

these notes are from

Chapter 7

Linear Programming and Reductions

in the class text

<http://algorithmics.lsi.upc.edu/docs/Dasgupta-Papadimitriou-Vazirani.pdf>

7.1.1 example: chocolate production

production * x_1 boxes/day of type P profit 1/box
* x_2 " N 6

demand * P at most 200 boxes/day
* N 300
* P+N 400

maximize $x_1 + 6x_2$

subject to $x_1 \leq 200$
 $x_2 \leq 300$
 $x_1 + x_2 \leq 400$
 $x_1, x_2 \geq 0$
 x_1, x_2 rational

.....

vector/matrix notation

c		1		x		x_1		A		1	0		b		200	
	6				x_2				0	1				300		
									1	1				400		

maximize $c^T x$
such that $A x \leq b$
 $x \geq 0$
 x rational

7.1.1 example: chocolate production

production * x₁ boxes/day of type P profit 1/box
* x₂ " N 6

demand * P at most 200 boxes/day
* N 300
* P+N 400

maximize x₁ + 6x₂

subject to x₁ ≤ 200
x₂ ≤ 300
x₁ + x₂ ≤ 400
x₁, x₂ ≥ 0
x₁, x₂ rational

solution: x₁ = 100, x₂ = 300, objective = 1900

proof of optimality?

$$\begin{array}{lll} x_2 & \leq 300 & 5*x_2 \leq 5*300 = 1500 \\ x_1 + x_2 \leq 400 & x_1 + x_2 \leq & 400 \\ \hline & & \\ \text{sum } x_1 + 6x_2 & \leq & 1900 \end{array}$$

magic LP trick :)

expanded problem: augment c, x, A, b

c		1		x		x1		A		1 0 0		b		200	
	6				x2				0 1 0				300		
	13				x3				1 1 1				400		

$$\begin{aligned} \text{max} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

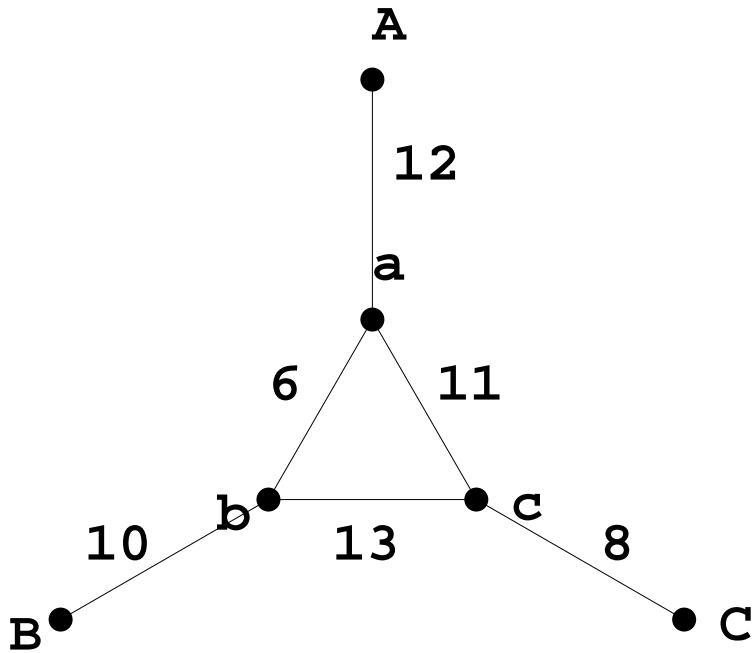
- evaluate at <https://sagecell.sagemath.org/>

```
p = MixedIntegerLinearProgram()
v = p.new_variable(real=True, nonnegative=True)
c, x, y, z = v["c"], v["x"], v["y"], v["z"]
p.set_objective(c)
p.add_constraint(c == x + 6*y + 13*z)
p.add_constraint(x <= 200)
p.add_constraint(y <= 300)
p.add_constraint(x + y + z    <= 400)
p.add_constraint(      y + 3*z <= 600)
round(p.solve(), 2)
p.get_values(c,x,y,z)
```

