Quiz 2-II Version G&H

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1. For each of (i)q=5 (ii)q=3 (iii)q=4, use the master method to give tight asymptotic bounds for the following recurrence relation[In each case, you must specify the values of the relevant parameters in the master method.]

   $$T(n) = \begin{cases} 3 & if\ n = 1 \\ 8T(n/q) + \sqrt{n^3}) & if\ n > 1 \end{cases}$$

   q=5 $a = 8, b = 5, f(n) = n^{3/2}$

   i) $\exists \varepsilon > 0$ s.t. $f(n) = n^{3/2} \in \Omega(n^{\log_5 8+\varepsilon})$.
   Because $\log_5 8 < \log_5 10 = \log_5 \sqrt{100} < \log_5 \sqrt{125} = 3/2$ and n^x is strictly increasing when $n > 1$. So $\exists c = 1 > 0$, $\forall n > 1$ s.t. $0 \leqslant cn^{\log_5 8+\varepsilon} \leqslant f(n)$ where $0 < \varepsilon < 3/2 - \log_5 8$.
   ii) $8f(n/5) = \frac{8}{\sqrt{125}} \times n^{3/2} \leqslant \frac{8}{\sqrt{125}} \times n^{3/2}$, so $af(n/b) \leqslant cf(n)\ \forall n \geqslant 1$ where $c = \frac{8}{\sqrt{125}} < 1$.
   So $T(n) \in \Theta(f(n)) = n^{3/2}$

   q=3 $a = 8, b = 3, f(n) = n^{3/2}$

   $\exists \varepsilon > 0$ s.t. $f(n) = n^{3/2} \in O(n^{\log_3 8-\varepsilon})$. Because $\log_3 8 > \log_3 6 = \log_3 \sqrt{36} > \log_3 \sqrt{27} = 3/2$ and n^x is strictly increasing when $n > 1$. So $\exists c = 1 > 0$, $\forall n > 1$ s.t. $0 \leqslant f(n) \leqslant c\dot{n}^{\log_3 8+\varepsilon}$ where $0 < \varepsilon < \log_3 8 - 3/2$.
   So $T(n) \in \Theta(n^{\log_3 8})$

   q=4 $a = 8, b = 4, f(n) = n^{3/2}$

   $f(n) = n^{3/2} \in \Theta n^{3/2} = \Theta(n^{\log_4 8})$ as $3/2 = \log_4 8$.
   So $T(n) \in \Theta(n^{\log_4 8} \lg n) = \Theta(n^{3/2} \lg n)$

2. In a heap with location 1 through n, what is *version H*) the smallest location which is a leaf? *version G*) the largest location which is not a leaf? Prove you answer is correct using heap properties.

   Proof:

   Version H A non-leaf node has at least one child which is $2i$. So if $2i > n$ the node i is a leaf node.

   $2i > n$

   $\implies i > n/2$

   $\implies i > \lfloor n/2 \rfloor$ because of i being an integer and the definition of $\lfloor x \rfloor$.

   $\implies i \geqslant \lfloor n/2 \rfloor + 1$ as i is an integer

   The smallest leaf node is $\lfloor n/2 \rfloor + 1$ as it is the accurate low bound of $i(min(i) = \lfloor n/2 \rfloor + 1)$.

   Version G A non-leaf node has at least one child which is $2i$. So i is a non-leaf node when $2i \leqslant n$.

   $2i \leqslant n$

   $\implies i \leqslant n/2$

   $\implies i \leqslant \lfloor n/2 \rfloor$ because of i being an integer and the definition of $\lfloor x \rfloor$.

   So $max(i) = \lfloor n/2 \rfloor$.