

acknowledgments

I first saw Hex around 1980 at a math colloquium after-party at Queen's University in Kingston hosted by my graph theory prof David Gregory. A few years later in Paris my Ph.D. supervisor Vašek Chvátal introduced me to Claude Berge, who showed me how to play.

In 2002 I started a computer Hex project at the University of Alberta in Edmonton funded by the Natural Sciences and Engineering Research Council of Canada. Thank you NSERC and Yngvi Björnsson, Michael Johanson, Maryia Kazakevich, Morgan Kan, Nathan Po, Jack van Rijswijck, Philip Henderson, Broderick Arneson, Aja Huang, Jakub Pawlewicz, Brad Thiessen, Henry Brausen, Jesse Huard, Kenny Young, Noah Weninger, Chao Gao, David Pankratz, Jacob Garber, Martin Müller and Jonathan Schaeffer for helping to build our bots and solvers.

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Ryan B. Hayward

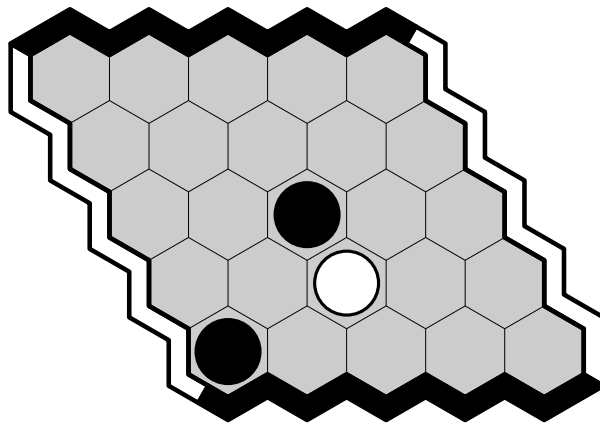
preface

Like chess and Go, Hex is a challenging two-player board game. Unlike chess and Go, the rules can be learned in a few seconds.

Hex rules are so simple that some game properties are easy to prove. For example, for any $n \times n$ board, the first player has a winning strategy. Can you prove that for chess or Go? Good luck, buddy!

This book is a quick introduction to Hex, the math behind the game, and its puzzles. Since 1942 — when Piet Hein invented the game — Hex has intrigued Jens Lindhard, Karen Thorborg, John F. Nash, David Gale, Claude Shannon, Martin Gardner, and many others.

Now it is your turn. Read on . . .



1

hex

Rule-Forty-two. All persons more than a mile high to leave the court.

The King, to Alice in Wonderland.

Here are the rules and some fundamental properties.

1.1 rules

The board is an n by m array of hexagons, usually with n equal to m . Two opposing board sides are black. Two opposing board sides are white. Figure 1.1 shows a 5×5 board.

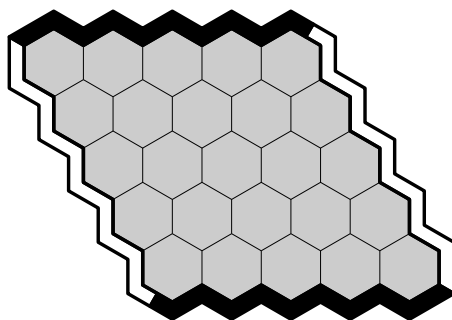


Figure 1.1. A 5×5 Hex board.

There are two players: Black uses black stones (or a black marker), White uses white stones (or a white marker). Players alternate turns. Black usually moves first. On a turn, a player puts a stone on (or marks) an empty cell. A player wins by joining the two sides of their color with a path of their stones (or marks). The path can twist and turn.

1.2 sample game

Figure 1.2 shows a sample 5×5 game. To start, Black opens in the center. In game notation, we write this as 1.B[c3], where 1 is the move number, B (Black) is the player who moved, and c3 is the cell where the move was made.

White tries to block Black with 2.W[c4]. Black takes the obtuse corner with 3.B[a5]. White splits Black's stones with 4.W[b4]. Black stops White from reaching the side with 5.B[a4]. White blocks Black's advance to the top with 6.W[b2], and the game continues.

1.B[c3] is actually a winning first move, but somehow Black blunders away the advantage, because this game ends with 22.W[e5]: White wins!

After move 1, would you have played this game differently? Black could have resigned after move 14: do you see why? How many errors in play can you find? We will answer these questions in a later chapter.