- you should get this output

```
[3100.0, 0.0, 300.0, 100.0]
```

7.1.3 example: optimum bandwidth allocation



- * users A B C
- * between each pair users, at least 2 units bandwidth
- * profit A-B \$3/unit bandwidth, B-C \$2/ub, A-C \$4/ub
- * inter-user communication routing:
direct (3 hops) or indirect (4 hops)

$$\max 3xAB + 3x'AB + 2xBC + 2x'BC + 4xAC + 4x'AC$$

$$\begin{aligned}
 \text{s.t. } & xAB + x'AB + xBC + x'BC && \leq 10 && \text{edge } (b,B) \\
 & xAB + x'AB && + xAC + x'AC && \leq 12 && \text{edge } (a,A) \\
 & && xBC + x'BC + xAC + x'AC && \leq 8 && \text{edge } (c,C) \\
 & xAB + && x'BC + && x'AC && \leq 6 && \text{edge } (a,b) \\
 & x'AB + xBC + && && x'AC && \leq 13 && \text{edge } (b,c) \\
 & x'AB + && x'BC + xAC && \leq 11 && \text{edge } (a,c) \\
 & xAB + x'AB && && && \leq 2 && \\
 & && xBC + x'BC && && \leq 2 && \\
 & & & && xAC + x'AC && \leq 2 && \\
 & xAB, x'AB, xBC, x'BC, xAC, x'AC && \geq 0 && && &&
 \end{aligned}$$

7.1.4 variants, standard inequality form

* max cx ...	can be max or min
* s.t. $3x_1 - 2x_9 \leq 3$	$\leq, \geq, =$
* $x_1 \geq 0$	$\geq 0, \leq 0$, unrestricted

less-equal form:	standard form:
max cx	max cx
s.t. $Ax \leq b$	s.t. $Ax = b$
$x \geq 0$	$x \geq 0$

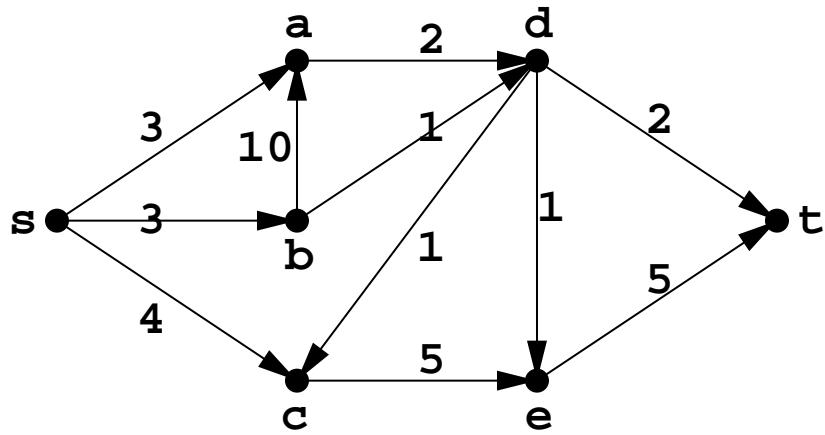
e.g. put this into standard less-equal form	$\min 3x_1 - 4x_2$
	s.t. $x_1 + x_2 = 7$
	$x_1 - x_3 \geq 8$
	$x_1, x_3 \geq 0$

max $-3x_1 + 4x_2$
s.t. $x_1 + x_2 \leq 7$
 $-x_1 - x_2 \leq -7$
 $-x_1 + x_3 \leq -8$
 $x_1, x_3 \geq 0$

max $-3x_1 + 4(x_4 - x_5)$
s.t. $x_1 + (x_4 - x_5) \leq 7$
 $-x_1 - (x_4 - x_5) \leq -7$
 $-x_1 + x_3 \leq -8$
 $x_1, x_3, x_4, x_5 \geq 0$

max $-3x_1 + 4x_4 - 4x_5$
s.t. $x_1 + x_4 - x_5 \leq 7$
 $-x_1 - x_4 + x_5 \leq -7$
 $-x_1 + x_3 \leq -8$
 $x_1, x_3, x_4, x_5 \geq 0$

