquad -\frac{1}{2^n} = \left\{ -\frac{1}{2^{n-1}} \;\middle|\; 0 \right\}$$

$$\frac{m}{2^n} = \underbrace{\frac{1}{2^n} + \ldots + \frac{1}{2^n}}_{m \text{ TERMS}}$$

$r \ge 0$, $n \ge 1$ INTS,

$$\left\{ \frac{r}{2^{n-1}} \;\middle|\; \frac{r+1}{2^{n-1}} \right\} = \frac{2r+1}{2^n}$$

WHAT GAME IS THIS?

?

$$G = \left\{ \frac{1}{4} \;\middle|\; \frac{1}{2} \right\} = \frac{3}{8}$$

THM: $a, b > 0$, DYAD. RAT., $\Rightarrow \{-a \mid \} = \{-a \mid \} = \{ \mid b\} = 0$

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/mar11.pdf

week 7. writing, theorem, quiz

writing

- links

https://webdocs.cs.ualberta.ca/~hayward/writing.html

- Cormac McCarthy

http://webdocs.cs.ualberta.ca/~hayward/papers/mccarthy-tips.pdf

- George Orwell

https://www.mhpbooks.com/6-writing-rules-from-george-orwell/

CF I: pruned dominated options (review)

if $G = \{S_L \mid S_R\}$ with $S_L = \{L_1, L_2, \ldots\}$ and $L_1 < L_2$

then $G = \{S_L \backslash L_1 \mid S_R\} \; = \; \{L_2, \ldots \mid S_R\}$

if $G = \{S_L \mid S_R\}$ with $S_R = \{R_1, R_2, \ldots\}$ and $R_1 > R_2$

then $G = \{S_L \mid S_R \backslash R_1\} \; = \; \{S_L \mid R_2, \ldots\}$

Proof (idea): use definition of equality.

example: simplify $G = \{0, *, 1, 2 \mid 0, *, 1, 2\}$

$0 < 1 < 2$ and $* < 1$

so $S_L = \{2\}$, $S_R = \{0, *\}$

so $G = \{2 \mid 0, *\}$

CF II: reversible options

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/wk8.pdf

week 6. partial order, canonical form

partial order

- review: does clobber(xxo) equal clobber(xo)?
- notation: $1*$ vs $*1$
- beyond $= 0$: < 0 > 0 $|| \, 0$
- beyond $G = H$: $G < H$ $G > H$ $G \, || \, H$
- transitive?
- recall: some games are numbers: 1, -1, $1/2$, $\ldots$

`http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/wk7.pdf`

canonical form

- games born on day 0 (depth 0) $\{\,|\,\}$ 0
- games born on day 1 (depth 1) $1, -1, *$
- games born on day 2 (depth 2) ?

- canonical form: notationally shortest
- can.form theorem part I: dominated options
- can.form theorem part II: reversible options

`http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/wk8.pdf`

Hearn's review

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/hearn-WW.pdf

some exercises

- what game is clobber(xo)? assume x is Left
- what game is clobber(xxo)? assume x is Left
- simplify $1*$
- 0 is the only game born on day 0.

  1, -1, $*$ are the only games born on day 1.

  Draw the Hasse diagram for these four games.

  For each pair of these games, prove how they are related:

  $A = B$? $\quad A < B$? $\quad A > B$? $\quad A || B$?
- prove $1 > \uparrow$

- clobber(xo) is $\{0 \mid 0\}$, called star, symbol $*$
- clobber(xxo) is $\{0 \mid *\}$, called up, symbol $\uparrow$
- $1* = \{*, 1 \mid 1\} = \{1 \mid 1\}$ since $1 > *$
- $1 > 0$

  why? def'n > 0
- $0 > -1$

  why? def'n $A > B$
- $1 > -1$

  why? def'n $A > B$, $-(-A) = A$
- $1 > *$

  why? def'n $A > B$, $-* = *$, $** = 0$
- $* > -1$

  why? def'n $A > B$, $-(-1) = 1$

  ...or prove $X > Y$ iff $Y < X$
- $1 > -1$

  why? def'n $A > B$, $2 > 0$

  ...or prove $>$ transitive
- $* \mid\mid 0$

  why? def'n $\mid\mid 0$
- $1 > \uparrow$

  def'n $>$, $\quad 1 + \downarrow = \{\downarrow, 1* \mid 1\}$

  L plays to $1* = \{1|1\}$, R forced to 1, L to 0

  R forced to 1, L to 0 $\qquad$ $1 + \downarrow$ is L-psn

week 5. impartial: equal, sum, nimbers, hex

announcements

- proj part 1 marked, if you want a bump in your mark, send me revisions by Friday
- quiz Thursday
- if you would like to help proofread a new book on Hex, let me know
- anything else ?

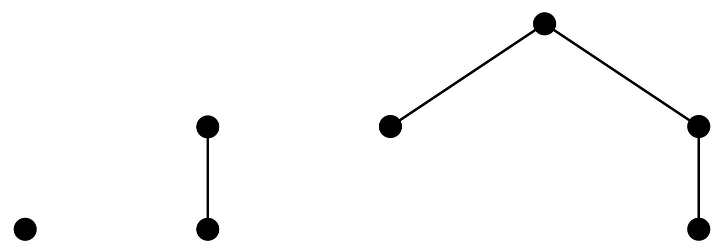
	partial (or impartial)	impartial
not'n	$G = \{\, \mathcal{G}^L \mid \mathcal{G}^{\mathcal{R}} \,\}$	$G = \{\, \mathcal{G}^{P1} \,\}$
oc	L, R, P, N	P, N
game 0	$\{ \mid \}$	$\{\ \}$
$G = 0$	G is P-psn	same
neg(G)	$\{\, -\mathcal{G}^R \mid -\mathcal{G}^L \,\}$	$\{\, -\mathcal{G}^{P1}\} = \{\, \mathcal{G}^{P1}\} = G$
$G + H$	$\{\, \mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R \,\}$	$\{\, \mathcal{G}^{P1} + H, G + \mathcal{H}^{P1} \,\}$
$G = H$	iff $G - H = 0$	iff $G + H = 0$

nimbers

def'n: $*n$ is 1-pile n-stone nim game

$*0, *1, *2, *3, \ldots$ are called nimbers

trees for $*0$ $*1$ $*2$



summing nim games

prove/disprove: $*1 + *2 = *3$

proof:

use def'n of game sum

then prove by induction: $*0 + *n = *n$

then finish proof

$$
\begin{aligned}
*1 + *2 &= \{\, *0 + *2, \quad *1 + *0, \quad *1 + *1 \,\} \\
&= \{\, *2, \quad *1, \quad *0 \,\} \\
&= \{\, *0, \quad *1, \quad *2 \,\} \\
&= *3
\end{aligned}
$$

prove/disprove: $*1 + *3 = *2$

use def'n of game sum

$$*1 + *3 = \{\, *0 + *3,\ \ *1 + *0,\ \ *1 + *1,\ \ *1 + *2 \,\}$$

$$*1 + *3 = \{\, *3,\ \ *1,\ \ *0,\ \ *3 \,\}$$

$$= \{\, *0,\ \ \ *1,\ \ \ *3 \,\}$$

$$= \,?$$

that didn't work ! why ?

