CMPUT 204 — Problem Set 1

Topics covered in Part I are mathematical background (Appendix A, pages 1058–1069; Appendix C, pages 1094–1126), mathematical induction and loop invariant, running time (function) comparison, and insertion sort; in Part II are algorithm analysis using insertion sort as an example and algorithm design using merge sort as an example, growth of functions, and asymptotic notations.

It is highly recommended that you read pages **1–61**, **1058–1069**, and **1094–1126** very carefully and do **all** the exercises. The following are some of them that you are **REQUIRED** to practice on.

Quiz questions are mostly based on this list, with some minor modifications necessary. Consult your instructor and TAs if you have any problem with this list.

Part I

1. The following theorem is attributed to Nicomachus (c. 100 A.D.):

$$\begin{array}{rcl} 1^3 & = & 1, \\ 2^3 & = & 3+5, \\ 3^3 & = & 7+9+11, \\ 4^3 & = & 13+15+17+19, \\ & \cdots & \text{et cetera } \cdots \end{array}$$

   (a) Give the expression of n^3 for general n.

   (b) Prove your expression.

2. The following proof by induction seems correct, but for some reason the equation for $n = 6$ gives $5/6$ on the left-hand side, and $4/3$ on the right-hand side. Can you find a mistake?

   THEOREM: $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)\times n} = \frac{3}{2} - \frac{1}{n}$.

   Proof: We use induction on n. For $n = 1$, $\frac{1}{1\times 2} = \frac{3}{2} - \frac{1}{n}$; and, assuming the theorem is true for n,

$$\begin{array}{rl} & \frac{1}{1\times 2} + \cdots + \frac{1}{(n-1)\times n} + \frac{1}{n\times (n+1)} \\ = & \frac{3}{2} - \frac{1}{n} + \frac{1}{n(n+1)} \\ = & \frac{3}{2} - \frac{1}{n} + (\frac{1}{n} - \frac{1}{n+1}) \\ = & \frac{3}{2} - \frac{1}{n+1}. \end{array}$$

3. How can $\log_2 n$ be computed on a pocket calculator that computes only natural logarithms of the form $\ln x$? Justify your answer. Start from the basic definition of logarithms.

4. Five distinct elements are randomly chosen from integers between 1 and 20, and stored in a sorted list $L[1]$, ..., $L[5]$. Using linear search we want to determine if an integer x (also chosen randomly from integers between 1 and 20) belongs to the list L.

   (a) What is the number of comparisons required on the average?

   (b) Give a similar analysis as in the first part if L has n elements and all numbers are selected from integers between 1 and m.

5. P13, Prob 1-1.

   Using functions $\lg n$, n, n^2, 2^n, and times 1 second, 1 day, 1 century.

6. P21, Ex 2.1-1.

7. P21, Ex 2.1-3.

8. P21, Ex 2.1-4.

Part II

1. Give, in big-O notation, the worst case number of times the operation $+$ is performed. You may assume that all variables are integers. Justify your answers.

   (a) Procedure $f_1(n)$, n is an integer

   ```
   for i ← 1 to n − 1 do
       for j ← i + 1 to n do
           for k ← 1 to j do
               x ← a + b + c
   ```

   (b) Procedure $f_2(n)$, n is an integer

   ```
   for i ← 1 to n do
       if odd(i) then
           for j ← 1 to n do
               x ← x + 1
           for j ← 1 to n do
               y ← y + 1
   ```

2. For each of the following statements give an adequate proof to show when it is true or an adequate counterexample to show when it is false. You may assume that all functions are positive non-decreasing.

   (a) If $f(n)+g(n) \in O(n^2)$ then $f(n) \in \Theta(n^2)$.

   (b) If $\frac{f(n)}{g(n)} \in \Theta(\lg n)$ then $g(n) \in o(f(n))$.

   (c) If $g_1(n) \in O(f_1(n))$ and $g_2(n) \in \Theta(f_2(n))$ then $g_2(n)^{g_1(n)} \in O(f_2(n)^{f_1(n)})$.

   (d) For any positive constant c, $f(cn) \in \Theta(f(n))$. (Hint: consider some of the fast-growing functions).

   (e) If $f(n) \in O(n^2)$ and $g(n) \in O(n^2)$ then $\frac{f(n)}{g(n)} \in O(1)$.

   (f) If $f(n) \in \Theta(n^3)$ and $g(n) \in \Theta(n^2)$ then $\frac{f(n)}{g(n)} \in \Theta(n)$.

   (g) $n^\alpha \in \Theta(n \;\; log\ n)$ for some $\alpha > 1$.

   (h) $n^k \in \Omega(n)$ for $k > 1$.

   (i) If $f(n) > g(n)$ then $f(n) + g(n) \in \Theta(f(n))$.

   (j) If $f(n) \in \Theta(n)$ and $g(n) \in \Theta(n)$ then $2^{f(n)} \in O(2^{g(n)})$.

   (k) $log\ n \in o(n^\alpha)$ for any $\alpha > 0$.

   (l) $log(n^{0.1}) \in o(log\ n)$.

   (m) If $f(n) \in \Theta(g(n))$ then $f(n) - g(n) \in \Theta(g(n))$.

   (n) $\sum_{i=2}^{n-1} log_2 i \in \Theta(n\ log\ n)$.

   (o) If $f(n) \in O(n^2)$ and $g(n) \in O(n^2)$ then $\frac{f(n)}{g(n)} \in O(1)$.

   (p) If $f(n) \in \Theta(n^3)$ and $g(n) \in \Theta(n^2)$ then $\frac{f(n)}{g(n)} \in \Theta(n)$.

   (q) $n^\alpha \in \Theta(n \;\; log\ n)$ for some $\alpha > 1$.

   (r) $n^k \in \Omega(n)$ for $k > 1$.

   (s) If $f(n) > g(n)$ then $f(n) + g(n) \in \Theta(f(n))$.

3. P27, Ex 2.2-2.
4. P27, Ex 2.2-3.
5. P36, Ex 2.3-1.
6. P36, Ex 2.3-3.
7. P36, Ex 2.3-4.
8. P36, Ex 2.3-6.
9. P38, Prob 2-2.
10. P50, Ex 3.1-1.
11. P50, Ex 3.1-5.
12. P58, Prob 3-3.

    Using functions n, $\lg^* n$, $\lg(n!)$, 2^n, $n!$, n^2, $\lg^2 n$, $\left(\frac{3}{2}\right)^n$, $2^{\lg n}$, $n \lg n$.