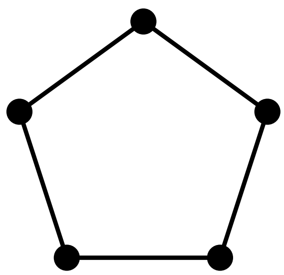
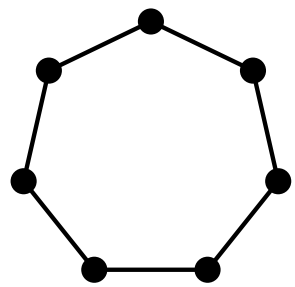
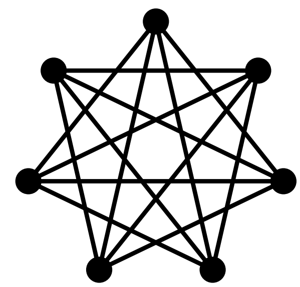
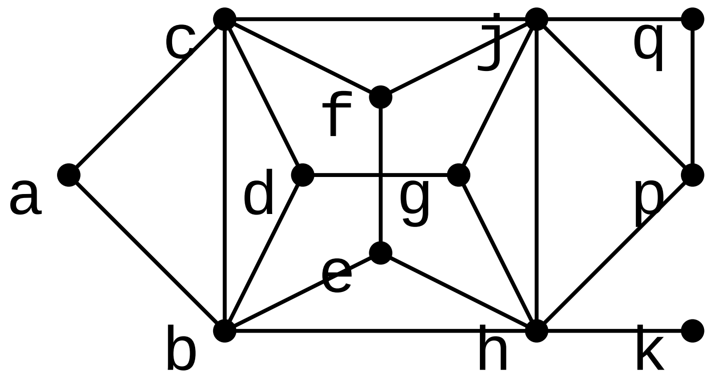
https://en.wikipedia.org/wiki/Perfect_graph

https://en.wikipedia.org/wiki/Strong_perfect_graph_theorem









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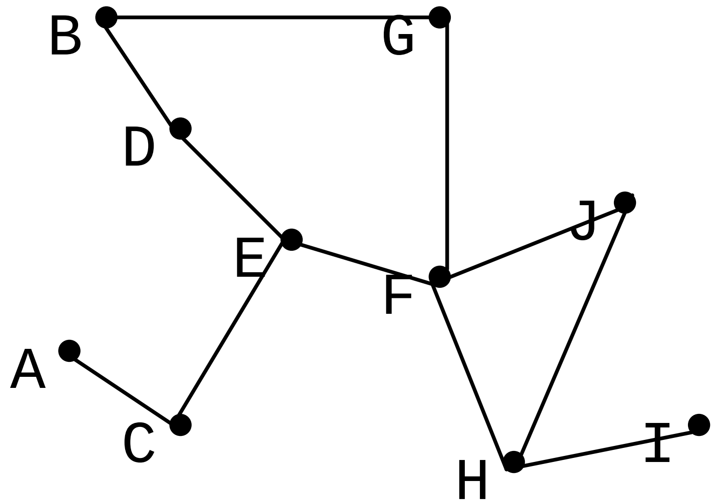
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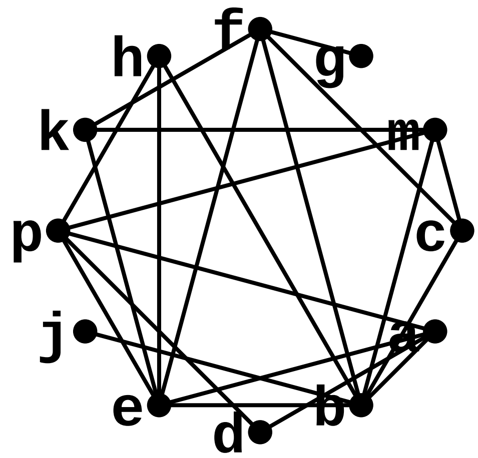
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* for a graph G = (V, E)

  VxV-E set of all unordered pairs of nodes of V that are not in E

  complement of G graph G = (V, VxV-E)

  chromatic_number min number colors needed to color nodes, s.t. adj't nodes get diff't colors

  clique_size size of largest clique

* for any subset S of V,

  E[S] all edges of E with both ends in S

  G[S] = (S, E[S]) subgraph of G induced by S

* G perfect for every subset S of V,

  chromatic number(G[S]) = clique_size(G[S])

* Strong Perfect Graph Theorem: G perfect iff

  neither G, comp(G) has induced odd cycle >= 5 nodes

Recall: a graph node is simplicial if its neighborhood is a clique.

observation

A simplicial node is not in an induced cycle with at least 5 nodes,

and not in the complement of such a cycle.

Proof of observation

$N(v)$ denotes the neighborhood of v.

Let v be in an induced odd cycle with at least 5 nodes. The cycle neighbors of v are non-adjacent and in $N(v)$ so v is not simplicial.

Let w be in the complement of an induced odd cycle with n nodes, where $n \geq 5$. If $n = 5$ then we are done by the previous argument, because the complement of a 5-cycle is a 5-cycle, so $n \geq 7$. Let the nodes of the cycle-complement C be $(\ldots u, v, w, x, y, \ldots)$.

In C all node pairs except for consecutive pairs are adjacent, e.g. v is non-adjacent to u and w and adjacent to x and y. Now x and y are non-adjacent and both in $N(v)$, so v is not simplicial.

exercise

1. which graphs on page 1 are perfect? justify each answer. also, for each graph, find its chromatic number ξ and clique size ω.

answers on the next page

1. You have two options: use the definition of perfect or use the Strong Perfect Graph Theorem (SPGT). If you use the definition, you will have to look at all non-empty induced subgraphs. There are $2^n - 1$ such subgraphs in each graphs with n nodes. It's usually less work to use the SPGT.

   The first three graphs are C_5 $(\xi 3, \omega 2)$, C_7 $(\xi 3, \omega 2)$, and $\overline{C}_7$, $(\xi 4, \omega 3)$, and so none of these are perfect.

   The next graph we saw before when we are talking about independent sets. Notice that a is simplicial, so not on an odd induced cycle with at least 5 nodes, nor the complement of such a cycle, so we can remove a from consideration. Similarly, k and q are simplicial so we can remove them. Now p is implicial in $G - a, k, q$, so we can remove p. Now you need to check that the remaining subgraph (induced by all the nodes except for a, k, q, p) contains no C_5, C_7, nor $\overline{C}_7$: the graph has only 8 nodes and so has no induced cycle or cycle-complement with 9 or more nodes. This graph is perfect ($\xi 3, \omega 3$).

The next graph we saw before when talking about TSP. It has a C_5 (B, D, E, F, G) so is not perfect ($\xi 3, \omega 3$).

The next graph we saw before when talking about isomorphism. It has a C_5 (c, f, e, p, m) so is not perfect.