rock paper scissors

play 5 games with your neighbour

minimax value of 5-game rps ? -5 (why?)

Rose payoff matrix	Colin action		
Rose action	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

matrix game 2-player, simultaneous-move, 0-sum

0-sum definition:

for each terminal state, P1-score + P2-score = 0

another matrix game: how is this 0-sum ?

Rose payoff matrix	Colin action		
Rose action	a	b	c
a	3	-2	1
b	0	4	-5
c	-1	-2	-3

answer: by definition   whever Rose wins x, Colin wins $-x$

Colin payoff matrix	Colin action			
Rose action	a	b	c	
a	-3	2		fill in the
b				missing
c			3	entries

stochastic algorithm

stochastic defn: with action(s) taken from a probability distribution

e.g. stochastic strategy S: play (rck, ppr, scr) with probabilities (.5, .3, .2)

- assume C plays S: what is R's best replying strategy T ?
- easy question: best 1-choice T ? analysis below

```
rck: expected R-winrate: .5 * 0 + .3 *-1 + .2 * 1  = -.1
ppr: expected R-winrate: .5 * 1 + .3 * 0 + .2 *-1  =  .3    <-- best
scr: expected R-winrate: .5 *-1 + .3 * 1 + .2 * 0  = -.2
```

- harder question: best stochastic T ?

theorem in a matrix game, every fixed stochastic strategy
has a best counter-strategy that is 1-choice

to find a best stochastic response (counter-strategy),
it suffices to consider only 1-choice responses   woo hoo :)

- $m \times n$ matrix gave has only
  m 1-choice strategies (for R)
  n 1-choice strategies (for C)

continue: best R-response to C-strat (rck .5, ppr .3, scr .2) ?

answer

- assume R plays (rock, paper, scissors) with probability (r, p, s)

so $0 \leq r, p, s \leq 1 \qquad r + p + s = 1$

- by theorem, sufficient to consider 1-choice strategies for R
- already seen: strats (1,0,0), (0,1,0), (0,0,1) R-exp-payoffs $-.1$, $.3$, $-.2$,

an R-best stochastic response is 1-choice strat (0, 1, 0)

ttt vs rps

2-player	2-player
alternate turn	simultaneous move
deterministic algorithms	stochastic algorithms
analysis: minimax	analysis: minimax ?

how to find matrix game minimax strategy ?

warmup: 2×2 matrix game

find matrix game value

- our story so far ...
- matrix game value a.k.a. *Von Neumann equilibrium*
- Von Neumann's theorem: every matrix game has a value ...
- today: how to use linear programming to find that value
- lecture assumes you have read and understood

  Game Theory, A Playful Intro (Kent/Devos), chapter 5.1

matrix game Kent/Devos GT:playful intro, Ch. 5

```
      C         what happens with 1-choice strat   r1 vs c1 ?

    2  -1                                          r2 vs c1 ?

  R 1   2                                          r1 vs c2 ?

                                                   r2 vs c2 ?
```

```
now consider R plays mixed strategy  x * r1 + y * r2,

   where x,y are probabilities (0 <= x,y <= 1    x + y = 1)


     case 1) x * r1 + y * r2 versus c1 ?

     case 2) x * r1 + y * r2 versus c2 ?


     case 1) R exp. payoff  x * 2 + y * 1

     case 2) R exp. payoff  x *-1 + y * 2
```

Rose wants minimax stoch. strat

- **minimax** should be called maximin
  - for a fixed stoch. R-strat (x, y), a C-best response?

    R-exp-payoff-minimizing (over all 1-choice C-strats)

    here `min{ 2x + y, -x + 2y }`
  - R wants max (over all R-strats) best C-response (R-exp-payoff minimizing)
  - sometimes called R's best guaranteed expected payoff
  - here: `max (over (x,y)) min{ 2x + y, -x + 2y }`

```
max (over (x,y)):  min{ 2x + y, -x + 2y }
```

