TWIST AND TURN: THE STORY OF HEX

THE CLASSIC 2-PLAYER CONNECTION GAME

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computing   university of alberta

2011 march

1 PRE-

2 EARLY

3 MID

4 RECENT

THANK YOU

- joint: Arneson, Henderson, Toft [Rex]
- also van Rijswijck, Bjänsson, Johanson, Kan
- also UofA GAMES, Schaeffer, Müller, Stewart
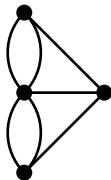- Natural Sciences and Engineering Research Council of Canada

OUTLINE

PRE-
EARLY
MID
RECENT

PREHISTORY
GRAPH THEORY

PREHISTORY

- 15 000 years ago   last ice age ends
- 10 000 years ago   agriculture
  hunt/gather $\dashrightarrow$ reap/sow
  stories around fire $\dashrightarrow$ games around hearth

OUTLINE
PRE-
EARLY
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RECENT

PREHISTORY
GRAPH THEORY

PREHISTORY

- 5 000 years ago   pre-??? Iraq
- 4 000 years ago   pre-go China
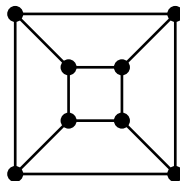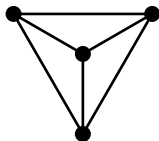- 3 400 years ago   pre-checkers Egypt
- 1 500 years ago   pre-chess India

OUTLINE
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PREHISTORY
GRAPH THEORY

GRAPH THEORY

1736 Euler   Königsberg bridge

OUTLINE
PRE-
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PREHISTORY
GRAPH THEORY

GRAPH THEORY

1736 Euler   Königsberg bridge

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PREHISTORY
GRAPH THEORY

GRAPH THEORY

1750 Euler   polyhedron $V - E + F = 2$

OUTLINE
PRE-
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PREHISTORY
GRAPH THEORY

GRAPH THEORY: 4-COLOR   [TO 1940]

1840 Möbius 5 princes

OUTLINE
PRE-
EARLY
MID
RECENT

PREHISTORY
GRAPH THEORY

GRAPH THEORY: 4-COLOR   [TO 1940]

| 1840 Möbius 5 princes | K5 not planar |

OUTLINE
PRE-
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PREHISTORY
GRAPH THEORY
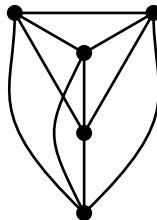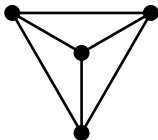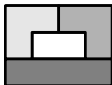
GRAPH THEORY: 4-COLOR [TO 1940]

1852 Guthrie –» brother –» de Morgan

1879 Kempe 'proof'

1890 Heawood: Kempe counterexample, 5 color theorem
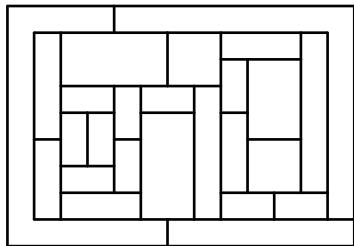
. . .

1940 known for maps with ≤ 35 regions

OUTLINE
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1942 HEIN GAME
1948 NASH GAME
1952 SHANNON COMPUTER
1957,58 GARDNER COLUMNS

HEIN'S GAME

RULES

- 2 players, alternate moves
- win: connect your two sides

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HEIN'S INSPIRATION

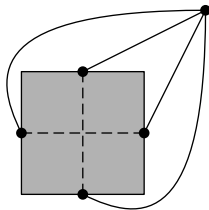## GAME CRITERIA

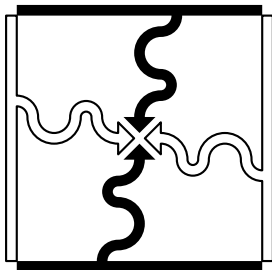- fair   *balanced/equal*
- progressive   *no state repetition*
- final   *finite*
- easy to comprehend   *clear state evaluation*
- strategic
- decisive   *no draw*
- no explicit math'l sol'n   *not easy to solve*

- !!! 2 opposing-side chords of quadrilateral must cross
- no draw ⇒ hexagonal grid

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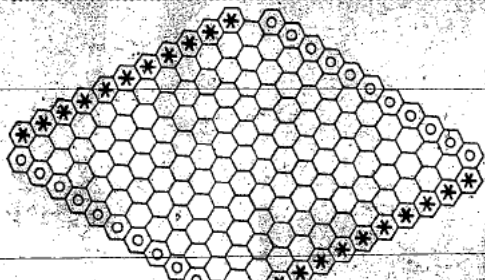
Hein's inspiration

HEIN'S INSPIRATION

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HEIN'S COLUMNS

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HEIN'S COLUMNS

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HEIN'S GRUKS

PROBLEMS

Problems worthy
        of attack
prove their worth
        by hitting back.