7.2.2 max flow



network input

- * source s, terminus t
- * on each arc, capacity

flow

- * on each arc, non-negative flow $f(\text{arc})$ (e.g. 73 litre/s) that does not exceed $\text{capacity}(\text{arc})$
- * at each node except for source and terminus,
 - * $\text{in-flow}(\text{node})$ [sum, over in-arcs j , of $f(j)$] = $\text{out-flow}(\text{node})$ [sum, over out-arcs k , of $f(k)$]

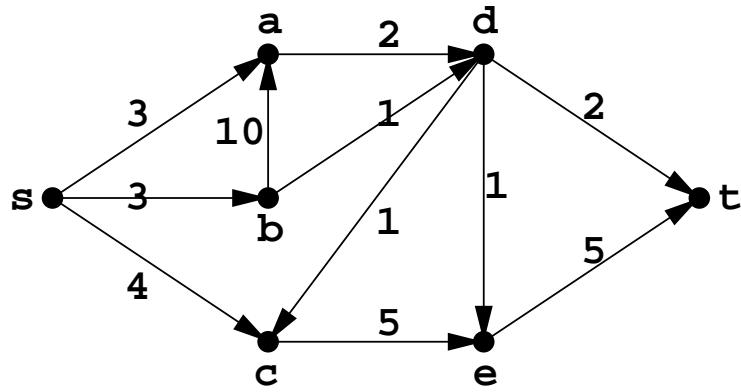
flow volume

- * $\text{out-flow}(\text{source})$, or $\text{in-flow}(\text{terminus})$

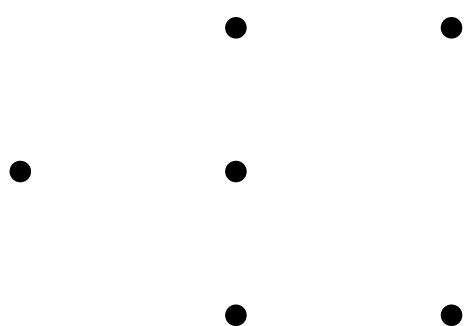
problem:

- * given network, find a max volume flow

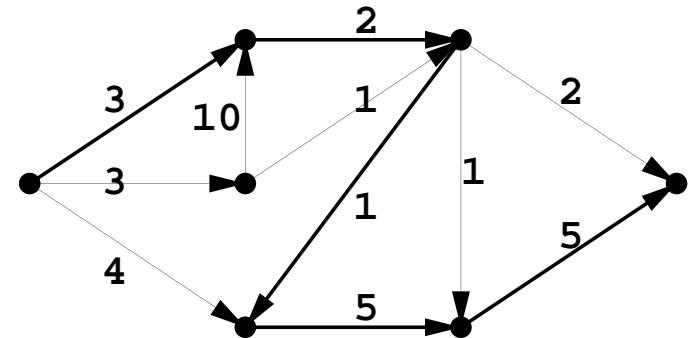
network



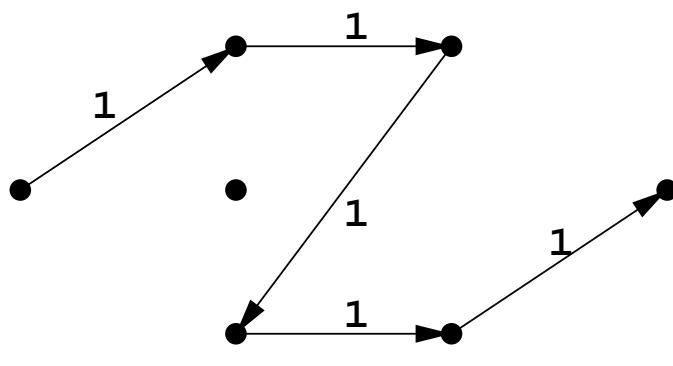
flow 0



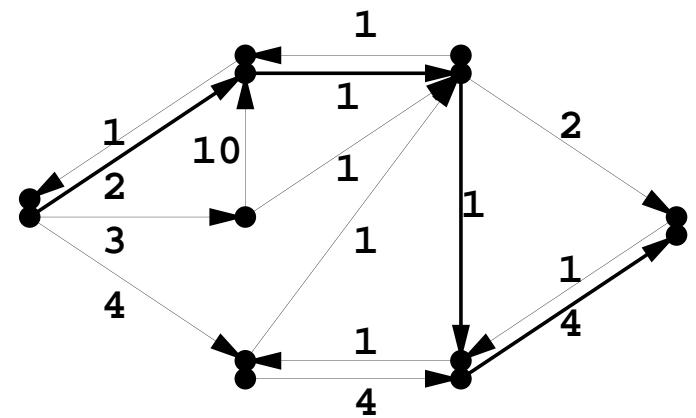
residual 0



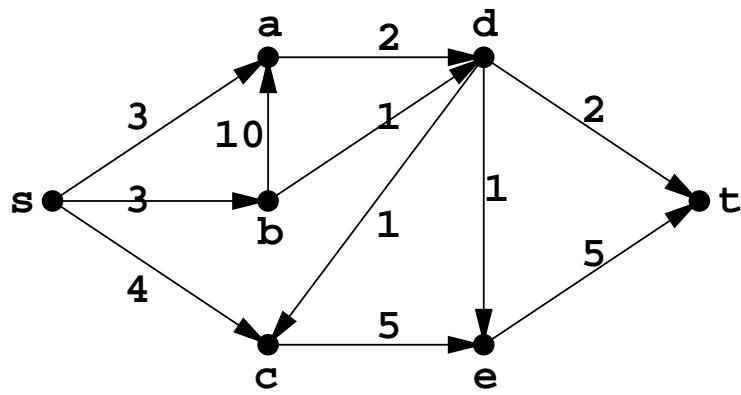
flow 1



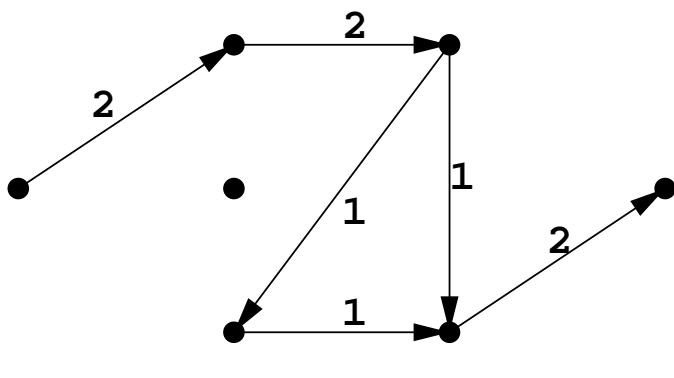
residual 1



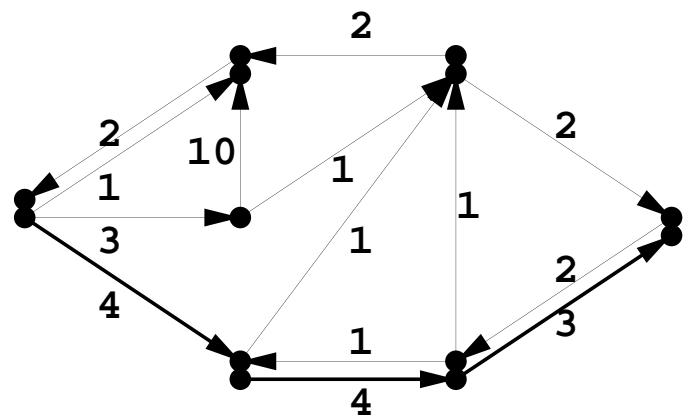
network



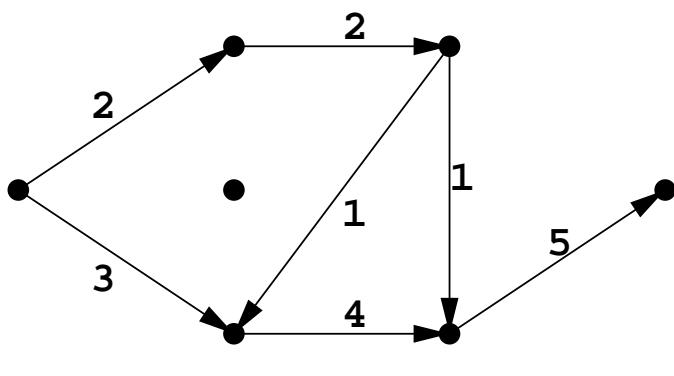
flow 2



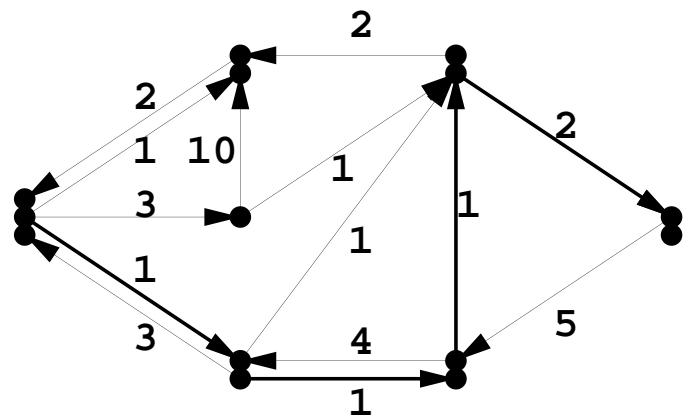
residual 2



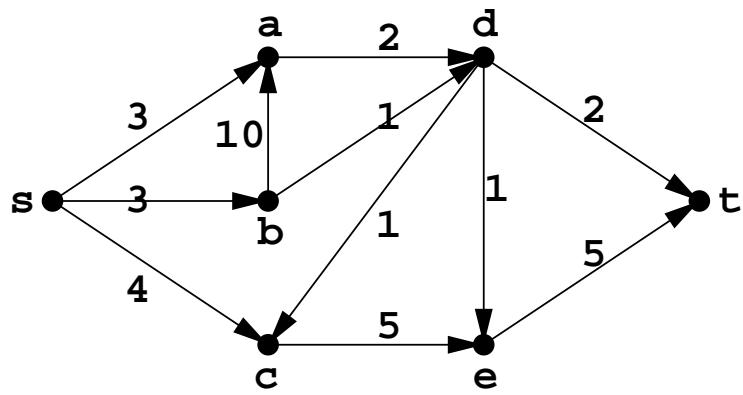
flow 3



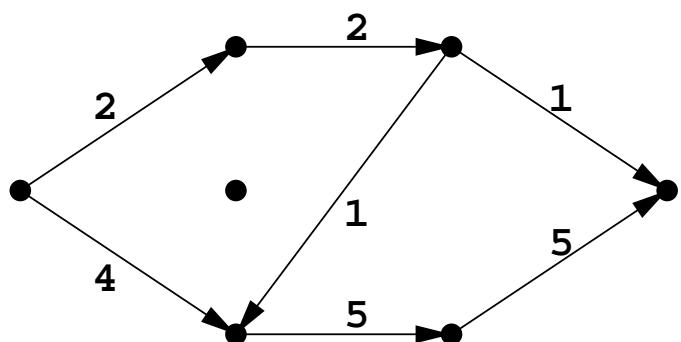