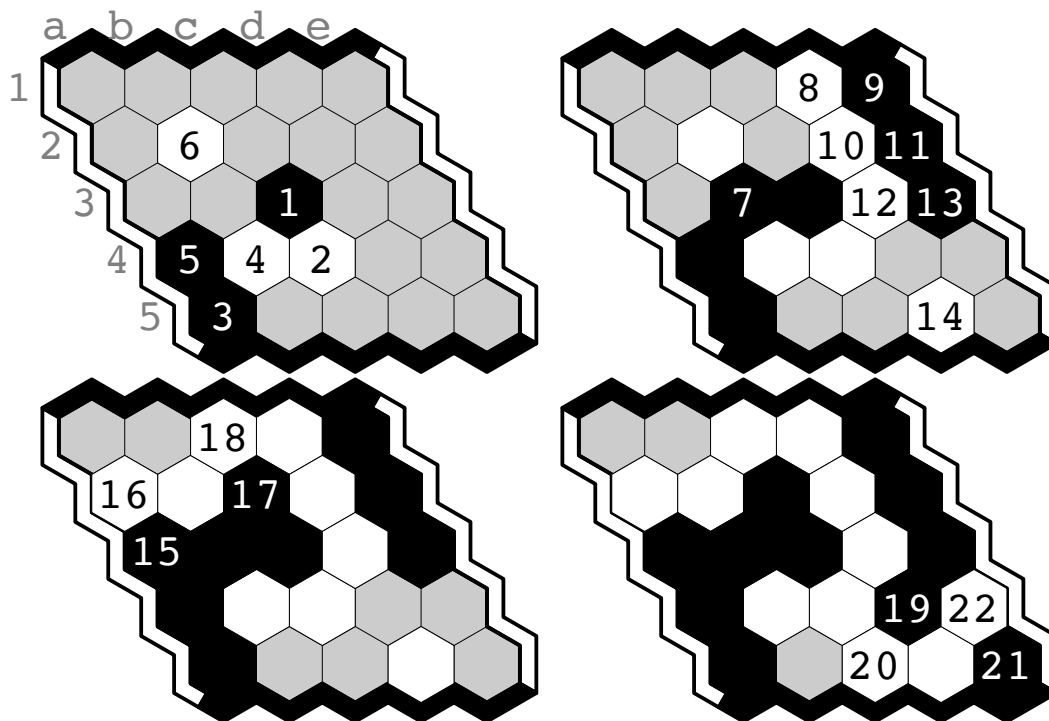


Figure 1.2. A sample game. White wins.

1.3 it's your turn

Find an opponent and play a game! Copy a board from Figure 1.1 or 1.3 or <https://webdocs.cs.ualberta.ca/~hayward/hexbook/hex.html#brd>. Or draw a simple board as in Figure 1.4. For that board, place stones on dots.

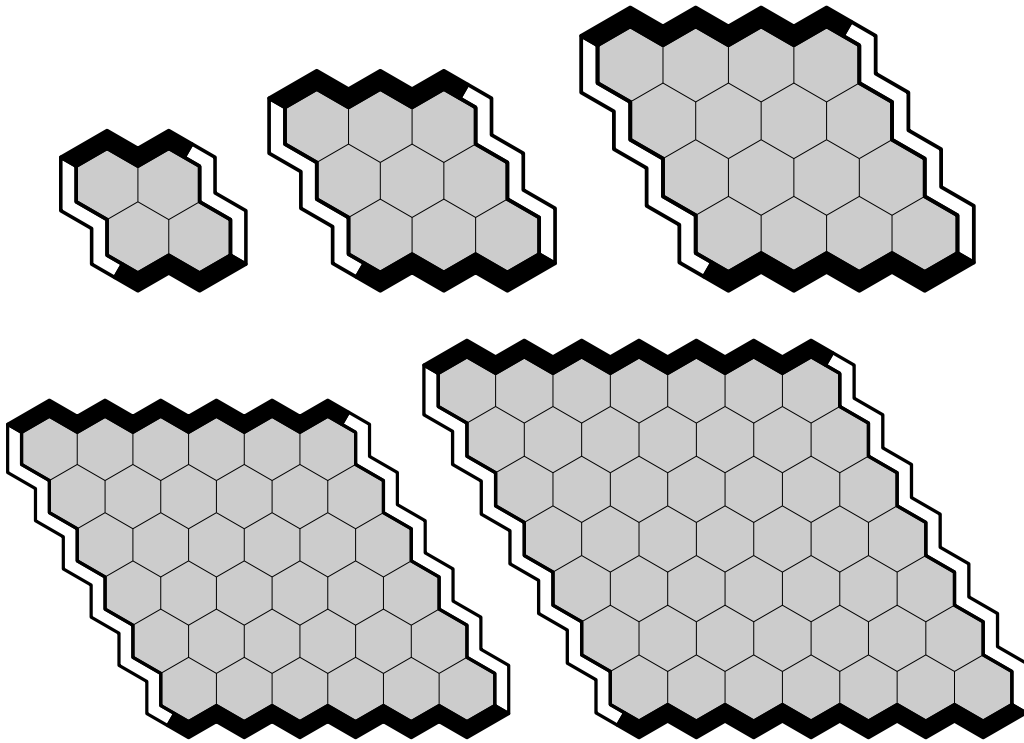


Figure 1.3. Hex boards.