prove: $*1 + *3 = *2$

use equality theorem: $G = H$ iff $G - H = 0$ iff (when impartial) $G + H = 0$

$*1 + *3 = *2$ iff $*1 + *3 + *2 = *0$ iff $*1 + *2 + *3 = *0$

so we want to show $*1 + *2 + *3$ is a P-psn.

we already know that, because $\text{nimsum}(1 + 2 + 3) = 0$.

nimber sum theorem

as a comb. game,

a sum of nimbers equals the nimber of

the nimsum of the corresponding integers

$$\begin{aligned} *1 + *2 &= *\text{nimsum}(1 + 2) \\ &= *\text{nimsum}(0\text{b } 01 + 0\text{b } 10) \\ &= *\text{nimsum}(0\text{b } 11) \\ &= *3 \end{aligned}$$

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$$\begin{aligned} *5 + *9 + *15 &= *\text{nimsum}(5 + 9 + 15) \\ &= *\text{nimsum}(0\text{b } 101 + 0\text{b } 1001 + 0\text{b } 1111) \\ &= *\text{nimsum}(0\text{b } 11) \\ &= *3 \end{aligned}$$

hex

- no draw
- extra stones
- 1pw
- 2pw (irregular)
- virtual connections, mustplay

hex talk

`https://webdocs.cs.ualberta.ca/~hayward/talks/twist2.pdf`

hex page on CMPUT 355

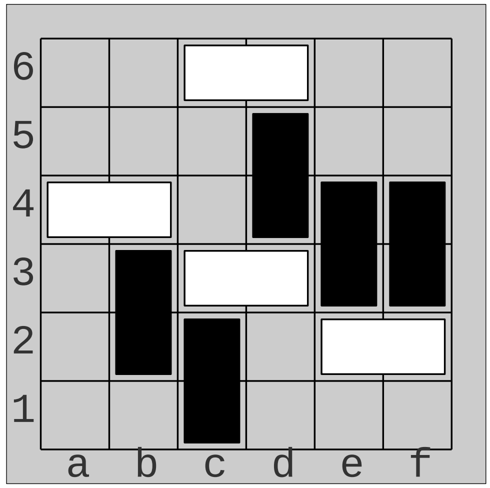
`https://webdocs.cs.ualberta.ca/~hayward/355/jem/hex.html`

hex-the-full-story via ualberta ccid

`https://www.taylorfrancis.com/books/hex-inside-ryan-hayward-bjarne-toft/10.1201/9780429031960`

week 4. dom, equal, iso, sum, chomp, hack

recall domineering



- R   right, white, horizontal
- L   left, black, vertical
- for position above
  - assume R-to-play, who wins ?
  - assume L-to-play, who wins ?
  - outcome class   P, N, L, R ?
- how to answer above questions easily?

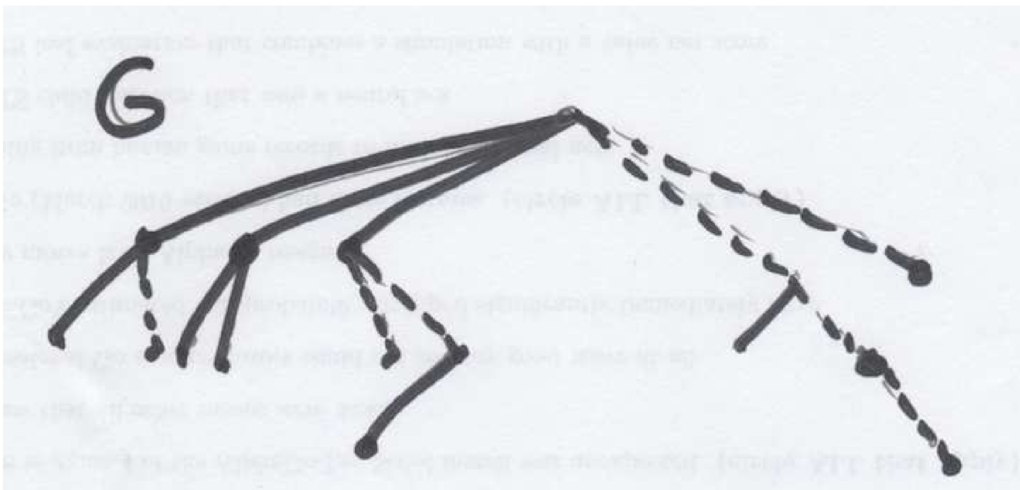
- answers
  - R
  - L
  - N

equals 0

- 0 is the game $\{ \mid \}$
- we care about outcome class
- we care about sums (decomposing a game into independent subgames)
- mot'n: 0 is useless in any sum
- def'n: $G = 0$ iff G is a P-position (2nd-player win)

CG tree

- given position is root
- each left option is left child
- each right option is right child



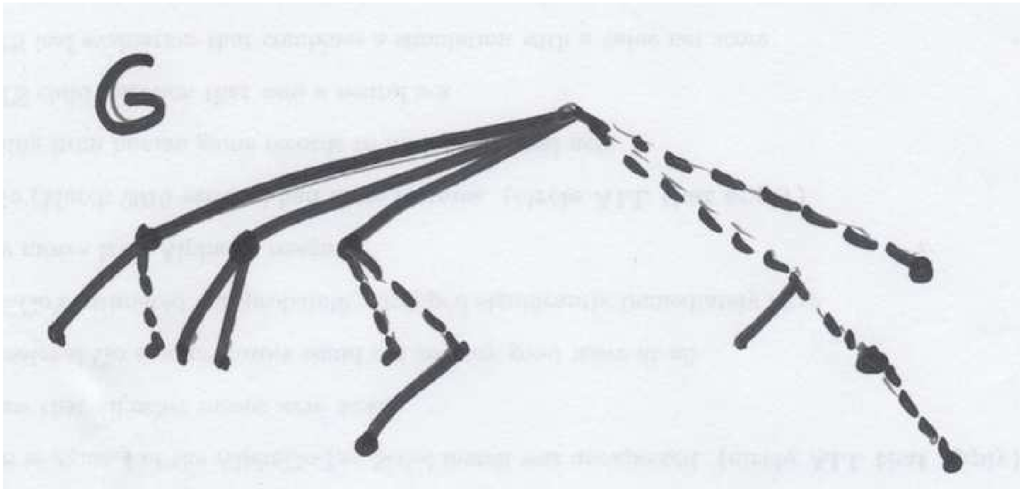
- how does this differ from
  game tree (of not-nec. comb. game) ?