residual 3



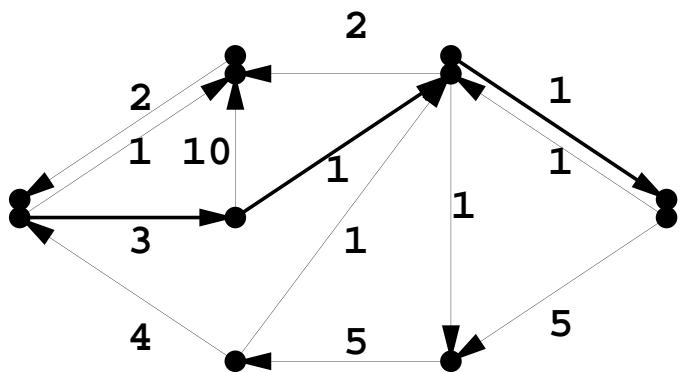
network



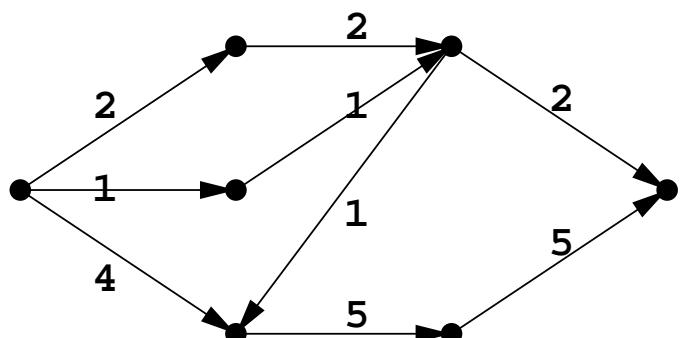
flow 4



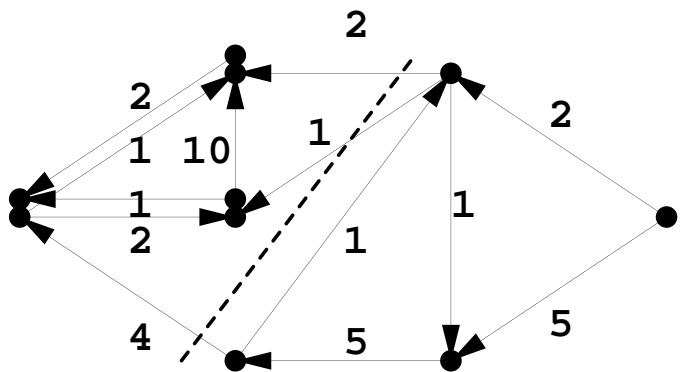
residual 4

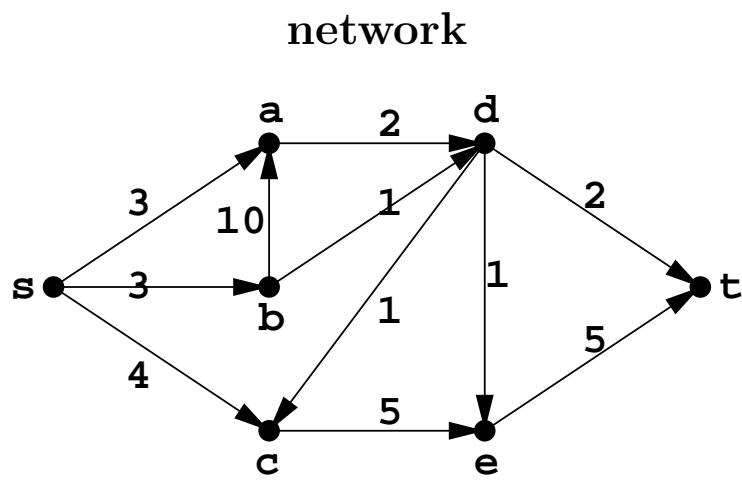


flow 5

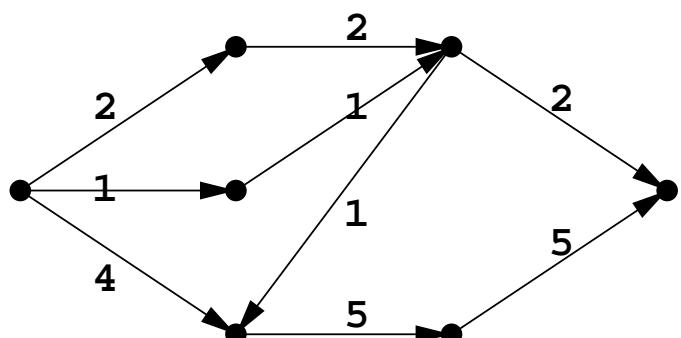


residual 5

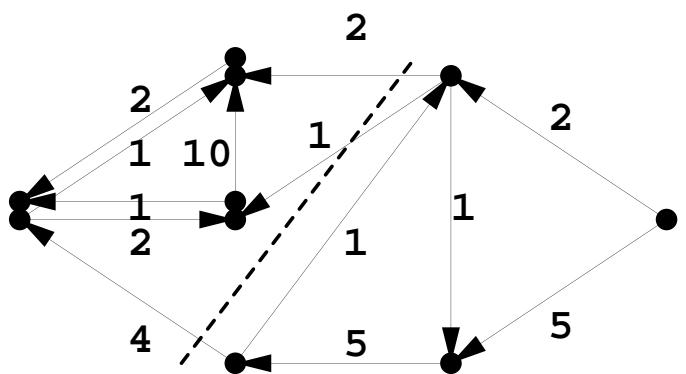




flow 5



residual 5



min cut?

in final residual network,

{ nodes reachable from s }	{ other nodes }
----------------------------	-----------------

here

{ s, a, b }	{ c, d, e, t }
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why?