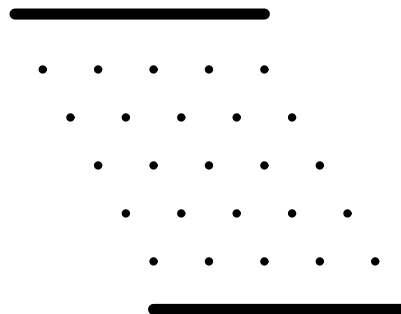


Figure 1.4. Draw a simple Hex board.

1.4 balancing the game

A strong player usually wins quickly against a weak player. To make games more fun, if your opponent is less experienced than you, allow them two or more moves to start, as in Figure 1.5. How would you finish this game?

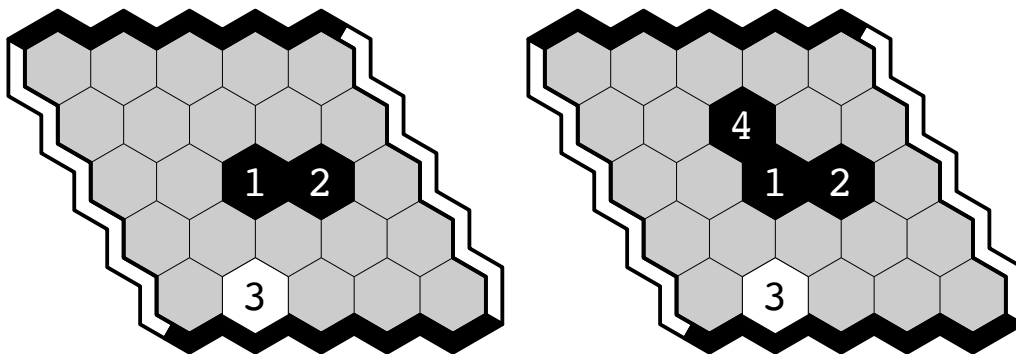


Figure 1.5. White allows Black two moves to start.

When two strong players play, the first player has an advantage. (We will prove this later!) To balance such games, you can use this extra *swap rule*:

- the first player makes the first move,
- the second player has the option — which they take if they think the first move is strong — to swap stones and sides with the first player,
- the player whose stone is not on the board makes the next move,
- from then, turns alternate as usual.

Figure 1.6 shows the start of a game of with the swap-rule. To start, Ann is Black, Kim is White. Ann opens 1.B[c3]. Now it is Kim's turn. Kim thinks that Ann's opening move is strong, so Kim swaps and the label at B[c3] changes from 1 (move 1) to S (swap here on move 2). Now Ann is White and Kim is Black and it is Ann's turn. Next Ann plays 3.W[b5]. In game notation, this game starts 1.B[c3] 2.W[swap] 3.W[b5].

Figure 1.2 shows a game in which the second player did not swap. In game notation, that game starts 1.B[c3] 2.W[d3] 3.B[a5].

The swap rule is often used in Hex tournaments, but in this book — unless we say otherwise — we will follow the usual Hex rules, so without swap.

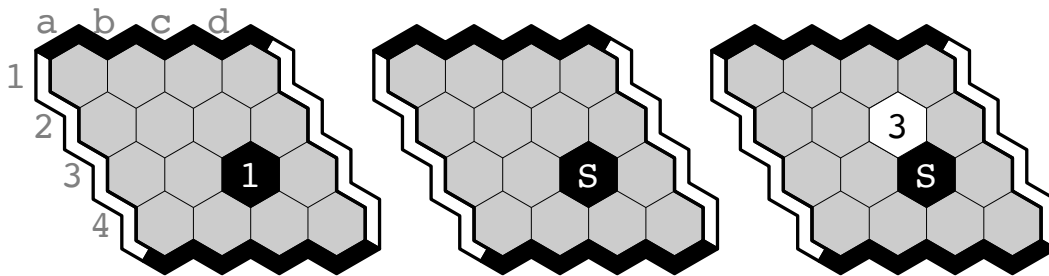


Figure 1.6. Start of a game with the swap rule.