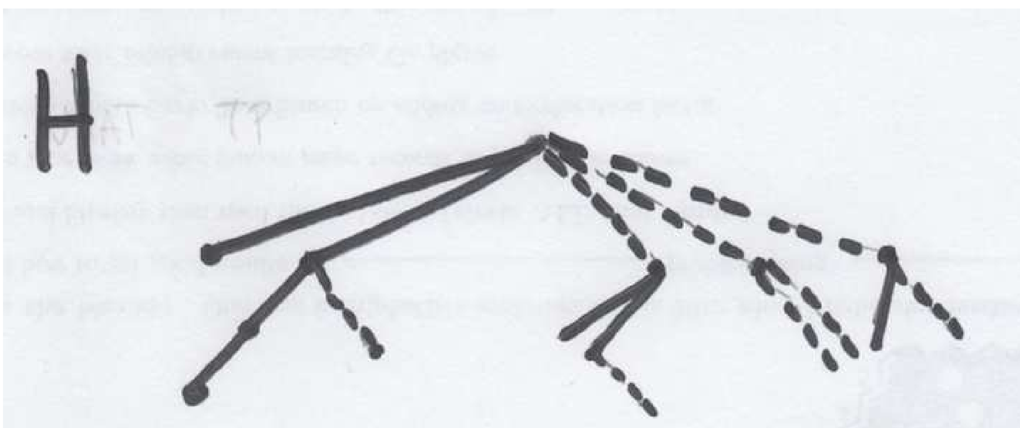
CG isomorphism

- G, H isomorphic iff
  their CG trees are isomorphic

CG negation

- how are G and H related?





- G + H = 0 proof: tweedledee tweedledum
- G = −H notation
- for $G =$ $\{\ \mathcal{G}^{\mathcal{L}}\ \mid\ \mathcal{G}^{\mathcal{R}}\ \}$

  define $-G$ as $\{-\mathcal{G}^{\mathcal{R}} \mid -\mathcal{G}^{\mathcal{L}}\}$

- example: say $G = \{A, B \mid C\}$, then
  $-G =$

CG equality

- oc(G)   outcome-class(G)
- mot'n

  G equals H iff they behave the same in any CG sum

- def'n

  $G = H$ iff

  for all CG X, $\text{oc}(G + X) = \text{oc}(H + X)$

- theorem.

  $G = H$ iff $G - H = 0$

  i.e. $G = H$ iff $G + -H$ is a P-psn

chomp

Fred Schuh 1952, David Gale 1974, Martin Gardner 1973

https://en.wikipedia.org/wiki/Chomp

https://www.win.tue.nl/~aeb/games/chomp.html

hackenbush

Conway 1970

`https://www.youtube.com/watch?v=DrtMWZbh1so`

`http://geometer.org/hackenbush/index.html`

`http://geometer.org/mathcircles/hackenbush.pdf`

Elwyn Berlekamp Hackenbush part 1: inequalities
`https://www.youtube.com/watch?v=omh4t8gZZcE`

exercises

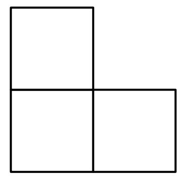
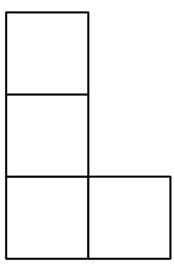
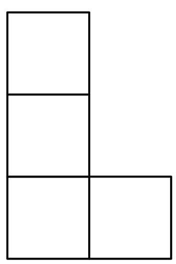
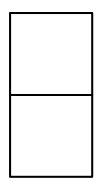
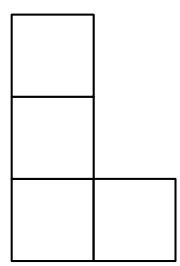
0 defined as $\{ \mid \}$

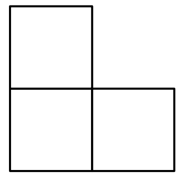
1 defined as $\{ 0 \mid \}$

-1 defined as $\{ \mid 0 \}$

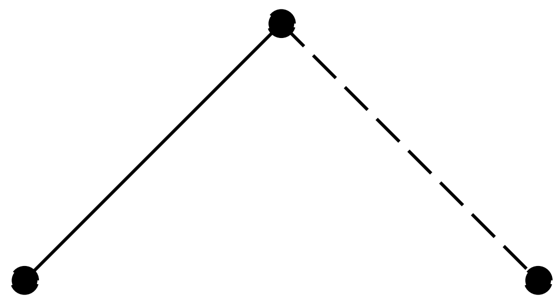
$*$ defined as $\{ 0 \mid 0 \}$

for each question, answer and prove

1. does this domineering game equal 0? 
2. for any game X, is $X + 0$ isomorphic to X?
3. for any games A and B,
   is $A + B$ isomorphic to $B + A$?
4. does $1 - 1 = 0$?
5. does  +  =  ?
6. find simplest CG tree whose game equals 

domineering game D 

D's CG tree



D in game notation $\{\, 0 \mid 0 \,\}$

is D a P-position?

no, it's the game $*$, an N-position

$D \neq 0$

is $X + 0$ isomorphic to X?

yes, argue by induction on depth of CG tree of X

(sketch of proof)

left options of $X + 0$ are the same as the left options of X, because there is no left option in the subgame 0

similarly, right options of $X+0$ are the same as the right options of X

so, yes, $X + 0$ is isomorphic to X

for any games A and B, is $A + B$ isomorphic to $B + A$?

yes (exercise for you :)

define Z as game $1 - 1$

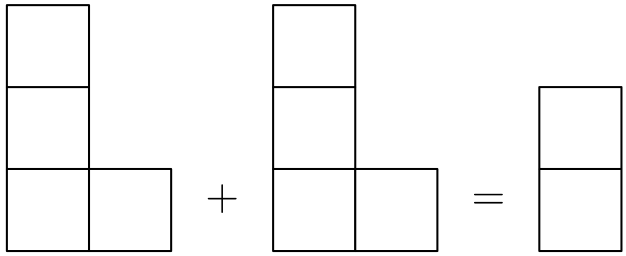
Z in game notation? $\{\, 0 - 1 \mid 1 - 0 \,\}$

iso to $\{\, -1 \mid 1 \,\}$

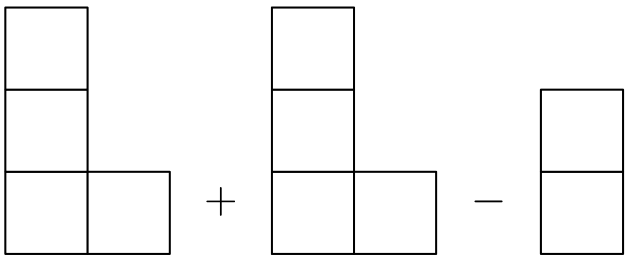
why?