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HEIN'S GRUKS

CONSOLATION GROOK

Losing one glove
is certainly painful,
but nothing
        compared to the pain,
of losing one,
throwing away the other,
and finding
        the first one again.

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NASH'S GAME

PRINCETON 1948-9

- Nash new game $\rightarrow$ Gale
- Gale board $\rightarrow$ Fine Hall

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NASH'S GAME

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Nash's game

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NASH'S THEOREM

N-BY-N HEX IS 1ST-PLAYER WIN

- lemma: extra X-cell ok for player X
- lemma: no draws in Hex
- suppose P2 has win strategy S2
- then P1 can move anywhere, forget move, and follow S2
- thus P1 has win strategy, contradiction □

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NASH AT RAND

PROJECT RAND

DOCUMENT

SOME GAMES AND MACHINES FOR PLAYING THEM

John Nash

D-1164

2 February 1952

Assigned to _Math R. Tucker_

OUTLINE
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N-BY-(N+1) HEX IS LARGER-SIDE WIN

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N-BY-(N+1) HEX IS LARGER-SIDE WIN

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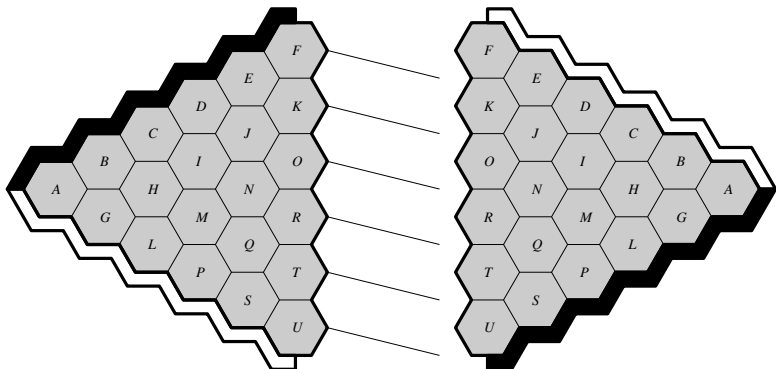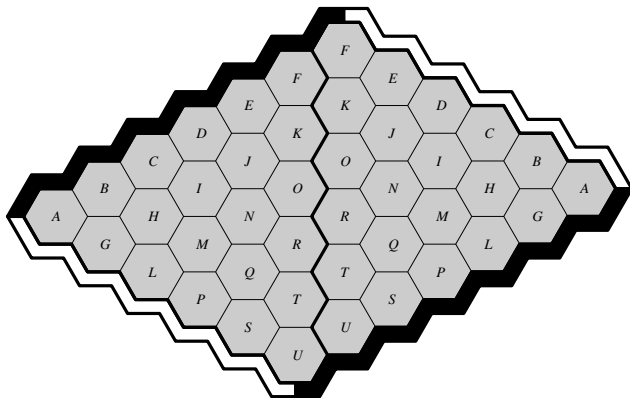
SHANNON'S ANALOG COMPUTERS

1951 BIRD CAGE (A.K.A. GALE, BRIDG-IT) MACHINE

- bird cage: players occupy *edges*
- board ⟷ electrical network
- apply black side-to-side voltage
- black edge ← short/contract edge
- white edge ← cut edge
- computer move: take edge with max voltage drop
- almost always won with first move

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SHANNON'S ANALOG COMPUTERS

1952 HEX MACHINE

- board ⟷ 2-dimensional potential field
- black cell ⟵ positive charge
- white cell ⟵ negative charge
- computer move: certain saddle point
- computer positionally strong, tactically weak