maximize z such that

$$z \leq 2x + y$$

$$z \leq -x + 2y$$

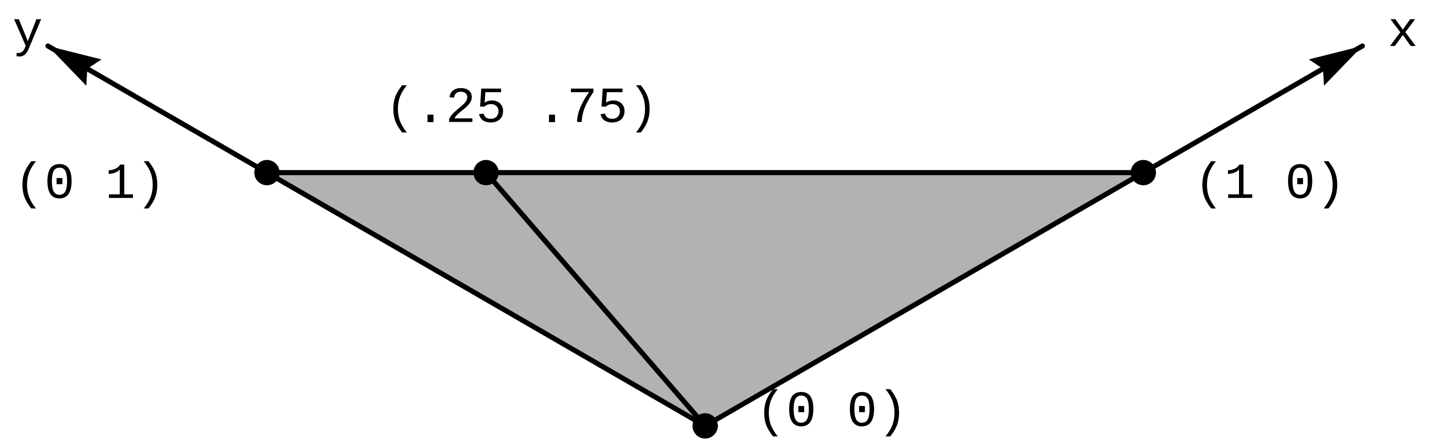
$$0 \leq x, y \leq 1$$

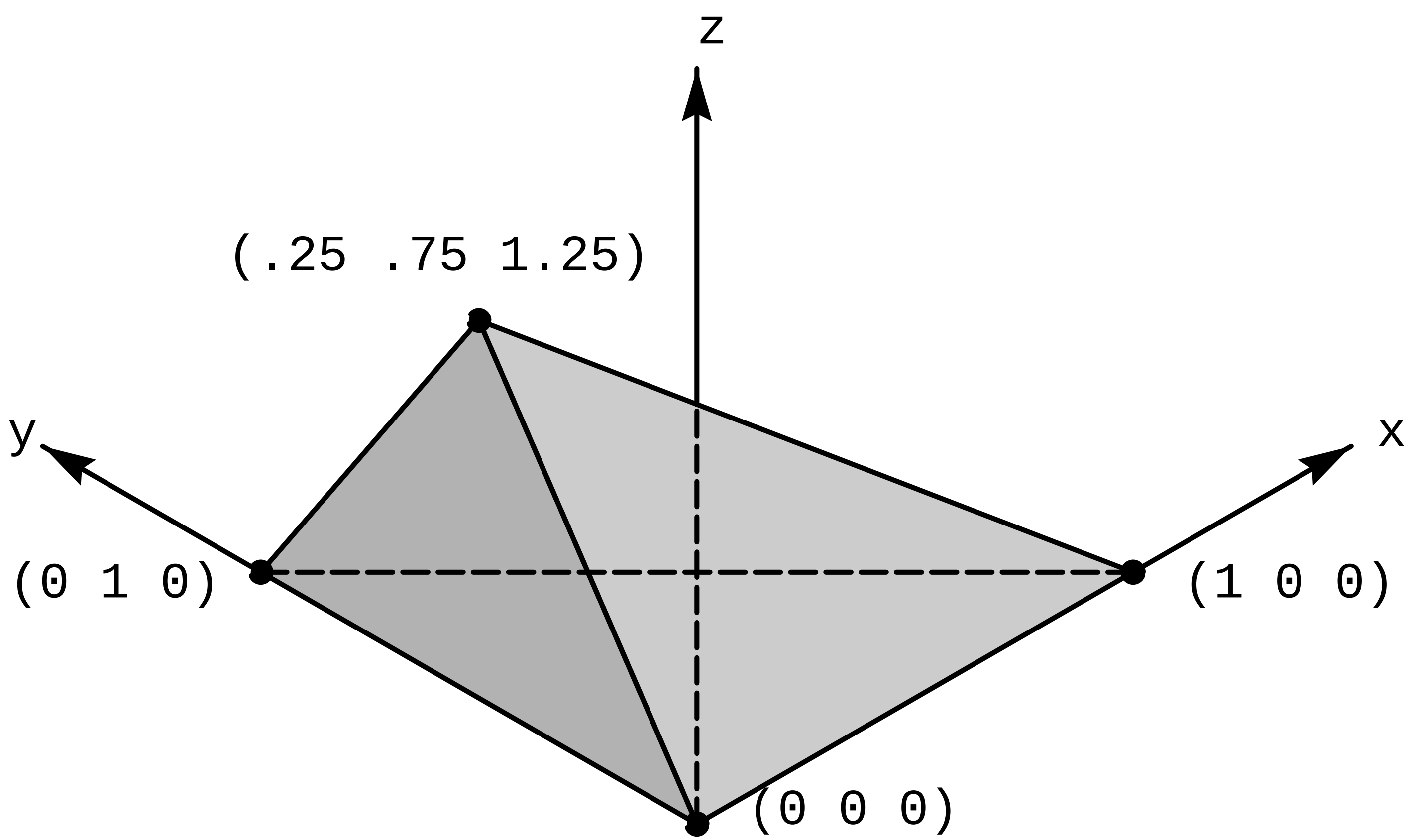
$$x + y = 1$$

how to solve 2-dimensional linear program

- try boundary of (x,y)-feasible region
- (0,0): $z = 0$
- (0,1): $z = \min \{2x + y = 1, -x + 2y = 2\} = 1$
- (1,0): $z = \min \{2x + y = 2, -x + 2y = -1\} = -1$
- (0,1) to (1,0) along $x + y = 1$?
- try $x + y = 1$ and $2x + y = -x + 2y$? see next page
- (1/4, 3/4): $z = \min \{2x + y = 5/4, -x + 2y = 5/4\} = 5/4$
- so R's minimax strat is (1/4, 3/4)