L's only option is to play on subgame 1, move subgame to 0, leaves game $0 - 1$, same game (by definition of our notation) as $0 + \ -1$, iso to $-1 + 0$, iso to -1

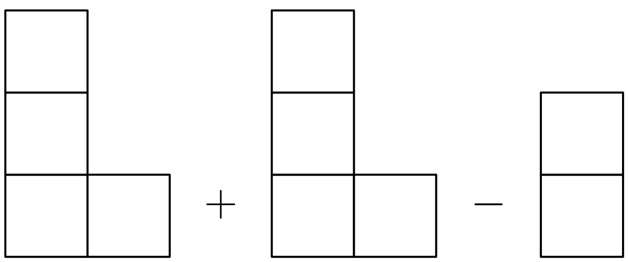
similarly, R's only option is to play on subgame -1, leave game 1



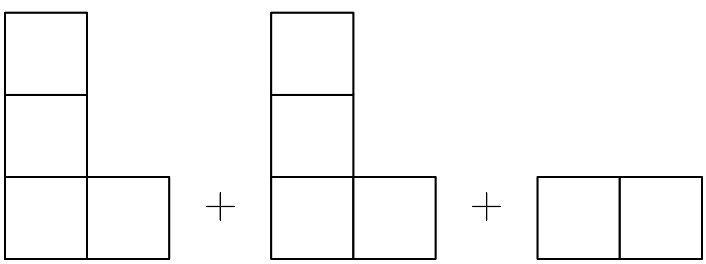
?

iff 

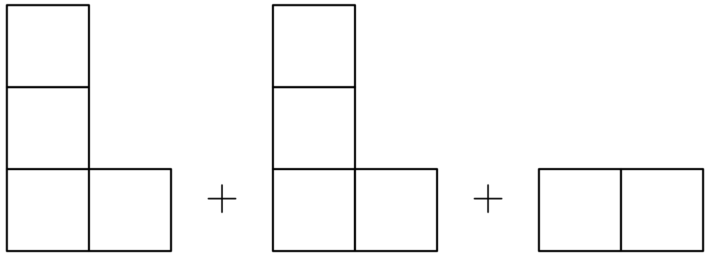
= 0

iff 

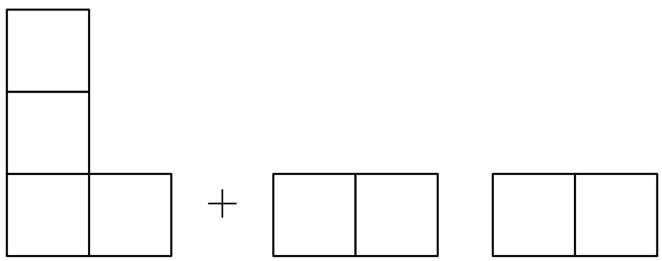
is a P-psn

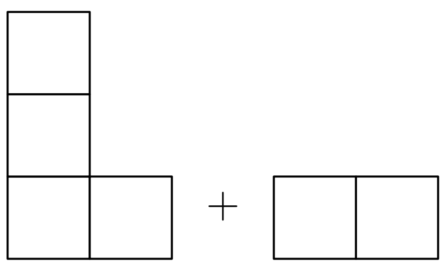
iff 

is a P-psn



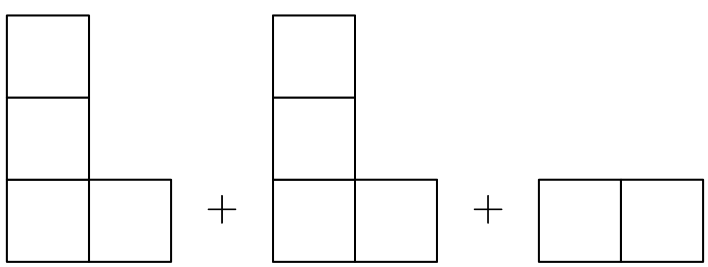
up to symmetry, L can play only to one of these games:

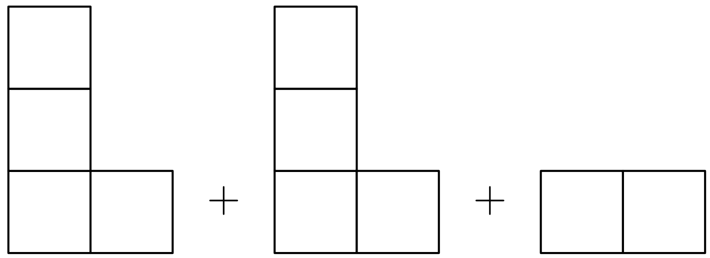




exercise: in each case, show R can now win

conclusion: if L plays first on this game, R wins





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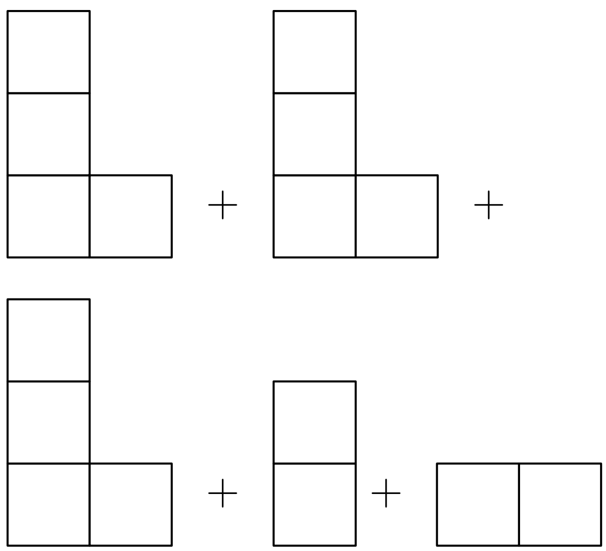
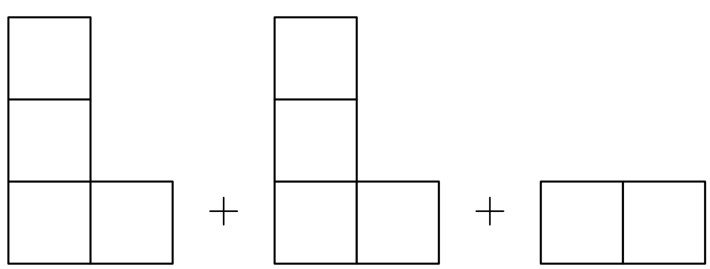
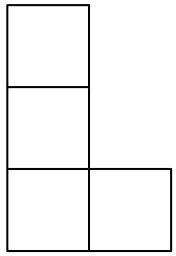
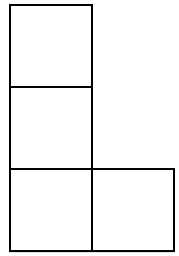
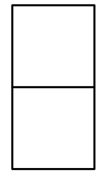
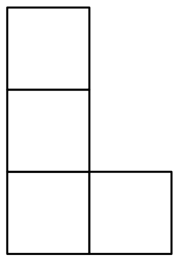
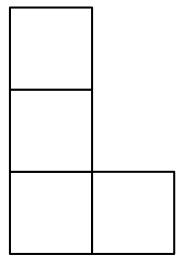