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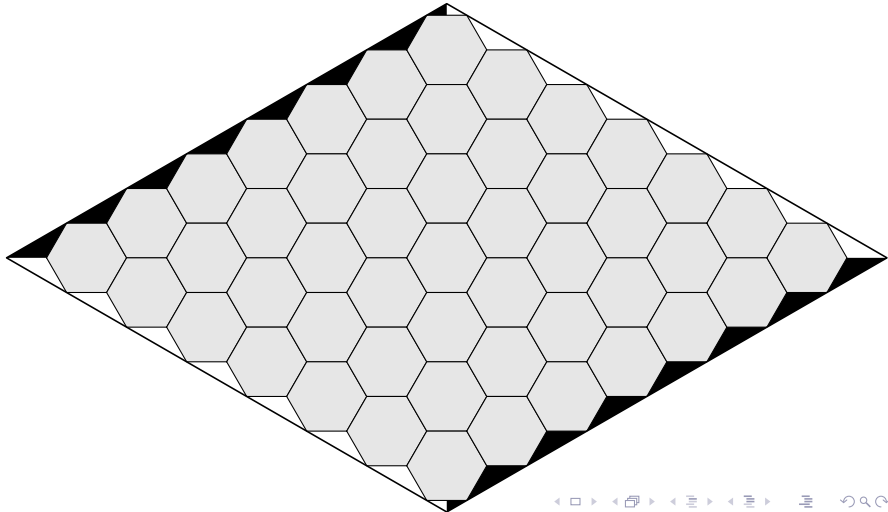1957,58 GARDNER COLUMNS

SHANNON'S ANALOG COMPUTERS

GAG MACHINE

- 7x8 board disguised to look regular
- played pairing strategy, always won

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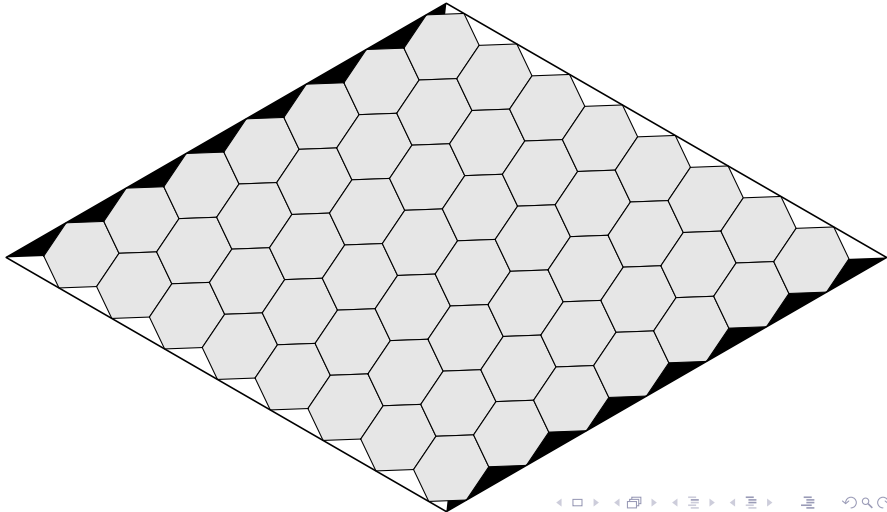1952 SHANNON COMPUTER
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SHANNON'S ANALOG COMPUTERS

SHANNON'S ANALOG COMPUTERS

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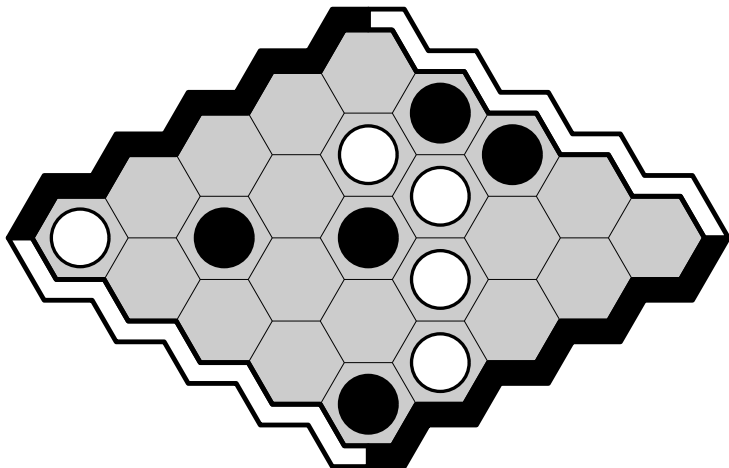
Beyond Hex: Shannon Switching Game