1.5 things to know

As you learn Hex, keep these properties in mind:

- Draws are not possible. (Do you believe this? Can you prove it?) If neither player has yet joined their two sides, then there must be at least one empty cell, so the game continues.
- On $n \times n$ boards, the first player has a winning strategy. Finding such a strategy is easy up to 5×5 boards. Can you do this?
- On $n \times m$ boards with n not equal to m , the player whose sides are closer together has a winning strategy, even if the opponent plays first. Can you find such a strategy?
- The centermost cell is a strong opening move. This is especially noticeable on small boards.
- For arbitrary Hex positions, finding which of the two players has a winning strategy is computationally hard.
- Hex is fun! (This property is best proved by example.)

We will prove some of these properties in a later chapter. For now, try the puzzles.

1.6 puzzles

Enough chat, let's play! Here are some simple puzzles. Can you find all winning moves? Solutions on the next page.

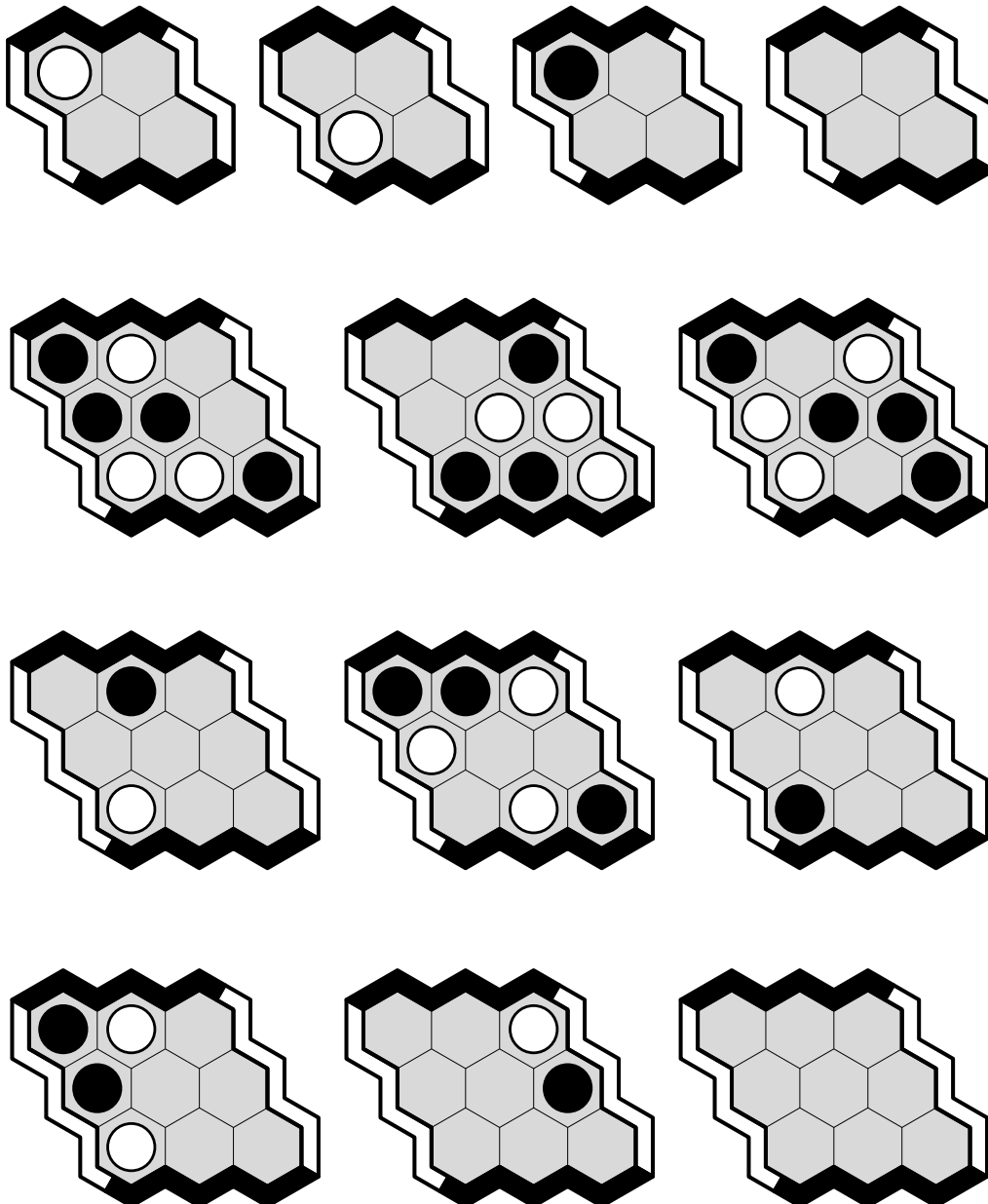


Figure 1.7. Puzzles. Black to play. For each puzzle, find all winning moves.