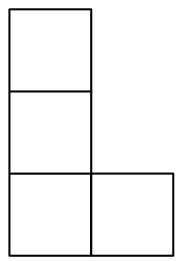
exercise: in each case, show L can now win

conclusion: this game is a P-psn   woohoo!

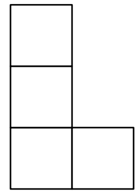


we have seen  +  = 

in other words,  +  = 1

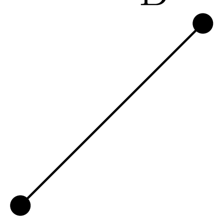
so what should we call  ?

we call this domineering game 1/2

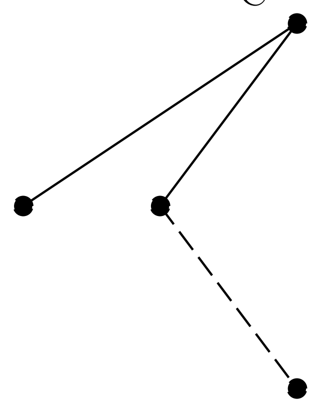
which is CG tree of  ? find each pair of trees whose games are equal. prove/disprove.

A

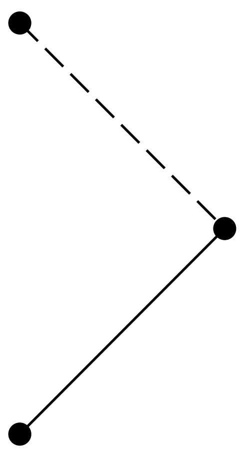
B



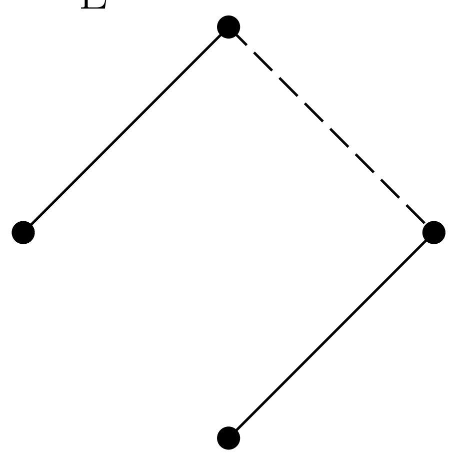
C



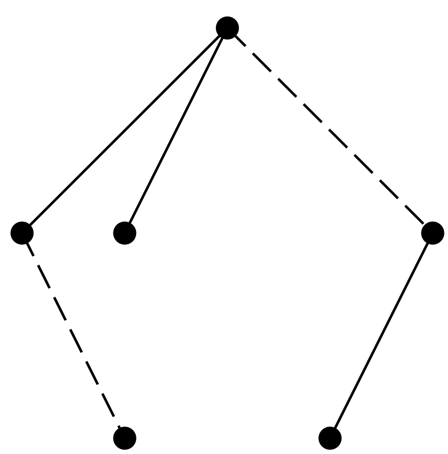
D



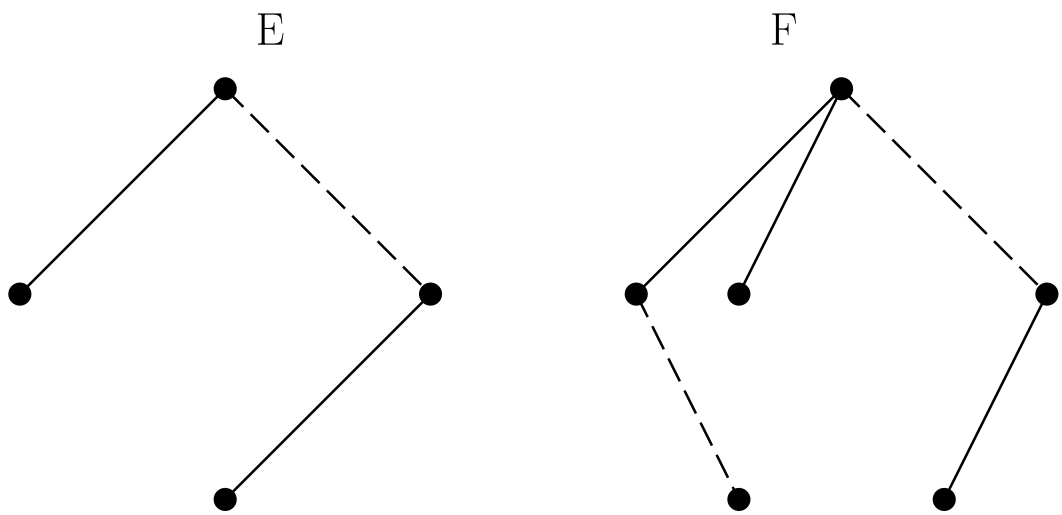
E



F



example. $E = F$. why?



informally:

in F, L's second option is always at least as good for L as L's first option

formally:

prove $E - F$ is a P-psn

week 3. cram, CG notation, FTCG, quiz

cram

- **cram** like domineering, except on each turn, each player can use either tile (horizontal, vertical)
- `https://en.wikipedia.org/wiki/Cram_(game)`
- e.g. who wins cram(2×3)? what is an optimal strategy?

```
                           . . .
                           . . .

        H H .              V . .         . V .
        . . .              V . .         . V .

H H V   H H .   H H .      V H H        V V .
. . V   H H .   . H H      V . .        V V .

                      . . .
                      . . .
                   /    |    \
                   .    . .    . + .
               . . .    . .    .   .
             /   |       |       |
           . .   0      . .     . .
```

notation

game graph means **game directed-acyclic-graph**

domineering game below: game graph?