- play on any graph
- two marked vertices
- black move: 'short' any vertex (make nbrs clique)
- white move: 'cut' any vertex (delete)
- black wins iff two marked vertices are shorted (connected)
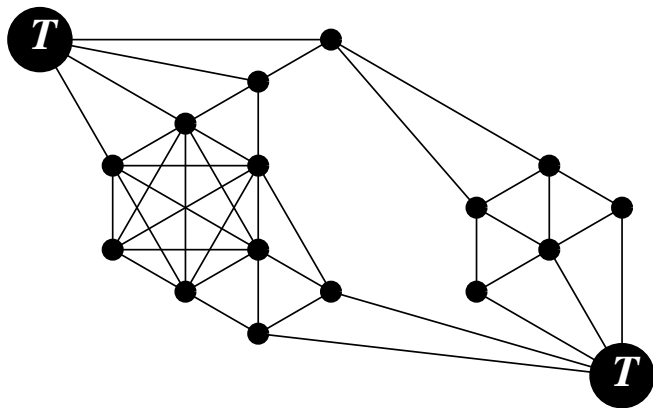
- generalizes Hex

OUTLINE
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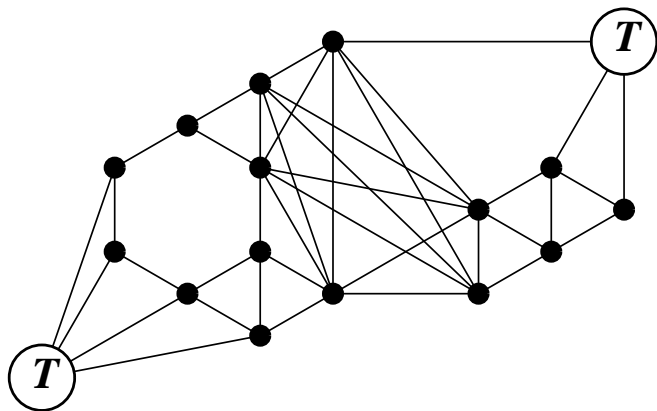**1952 SHANNON COMPUTER**
1957,58 GARDNER COLUMNS

BEYOND HEX: SHANNON SWITCHING GAME

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Beyond Hex: Y

Shannon, Milnor, Schensted & Titus

- connect all 3 sides to win

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BEYOND HEX: Y

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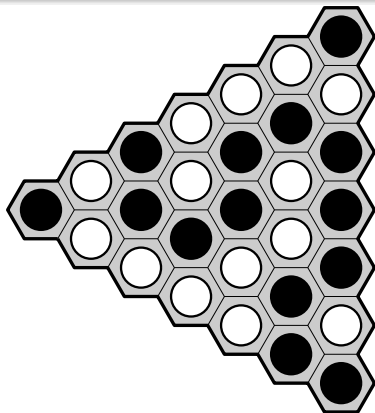
GARDNER'S COLUMNS

1957 JULY SCIENTIFIC AMERICAN MATHEMATICAL GAMES

- concerning the game of Hex, which may be played on the tiles of the bathroom floor

1958 OCT SCIENTIFIC AMERICAN MATHEMATICAL GAMES

- 4 mathematical diversions involving concepts of topology

OUTLINE
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1963 Lehman solution
1969 Beck opening
1975 Schensted & Titus book
1975 Even & Tarjan PSPACE-complete
1977 Berge problems
1979 Gale no-draw proof

Lehman's Bridg-It solution

- poly-time algorithm to find winning move
- maintain 2 edge-disjoint spanning trees
- Bridg-It less fun