7.2.2 max flow transforms to LP

```
variables sa sb sc ad ba bd ce dc de dt et  
  
max      sa + sb + sc          flow out of source  
                  flow conservation  
s.t.      sa + ba = ad          node a  
          sb = ba + bd          b  
          sc + dc = ce          c  
          ad + bd = dc + de + dt    d  
          ce + de = et          e  
                                capacities  
          sa <= 3  
          sb <= 3  
          sc <= c  
          ad <= 2  
          ba <= 10  
          bd <= 1  
          ce <= 5  
          dc <= 1  
          de <= 1  
          dt <= 2  
          et <= 5  
  
          sa,sb,sc,ad,ba,bd,ce,dc,de,dt,et >= 0
```

conclusion: max flow polytime transforms into LP

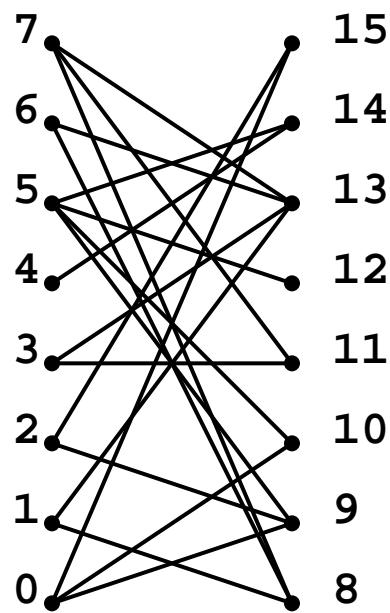
max flow integrality observation:

if all capacities are integer, then there exists
an all-integer max flow (why?)

7.3 max bipartite matching

instance: bipartite graph $G = (V_0, V_1, E)$

query: what is the size of a maximum matching in G ?



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