we care only about outcome: when drawing game graph, prune isomorphic siblings (leave one of each iso pair)

```
    x
x x x      L to x x     R to  x   or    0
                                        x

x x = -1
           L no options   R to 0

x   =  1
x          L to 0         R no options
```

CG game notation $G = \{G^L \mid G^R\}$

alternative to game graph

above domineering game in this notation?

game 0 is $\{ \mid \}$ game 1 is $\{ 0 \mid \}$ game -1 is $\{ \mid 0 \}$

so game above $G = \{ -1 \mid 1, 0 \}$

game sums

define $G + H = \{ G^L + H, G + H^L \mid G^R + H, G + H^R \}$

eg. domineering sum G + H below?

```
G       x    H  x x       A  x x   B  x
     x x x      x x                   x

G + H =  { -1 + H, G + 1  |  1 + H, 0 + H, G + -1 }

      =  { H - 1, G + 1   |   H + 1,   H,   G - 1 }
```

partisan, impartial

comb-game **impartial** if, for every position, the move options are the same (yield the same new positions) for both players

comb-game **partisan** if not impartial

e.g. nim impartial, Hex partisan Go? cram?

FTCG

assume normal play

- P-psn: 2nd-player (prev) has win-strat
- N-psn: 1st-player (next) has win-strat
- L-psn: L has win-strat, regardless of who plays next
- R-psn: R has win-strat, regardless of who plays next
- e.g. nim(3,2,1) ? nim(3,3,3)? nim(3,3,4)?
- e.g. domineering(2×3)? e.g. domineering(3×2)?
- **FTCG:** every finite game is exactly one of P-position, N-position, L-position, R-position
- proof? argue similar to Z's theorem (or use Z's theorem)

week 2. strategy as dag, Z's thm, CG intro

strategy as dag

earlier we defined strategy as a function. we can define strategies using the game tree, or directed acyclic graph if we want to save space and so acknowledge that a fixed state can be reached in more than one way, of all possible continuations of the game.

e.g. transpositions in tic-tac-toe, hex, any game where history is irrelevant

strategy for a player A (with opponent B) from state $S_0 = (P, X)$, a strategy is a subdag $T = T(P, A)$ of the game dag with root S_0, such that for each node $S = (p, X)$ in T,

- if $X = A$ then the only child of S is $f(S)$,
- if $X = B$ then, for each legal move m by B from S, the state S_m reached after move m is a child of S.

Notice: in T,

- every node $S = (p, A)$ either is terminal or has exactly one child;
- every node $T = (q, B)$ has one child for each legal move that B can make from q.

recall: *winning strategy* is strategy with minimax value win. defn *drawing strategy, losing strategy* similarly. exercise: prove strategy is drawing iff each leaf is a winning or drawing position and at least one leaf is not winning

examples

for a state S and a player X, we say X *wins* S (resp. draws) whenever X has a winning (resp. drawing) strategy for S, and X *loses* S whenever X has no winning and no drawing strategy for S (i.e. for any X-strategy, at least one leaf is X-losing; equivalently, the opponent has a winning strategy).

is this 1st-player strategy for nim{1,2} winning?

no: there is a leaf where 1st-player loses

```
       {1,2}
      /
   {2}
  / |
{} {1}
    |
   {}
```

is this 2nd-player strategy for nim{1,2} winning?

no: there is a leaf where 2nd-player loses

```
       {1,2}
      /  |  \
   {2} {1,1} {1}
  /      |     \
{}      {1}     {}
         |
         {}
```

for nim{2,3}, who wins? show winning strategy graph

Zermelo's theorem for WLD games

mot'n: chess, who wins? Bouton 1901, Zermelo 1913

assume alternate-turn 2-player win-loss-draw game, e.g. chess. name the two players: **L (left)**, **R (right)**

recall: **L wins** means L has a winning strategy (minimax value win: wins against all possible opponent strategies)

recall: **L draws** means L has a non-losing strategy (minimax value draw: wins or draws against all possible opponent strategies)

Z's theorem for any WLD AT 2p game G, exactly one of these: L wins, R wins, or both players draw

proof by induction on the depth of T, game dag of G.

depth 0: root is terminal node, so value is L-win-R-lose or L-lose-R-win or draw

depth $n > 0$, assume holds for shorter dags, root $S = (P, X)$, assume WLOG $X = L$, child subdags $T_1 \ldots T_k$

case 1. some T_j is L-win

case 2. every T_j is R-win

case 3. not (1 or 2), i.e. no T_j L-win and some T_j not R-win (so draw).

exercise: finish proof

corollary pruning T yields win-strat for L or win-strat for R or draw-strat for both

CG motivation, go

recall: what we now call CG started in 70s when Conway, watching go players, noticed that endgame often decomposed into independent subgames

go rules

warning: there are different go rulesets. e.g. three largest pro go associations: China, Korea, Japan each have different rulesets. we follow a version of Tromp-Taylor rules, because they are well-defined and easy to implement

`http://tromp.github.io/go.html`

- we allow any rectangular boardsize, e.g. 1×25, 3×4
- we usually do not allow suicide
- *clearing* corresponds to capturing
- unless stated otherwise, we follow TT's *positional superko*. other variants include situational superko.
- TT rules are close to Chinese rules
- Japanese rules are more restrictive, e.g. you lose points if you play in your captured territory
- when humans play, they agree on which stones are captured before scoring: if they don't agree, they keep playing. with TT rules, you just score the final position.

learn some go

https://webdocs.cs.ualberta.ca/~hayward/355/jem/go.html

https://webdocs.cs.ualberta.ca/~hayward/355/jem/go.html#learn

go

go can be played on any $n \times m$ board. we follow Tromp-Taylor no-suicide positional superko rules. B (Black) versus W (White). A move is made by placing a stone on an empty point (intersection of horizontal and vertical line) and then removing each opponent group that is captured. a *group* is a set of stones of the same color that is connected (you can go between any two stones in the group following a path of adjacent stones in the group) and maximal (not a proper subset of any larger connected stone set). the *liberties* of a group are all empty points that touch some stone in that group. to be legal, the group that contains the stone just played must have at least one liberty.

A *legal go position* is an assignment of stones to some points so that, for each player, each group (maximal connected set of stones of that color) has at least one liberty (adjacent empty point).