Beck opening: n-by-n acute corner loses

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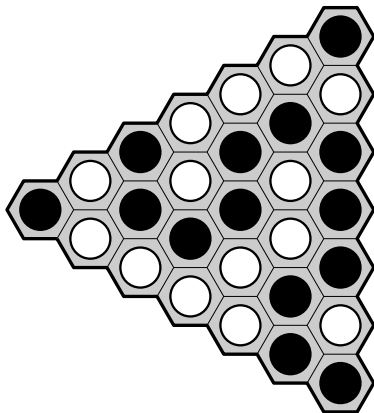1977 Berge problems
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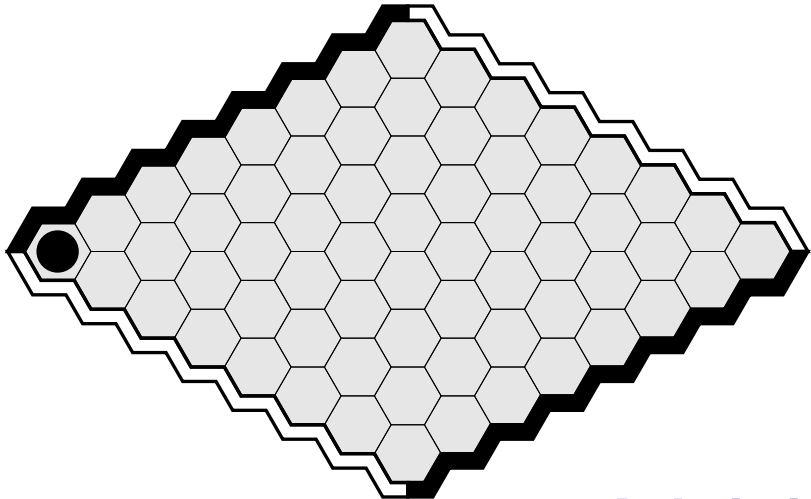
# Schensted & Titus   Mudcrack Y & Poly-Y

PROVERBS

- play the best ... 1st player wins ... handicapping
- two-way stretch ... best offense is a good defense
- be relevant ... double trouble
- you can't see the whole sky through a bamboo tube
- waste not, want not ... shun the worthless triangle
- beware the square ... don't trust the pentagon
- luck is a many-sided region ... breaking mirrors is bad luck
- never try to cut a bamboo joint
- equivalent patterns ... the aim of the game
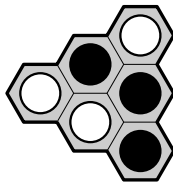
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Schensted's Y reduction: no draws

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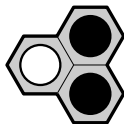1977 BERGE PROBLEMS
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SCHENSTED'S Y REDUCTION: NO DRAWS

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SCHENSTED'S Y REDUCTION: NO DRAWS

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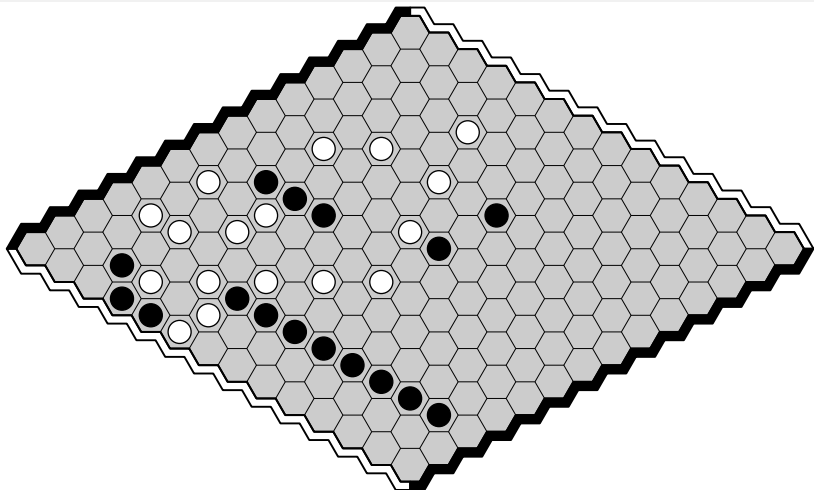1979 GALE NO-DRAW PROOF