1.7 solutions

Here are all Black-winning moves. If no move wins, ? shows a move that most prolongs the game. The next chapter helps you find these solutions.

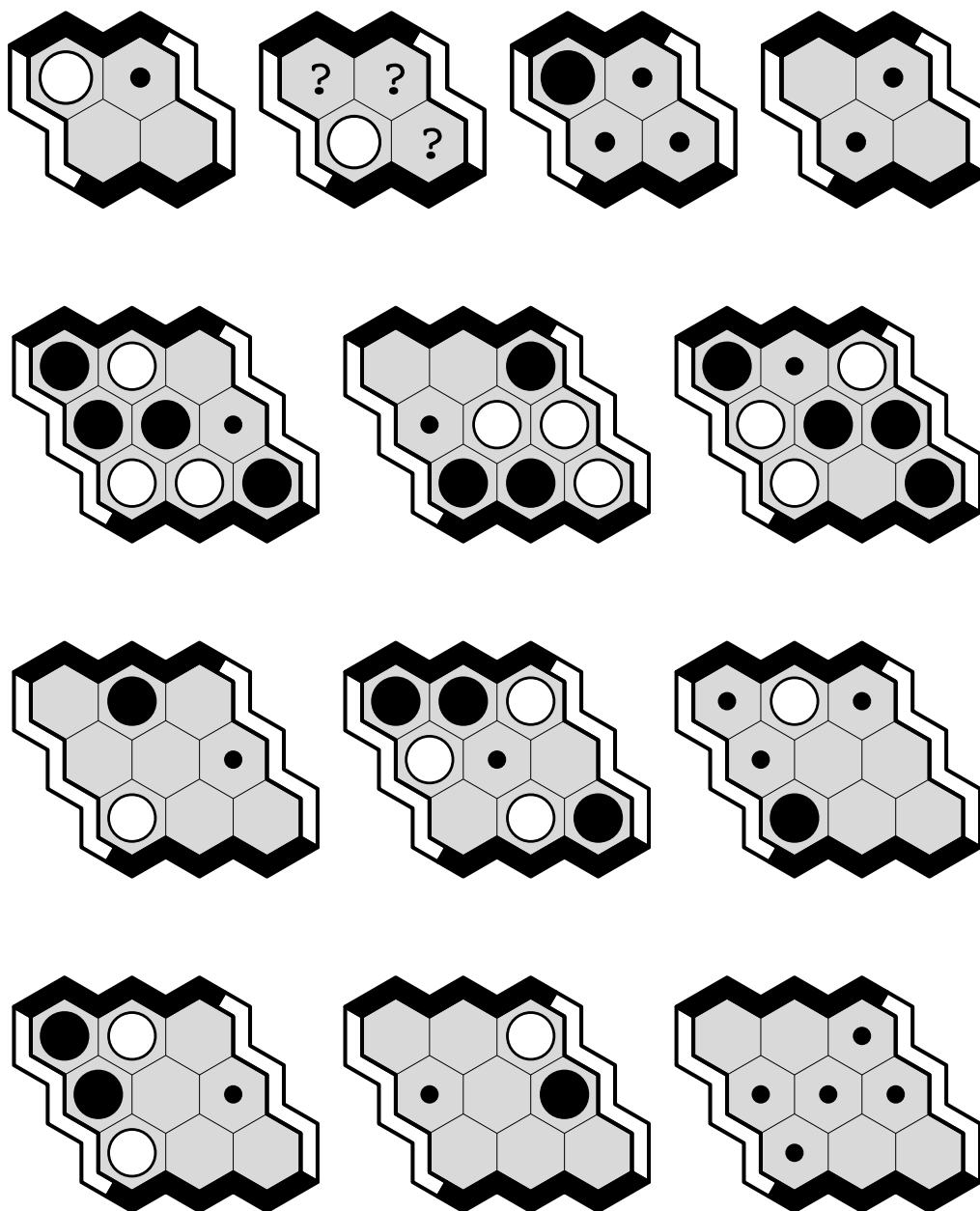


Figure 1.8. All Black-winning moves.