E.g. legal 2x3 go positions:

```
.*.   .*o   .*o   .*o   **o   .*o   .*o   .o.
...   .*.   o*.   **.   **.   oo.   **.   o.o
```

E.g. illegal 2x3 go positions:

```
o*o   .*o   o*o   o*o   **o   **o   .*o   .o.
.o.   .*o   o*.   *..   **o   oo.   **o   o*o
```

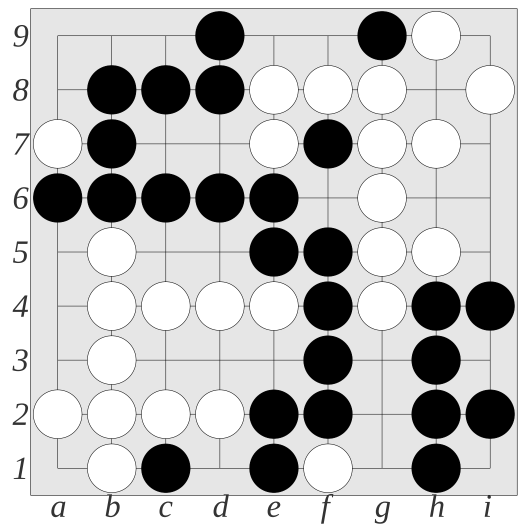
Procedure to determine whether a move is legal:

1. place stone on empty point
2. for each opponent group whose last liberty was just removed, remove all the stones in that group (they have been captured)
3. does the player's group containing the stone just placed have any liberties? if no, return **illegal move** (no liberties)
4. has the new board position occurred previously in the game? if yes, return **illegal move** (positional superko violation)
5. return **legal move**

examples?

cg motivation: go endgames

Figure 3.1, *Mathematical go: Chilling Gets the Last Point*, Berlekamp and Wolfe. White to play. best move? final score?



one difference between chess and go? subgame decomposition

in cg, we want to answer this question: how do we decide which subgame to play in?

notice that the overall game is alt-turn, but that we are interested in (sub)game sums. so, when we consider a game, we think of it as a position with player-to-move unknown

solving hex

- for all board sizes up to 9×9, win/loss values of all empty-board first-moves is known
- for 10×10, win/loss values of 8 empty-board first-moves is known (4 losses using proof by contradiction, 4 wins by computer search)
- 11×11 ?

http://webdocs.cs.ualberta.ca/~hayward/talks/hex.someques.pdf

https://webdocs.cs.ualberta.ca/~hayward/355/jem/hex.html

solving go

- 1×n go solved up to $n = 9$
- for $2 \leq m \leq n$, $m \times n$ go solved up to $m{\times}n$ around 30
- our group has a 3×3 solver
- 6×6 ?

http://erikvanderwerf.tengen.nl/5x5/5x5solved.html

https://webdocs.cs.ualberta.ca/~hayward/355/jem/go-solve.html

week 1: intro, nim

intro

motivation. nim. tree of all continuations. some definitions. how to define strategy. tree of all continuations. nim sum. Bouton's theorem. using Bouton's theorem.

combinatorial game

a *combinatorial game* is two-player, alternate-turn, perfect-information, zero-sum, deterministic, finite. the two players are Left and Right. when colors are black (e.g. go, hex, chess), white Left is bLack, righT is whiTe. a CG is *normal form* if it is win-loss and the losing player is whoever cannot make a move. e.g. nim is normal form. (see `https://en.wikipedia.org/wiki/Combinatorial_game_theory` for another definition of combinatorial game)

motivation

board-game math. nim, hex, go, chess: what math arises when we try to solve these games (play perfectly) on small boards (large board search space intractible).

Conway, Berlekamp, Guy. *On numbers and games. Winning ways for your mathematical plays.* Albert, Nowakowski, Wolfe. *Lessons in Play.* Siegel *Combinatorial Game Theory.* DeVos, Kent. *Game Theory: A Playful Intro*

recreational math. Gardner. research problems: 6x6 go, 10x10 hex.

nim

two players. a *pile* is a non-empty set of stones. a *position* is a (possibly empty) set of piles of stones. on a turn, the player-to-move removes a positive number of stones from some pile. whoever cannot make a move loses and the other player wins.

we can represent a position as a (possibly empty) multiset of positive integers. here is a game between A and B starting from `{3,5,5}` :

```
  {3,5,5}
A {3,1,5}={1,3,5}
B {1,3}
A {1,1}
B {1}
A {}
B loses
```

https://en.wikipedia.org/wiki/Nim

https://webdocs.cs.ualberta.ca/~hayward/355/jem/nim.html

normal play end condition: whoever cannot move is loser

misère end condition: whoever cannot move is winner

N (next) is the player-to-move next. **P (previous)** is the player who is not N, i.e. the player-who-moved previously, if there was a previous move.

who wins nim {0,0}?

N has no legal moves, so loses, so P wins.

who wins nim $\{0{,}1\} = \{1{,}0\}$?

N: she can move to the P-position {0,0}

who wins nim $\{0{,}n\}$ for any positive n? N

who wins nim $\{n{,}n\}$ for any positive n? P

defn: tweedledee-tweedledum strategy a 2nd-player strategy for player B in which B always mirrors A's move. e.g. in 2-pile nim from $\{n{,}n\}$, here is such a strategy for second player B: if A just removed t stones from a pile, remove t stones from the other pile how many moves from nim (1, 3, 3)? $1+3+3=7$

how many *pairwise non-isomorphic* moves from nim (1, 3, 3)?

4: to any of {0,3,3}, {1,2,3}, {1,1,3}, {0,1,3}

how many moves from nim {1, 3, 3}? 4

who wins nim {1, 2, 3}? solve from bottom up

{0, 0, 0}	P-position	no legal moves
{0, 0, 1}	N-position	exists move to P-position {0,0,0}
{0, 0, 2}	N	exists move to P-position
{0, 1, 1}	P	only move is to N-position
{0, 0, 3}	N	exists move to P-position
{0, 1, 2}	N	exists move to P-position
{1, 1, 1}	N	exists move to P-position
{0, 1, 3}	N	exists move to P-position
{0, 2, 2}	P	each move is to N-position
{1, 1, 2}	N	exists move to P-position
{0, 2, 3}	N	exists move to P-position
{1, 1, 3}	N	exists move to P-position
{1, 2, 2}	N	exists move to P-position

{1, 2, 3} ?

nim-sum of multiset of non-neg. integers is xor-sum

e.g. $1 \bigoplus 3 \bigoplus 6 \bigoplus 7 = 3$

```
1                        1
3                      1 1
6                    1 1 0
7                    1 1 1
                     -----
          nim-sum    0 1 1
```

multiset $M = \{6, 13, 13, 24, 30\}$ has nim-sum 0:

```
 6                       1 1 0
13                     1 1 0 1
13                     1 1 0 1
24                   1 1 0 0 0
30                   1 1 1 1 0
                       -----
          nim-sum    0 0 0 0 0
```

if we change (decrease, increase) one integer in M?