HEX PSPACE-COMPLETE

- 1975 Even & Tarjan   Shannon v-switching PSPACE-complete
- 1981 Stefan Reisch   *Hex ist PSPACE-vollstandig*
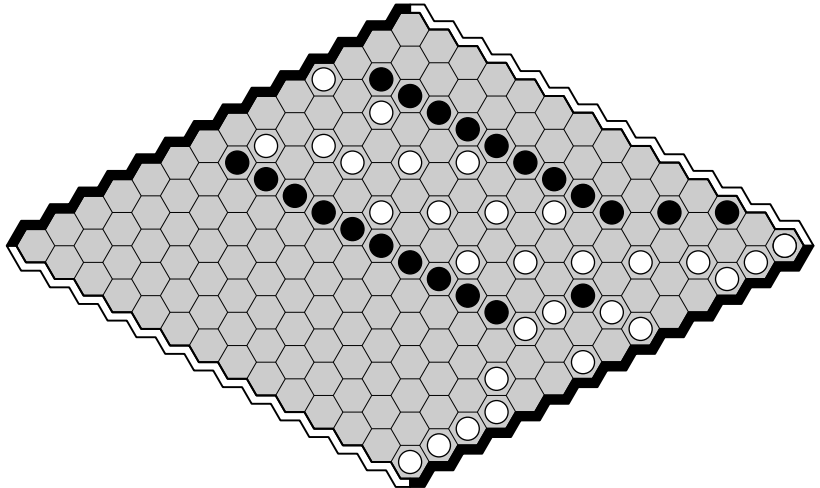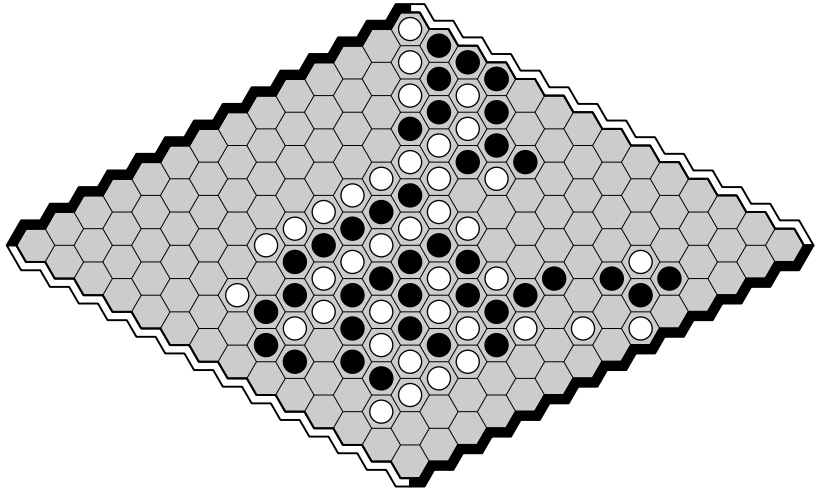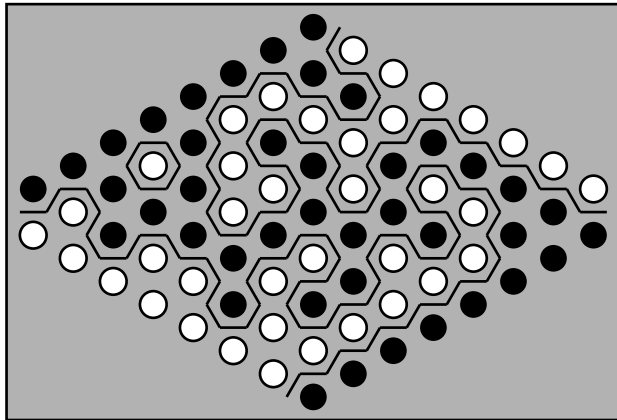- 2000 Clay Math Inst $1 000 000 P vs NP

BERGE L'ART SUBTIL DU HEX

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Berge L'Art Subtil du Hex

BERGE L'ART SUBTIL DU HEX

GALE'S NO-DRAW PROOF

Gale's no-draw proof

OUTLINE
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2000– COMPUTER PLAYERS
SOLVING HEX OPENINGS
REFS
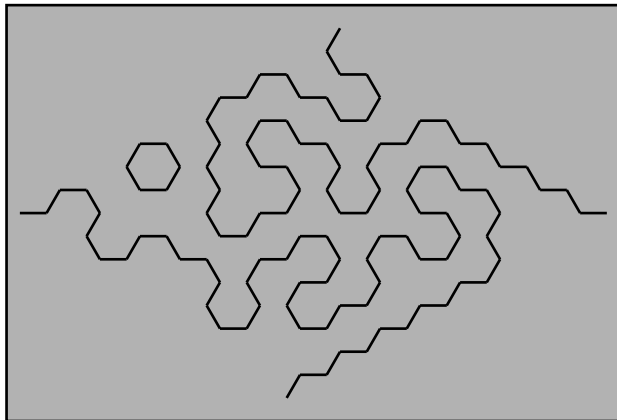
COMPUTER PLAYERS