$$
\begin{aligned}
2x + y &= -x + 2y \\
3x &= y \\
x + y &= 1 \\
x + 3x &= 1 \\
4x &= 1 \\
x, y &= 1/4, 3/4
\end{aligned}
$$


y
x
(.25 .75)
(0 1)
(1 0)
(0 0)


z
(.25 .75 1.25)
y
x
(0 1 0)
(1 0 0)
(0 0 0)

2x2 matrix game conclusion

- R has a minimax strat, expected payoff 5/4
- exercise: show C has a minimax strat (3/4, 1/4), exp.-R-payoff 5/4
- so R can play so that R has exp payoff at least 5/4
- so C can play so that R has exp payoff at most 5/4
- so 5/4 is called the (minimax) value of this game
- this pair of minimax strats called Von Neumann equilibrium

every matrix game has a VN equilibrium

to solve n-dimensional linear program, use LP-solver

- evaluate at `https://sagecell.sagemath.org/`

```
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
x, y, z = v['x'], v['y'], v['z']
p.set_objective(z)
p.add_constraint(z <= 2*x +   y)
p.add_constraint(z <=  -x + 2*y)
p.add_constraint(x + y == 1)
p.add_constraint(x >= 0)
p.add_constraint(y >= 0)
p.solve()
p.get_values(z,x,y)
```

`you should get this output [1.25, 0.25, 0.75]`

- this tells us that R is guaranteed expected payoff 1.25 when she plays row 1 with probability .25 and row 2 with probability .75
- check this: her expected payoff against 1-choice strat column 1? column 2?
- can we verify that this is her maximum guaranteed expected payoff?
- yes: Von Neumann's theorem, which tells us that there will be a mixed strategy for Colin with guaranteed expected payoff (owing to Rose) at most 1.25
- let's use the same method as above to find an optimizing mixed strategy for Colin

- Colin wants to minimize his guaranteed expected payoff
- for any mixed Colin-strat (s,t) we assume Rose will play the maximizing 1-choice R-strat
- Colin wants to minimize `max{ 2s - t, s + 2t }`
- reformulate this as a maximization problem (for SageMath)
- Colin wants to maximize `min{ -2s + t, -s - 2t }`

$$\text{maximize } z \text{ such that}$$

$$z \leq -2s + t$$

$$z \leq -s - 2t$$

$$0 \leq s, t \leq 1$$

$$s + t = 1$$

- evaluate this program at `https://sagecell.sagemath.org/`

```
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=False)
s, t, z = v['s'], v['t'], v['z']
p.set_objective(z)
p.add_constraint(z <= -2*s + t)
p.add_constraint(z <=  -s - 2*t)
p.add_constraint(s + t == 1)
p.add_constraint(s >= 0)
p.add_constraint(t >= 0)
p.solve()
p.get_values(z,s,t)
```

- you should get this output

```
[-1.25, 0.75, 0.25]
```

- this tells us that C is guaranteed expected payoff -1.25 when he plays col 1 with prob .75 and col 2 with prob .25
- check this: his expected payoff against 1-choice strat row 1?   row 2?
- can we verify that this is his guaranteed expected payoff?
- yes, because R has a guaranteed expected payoff exactly the negative of this amount
- we have found, and verified, that value for this game is 1.25
  - in expected value, by following her mixed (.25, .75) strategy, R is guaranteed to win at least this amount against any 1-choice col strat
  - in expected value, by following his mixed (.75, .25) strategy, C is guaranteed to lose at most this amount against any 1-choice row strat

- another example

```
        C
    1   2   1
R   1   0   2
    3   1   0
```

- R wants to maximize her guaranteed expected payoff
- for any mixed R-strat (a,b,c), assume C plays a minimizing 1-choice C-strat
- R wants to maximize `min{ a + b + 3c, 2a + c, a + 2b}`
- R wants to

$$\text{maximize } z \text{ such that}$$

$$z \leq a + b + 3c$$

$$z \leq 2a + c$$

$$z \leq a + 2b$$

$$0 \leq a, b, c \leq 1$$

$$a + b + c = 1$$

- use sagemath to find
  - value of this matrix game
  - a minimax strategy for R
  - a minimax strategy for C

- another example

```
          C

     0    1  -2

R   -1    0   1

     1   -1   0
```