nim-sum will no longer be 0. do you see why?

```
 6                       1 1 0
10                     1 0 1 0   <- changed
13                     1 1 0 1
24                   1 1 0 0 0
30                   1 1 1 1 0
                       -----
          nim-sum    0 0 1 1 1   <- no longer 0
```

Bouton's theorem

nimsum of a multiset of positive integers is the columnwise mod-2 sum of the binary representations. **Bouton's theorem**: a nim position is winning if and only if its nimsum is positive.

e.g. nimsum of {3,5,5} is `0b 011` which is 3 which is positive, so Bouton's theorem says that this is position is winning (for the player-to-move), i.e. that the ptm has a winning strategy.

```
3         0 1 1
5         1 0 1
5         1 0 1
nimsum    0 1 1
```

We can use Bouton's theorem to find all winning moves. nim{3,5,5} is winning, so there must be a move that leaves a losing position, i.e. a position with nimsum 0. here are all positions that ptm can move to from nim{3,5,5}: 55 155 255 35 135 235 335 345. compute the nimsum of each. which have nimsum 0?

using Bouton's theorem to find all winning moves

Here is a faster way to use Bouton's theorem to find all winning moves from a nim position:

- in the nimsum array, find the left-most column c whose mod-2 sum is positive (if the nimsum is positive, why must such a column exist?)

- pick any row r with a one in column c. (why must such a row exist?)
- change the 1 at (row r, column c) to 0
- for each location Z = (r,c') with c' ¿ c, set Z to be the mod-2 sum of all other elements of that column
- why does this process leave a position with nimsum 0?
- why does process correspond to a legal move in nim?

e.g. nim{3,5,5}, the only column with mod-2 sum 1 is the middle column, so c is 2 (middle column). the only row with a 1 in that column is column 1 (top), we we set that location to 0. now we set location (row 1, column 3) to the mod-2 sum of the rest of that column which is +=0*. So the only winning move from {3,5,5} is to {5,5}, i.e. from the pile with 3 stones, remove 3 stones.

```
3          1 1    -->     0 0     0
5        1 0 1          1 0 1     5
5        1 0 1          1 0 1     5
nimsum   0 1 0          0 0 0
```

another example: from {3,5,7}, three winning moves

```
3    1 1           1 1   3          1 1  3  -->    1 0   2
5  1 0 1         1 0 1   5   --> 1 0 0  4        1 0 1   5
7  1 1 1   -->   1 1 0   6       1 1 1  7        1 1 1   7
   0 0 1         0 0 0           0 0 0           0 0 0
```

nim-sum theorem (Bouton's theorem) nim state winning iff its nim-sum is positive

- Let M be a multiset of non-negative integers including a positive integer m. Let M' be the multiset obtained from M by replacing m (just one instance) with an integer $0 \leq m' < m$. If nim-sum$(M) = 0$, then nim-sum$(M') \neq 0$.

- Let M be a multiset of non-negative integers such that nim-sum$(M) \neq 0$. Then some m in M can be replaced with a smaller integer m' so that nim-sum$(M') = 0$.

  How to find m and m'? Let j be the position of the leftmost 1 in binary(nim-sum(M)). Let m be any integer in M whose position-j digit in binary(m) is also 1. Let $M - m$ be the multiset obtained by removing (one instance of m) from M. Let $q =$ nim-sum$(M - m)$. Let M' be the multiset obtained from M by replacing (one instance of) m with q. Then nim-sum$(M') = 0$.

example of second part of theorem.

```
 6                  1 1 0
10                1 0 1 0
13                1 1 0 1   <- change
24              1 1 0 0 0
30              1 1 1 1 0
                  -----
        nim-sum 0 0 1 1 1
                    j
```

j is position 3 (counting from right), so m can be 6, 13, or 30. Pick $m = 13$. Then $M - m = \{6, 10, 24, 30\}$, nim-sum$(M - m) =$ `01010_b` $= 10$, $M' = \{6, 10, 10, 24, 30\}$.

```
 6                  1 1 0
10                1 0 1 0
10                1 0 1 0
24              1 1 0 0 0
30              1 1 1 1 0
                  -----
        nim-sum 0 0 0 0 0
```

nim: Bouton's corollary

recall Bouton's theorem:

nimsum-0, all moves to nimsum-pos

nimsum-pos, some move to nimsum-0

corollary nim position winning iff nimsum-pos

proof induction on pile-sum.

tree of all continuations, a.k.a. game tree

when analyzing a game, it can help to consider the *tree T of all possible continuations.*, also called the *game tree*. The root is the initial position. For each node v in the tree, the children of v are all states that the player-to-move can move to from v. e.g. here is the game tree for nim{1,2}:

```
            (1,2)
           /  |  \
        (2) (1,1) (1)
       / |    |     \
     () (1)  (1)     ()
         |    |
        ()   ()
```

tic

like tic-tac-toe, except 1x3 board instead of 3x3, win by getting 2-touching-in-a-row.

assume x plays first: here is part of the game tree (add edges yourself)

```
                  . . .
            /          |    \
  x . .              . x .   . . x  <- has iso. sibl.
  /   \             /   \              so prune this
x o .   x . o    o x .   . x o         node
  |       |        |       |
x o x   x x o    o x x   x x o
draw    x-win    x-win   x-win
```

strategy

state $S = (X, A)$ is position X and player-to-move A.

for $S = (X, A)$, *strategy* is a function $f(s)$ that, for each state $T = (X', A)$ reachable by a sequence of legal moves from X, gives the move that A makes at X'.

a player's strategy is *winning* if it wins against all possible opponent strategies.

a player's strategy is *minimax* if it maximizes the minimum score it achieves over the set of all possible opponent strategies.

e.g. consider a particular state $S = (X, A)$ with B the opponent of A for which B has possible strategies T_1, T_2, T_3 and A has possible strategies S_1, S_2. Assume that, against B's three strategies, S_1 scores result win, win, loss respectively and S_2 scores result draw, win, draw. Then A's minimax strategy is S2, because the min of {win,win,loss} is loss, and the min of {draw,win,draw} is draw, and draw is better for A than loss.

e.g. here is a nim strategy: always remove all stones from a largest pile. notice that this is a winning strategy for nim with 1 pile. can you find a nim position with 2 piles for which this is a winning strategy? losing? can you find a nim position with 3 piles for which this is a winning strategy? losing?

for nim with at most two piles, can you find a winning strategy?

hint: tweedle

for nim{3,5,7}, find a minimax strategy for the first player.

hint: Bouton

http://webdocs.cs.ualberta.ca/~hayward/cgt/asn/21/wk10.pdf