ICGA CGO HEX TOURNAMENTS

- 2000 London   Hexy Queenbee Killerbee
- 2003 Graz   Six Mongoose
- 2004 Ramat-Gan   Six Mongoose
- 2006 Turin   Six Wolve HexKriger
- 2008 Beijing   Wolve MoHex Six Yopt
- 2009 Pamplona   MoHex Wolve Six Yopt
- 2010 Kanazawa   MoHex Wolve MIMHex Yopt

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COMPUTER PLAYERS

SHANNON TO WOLVE

- Shannon bird-cage eval'n
- Hexy: + limited search, and-or virtual connection algebra
- Six: + vc lazy queue processing (+ vcs through edge)
- Wolve: + inferior cell engine, capture in vc engine, solver

MOHEX

- same ic engine, vc engine, solver as Wolve
- Monte Carlo tree search

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SOLVING HEX OPENINGS

HUMAN PROOFS

- 2001 Yang   17/49 7x7
- 2002 Yang   8x8
- 2003 Yang   9x9
- 2004 Noshita   7x7
- 2005 Noshita   8x8
- 2006 Mishima   8x8

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2000– COMPUTER PLAYERS
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SOLVING HEX OPENINGS

COMPUTER PROOFS

- 1995 Enderton   6x6
- 2000 van Rijswijck   6x6
- 2003 H Björnsson Johanson Kan Po van Rijswijck   all 7x7
- 2007 Rasmussen Maire Hayward   all 7x7
- 2009 Arneson H Henderson   all 8x8
- 2010 A H H   53/31 9x9

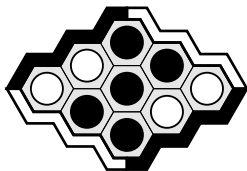OUTLINE
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SOLVING HEX OPENINGS

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2000–ᴄᴏᴍᴘᴜᴛᴇʀ ᴘʟᴀʏᴇʀꜱ
Sᴏʟᴠɪɴɢ Hᴇx Oᴘᴇɴɪɴɢꜱ
ʀᴇꜰꜱ

Sᴏʟᴠɪɴɢ Hᴇx Oᴘᴇɴɪɴɢꜱ

OUTLINE
PRE-
EARLY
MID
RECENT

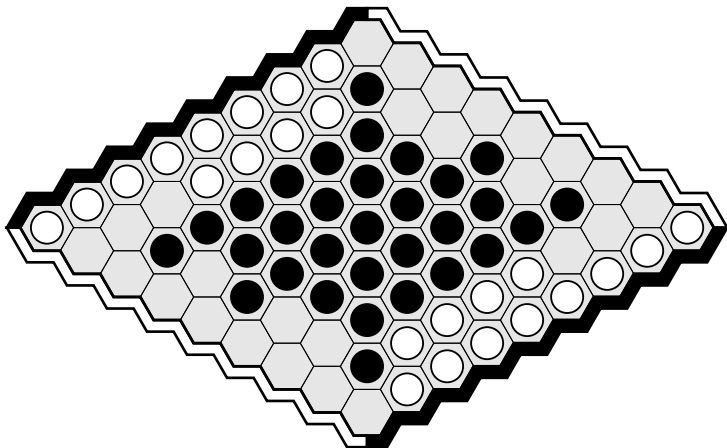2000–   COMPUTER PLAYERS
SOLVING HEX OPENINGS
REFS

READING

- *Hex Strategy . . .*   Browne
- *Connection Games . . .*   Browne
- *Hexaflexagons, Probability, Paradoxes . . .*   Gardner
- *2nd SciAm Bk Math'l Puzzles & Diversions. . .*   Gardner
- *Politiken* columns   Hein
- *Everything You Always Wanted . . . Hex . . .*   Maarup
- *A Beautiful Mind*   Nasar
- *Mudcrack Y and Poly-Y*   Schensted & Titus
- *Four Colors Suffice*   Wilson